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# **An incompressible diffuse model with phase transition in the sharp interface limit**

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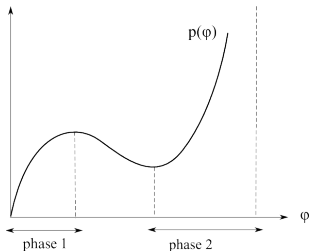
## Outline

- Description of the model
- Asymptotic setting
- Sharp interface limit:
  - Euler regime
  - Navier-Stokes regime

## Introduction of the model

We consider an incompressible Navier-Stokes-Korteweg model for  $(\varphi, \mathbf{v}, \lambda)$

$$\begin{aligned}\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) &= c_+ (m_j \Delta - m_r) (c_+ \mu(\varphi) + c_- \lambda), \\ \rho(\varphi) (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) + \frac{1}{M^2} \nabla p(\varphi) + \nabla \lambda &= \frac{1}{\operatorname{Re}} (\operatorname{div}(\eta \operatorname{div} \mathbf{v} \mathbf{1} + \hat{\eta} (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ &\quad + \frac{C}{M^2} \varphi \nabla \Delta \varphi), \\ \operatorname{div} \mathbf{v} &= c_- (m_j \Delta - m_r) (c_+ \mu(\varphi) + c_- \lambda),\end{aligned}$$



Domain:  $(t, \mathbf{x}) \in \Omega \times (0, T)$ ,  $\Omega \subset \mathbb{R}^d$

Boundary conditions:

$$\mathbf{v}|_{\partial\Omega} = 0$$

$$\partial_n (c_+ \mu + c_- \lambda)|_{\partial\Omega} = \partial_n \varphi|_{\partial\Omega} = 0$$

Initial conditions:

$$(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_I, \varphi_I)$$

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The compressible Navier-Stokes-Korteweg model for  $(\rho, \mathbf{v})$ :

$$\begin{aligned}\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) + \frac{1}{M^2} \nabla p(\rho) &= \frac{1}{\operatorname{Re}} (\operatorname{div}(\eta \operatorname{div} \mathbf{v} \mathbf{1} + \hat{\eta}(\nabla \mathbf{v} + \nabla \mathbf{v}^T))) + \frac{C}{M^2} \rho \nabla \Delta \rho.\end{aligned}$$

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**Model H** by Hohenberg and Halperin'77 for  $(c, \mathbf{v}, \lambda)$ :

$$\begin{aligned}\partial_t c + \operatorname{div}(c \mathbf{v}) &= m_j \Delta \mu(c), \\ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla \lambda &= \frac{1}{\operatorname{Re}} (\operatorname{div}(\hat{\eta}(\nabla \mathbf{v} + \nabla \mathbf{v}^T))) - \frac{C}{M^2} \operatorname{div}(\nabla c \otimes \nabla c), \\ \operatorname{div} \mathbf{v} &= 0.\end{aligned}$$

We consider two incompressible true mass densities:

$$\tilde{\rho}_1 = \text{constant} \quad \tilde{\rho}_2 = \text{constant}$$

Then the mixture has a mass density:

$$\rho(\varphi) = \tilde{\rho}_1 \frac{1 + \varphi}{2} + \tilde{\rho}_2 \frac{1 - \varphi}{2}$$

This is motivated from the partial mass densities for an homogenous mixture of two constituents:

$$\rho = \rho_1 + \rho_2 = \frac{m_1}{V} + \frac{m_2}{V} = \tilde{\rho}_1 \frac{V_1}{V} + \tilde{\rho}_2 \frac{V_2}{V}$$

Here the phase field variable  $\varphi$  is a quantity which is volume fraction related:

$$\frac{V_1}{V} = \frac{1 + \varphi}{2}.$$

Balance equations for the basic variables  $(\varphi, \mathbf{v}, \lambda)$ :

$$\begin{aligned}\frac{\partial \varphi}{\partial t} + \operatorname{div}(\varphi \mathbf{v} + c_+ \mathbf{j}) &= c_+ r, \\ \rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v} \times \nabla) \mathbf{v}) - \operatorname{div}(\boldsymbol{\sigma}_{NS}) &= \operatorname{div}(\boldsymbol{\sigma}_K) \\ \operatorname{div}(\mathbf{v}) &= c_-(r - \operatorname{div}(\mathbf{j}))\end{aligned}$$

Constitutive relations complete the system:

$$\begin{aligned}\mathbf{j} &= -m_j \nabla(c_+ \mu + c_- \lambda), \\ r &= -m_r (c_+ \mu + c_- \lambda), \\ \mu &= \left( \frac{\partial f}{\partial \varphi} - \nabla \cdot \frac{\partial f}{\partial \nabla \varphi} \right), \\ \boldsymbol{\sigma}^K &= -\nabla \varphi \otimes \frac{\partial f}{\partial \nabla \varphi} + (f - \varphi \mu) \mathbf{1}, \\ \boldsymbol{\sigma}^{NS} &= -\lambda \mathbf{1} + \eta(\varphi) \operatorname{div}(\mathbf{v}) + \hat{\eta}(\varphi) (\nabla \mathbf{v} + \nabla \mathbf{v}^T).\end{aligned}$$

The constants:

$$m_j, m_r, c_{\pm} \geq 0$$

And the free energy:

$$f = f(\varphi, \nabla \varphi)$$

## Free energy and the Maxwell points

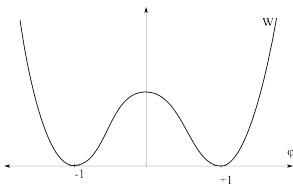
We consider a scaled free energy density in terms of the phase fraction

$$\varphi : [0, \infty) \times \Omega \rightarrow (-\varepsilon - 1, 1 + \varepsilon):$$

$$f(\varphi, \nabla\varphi) = W(\varphi) + \frac{\gamma\varepsilon^2}{2} |\nabla\varphi|^2,$$

with

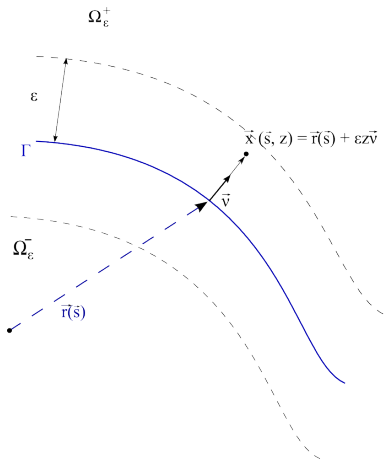
- $W(\varphi) \geq 0$  is a double-well potential,
- $W : \mathbb{R} \rightarrow [0, \infty)$  with  
 $W(-1) = W(1) = 0$



### Maxwell construction.

Due to the choice of  $W$  above the Maxwell points of  $W$  are given by  $\varphi^1 = -1$  and  $\varphi^2 = 1$ .

This scaling of the free energy characterizes an interfacial region with a thickness of order  $\varepsilon$ .



- The set of  $C^{1,2}$ -hypersurfaces:

$$\Gamma(t; \varepsilon) = \{\mathbf{x} \in \Omega : \varphi_\varepsilon(t, \mathbf{x}) = 0\}.$$

↓ zero level set limit

$\Gamma$

- A local parametrization of  $\Gamma$ :

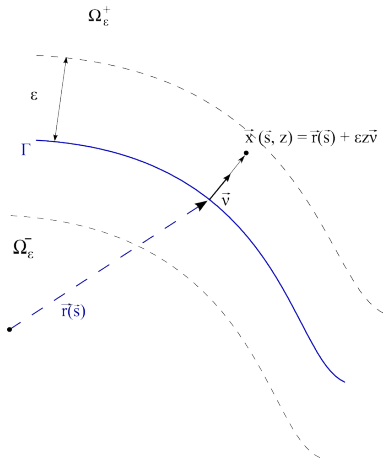
$$\mathbf{r}(t, \mathbf{s}) : I \times U \rightarrow \mathbb{R}^d$$

- A new coord. in the neighbourhood of  $\Gamma$ :

$$\mathbf{x}(t, \mathbf{s}, z) := (t, \mathbf{r}(t, \mathbf{s}) + \varepsilon z \boldsymbol{\nu}),$$

- The normal component of the interface velocity :

$$w_\nu = \partial_t \mathbf{r} \cdot \boldsymbol{\nu}.$$



- Bulks  $\Omega_\varepsilon^\pm \iff$  outer setting

$$(\varphi_\varepsilon(t, \mathbf{x}), \mathbf{v}_\varepsilon(t, \mathbf{x}), \lambda_\varepsilon(t, \mathbf{x}))$$

$\Downarrow$

expansion in  $\varepsilon$

- For a general field  $f$

$$f(t, \mathbf{x}) = F(t, \mathbf{s}, z)$$

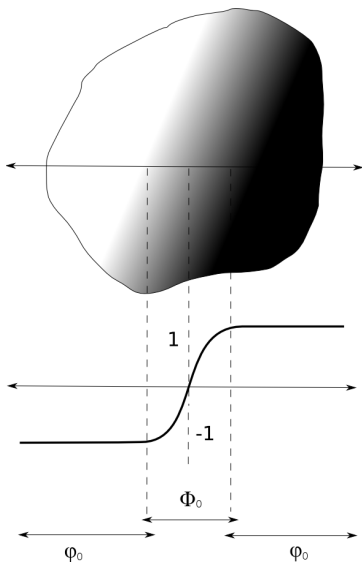
- Interfacial layer  $\Gamma_\varepsilon \iff$  inner setting

$$(\Phi_\varepsilon(t, \mathbf{s}, z), \mathbf{V}_\varepsilon(t, \mathbf{s}, z), \Lambda_\varepsilon(t, \mathbf{s}, z))$$

$\Downarrow$

expansion in  $\varepsilon$

- Also expand  $(\mu_\varepsilon, p_\varepsilon)$  and  $(M_\varepsilon, P_\varepsilon)$ .



### ■ Matching:

Large  $z$  behavior of inner variables

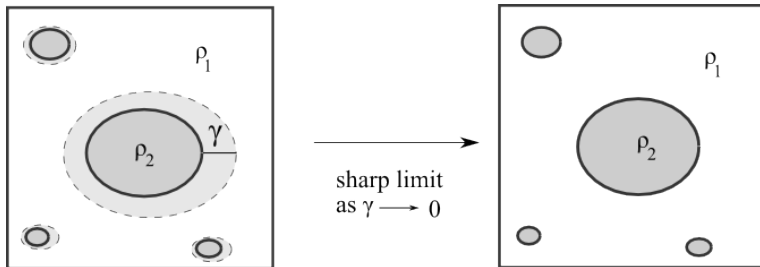
$\approx$

Close to  $\Gamma$  behavior of outer variables

### ■ Term-by-term matching:

$$\Phi_0(\mathbf{s}, \pm\infty) \sim \varphi_0|_{\Gamma}$$

$$\Phi_1(\mathbf{s}, \pm\infty) \sim \varphi_1|_{\Gamma} + \nabla\varphi_0|_{\Gamma} \cdot \boldsymbol{\nu}$$



Outer setting  $\implies$  PDEs in bulks for  $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \mu_0, \mu_1, p_0, p_1)$ .

Inner setting  $\implies$  **solvable** ODEs for  $(\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0, M_0, M_1, P_0, P_1)$

Solvability cond. for the inner setting + matching cond.  $\implies$  interface conditions.

Thus we end up with **PDEs for bulks** and **boundary conditions at the interface**.

## Diffuse model in Euler (E) regime

We consider low Mach number and low viscosity (high Reynolds number) regime

$$M \sim \varepsilon^{1/2}, \quad \frac{1}{\text{Re}} \sim \varepsilon.$$

Scaled Navier-Stokes-Korteweg/Cahn-Hilliard system reads

$$\begin{aligned} \partial_t \varphi + \text{div}(\varphi \mathbf{v}) &= \frac{1}{\varepsilon} c_+ (m_j \Delta - m_r) (c_+ \mu(\varphi) + \varepsilon c_- \lambda), \\ \rho(\varphi) (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) + \frac{1}{\varepsilon} \nabla p(\varphi) + \nabla \lambda &= \varepsilon \nabla (\eta(\varphi) \text{div} \mathbf{v}) + \varepsilon \text{div}(\hat{\eta}(\varphi) (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ &\quad + \gamma \varepsilon \varphi \nabla \Delta \varphi, \\ \text{div} \mathbf{v} &= \frac{1}{\varepsilon} c_- (m_j \Delta - m_r) (c_+ \mu(\varphi) + \varepsilon c_- \lambda), \end{aligned}$$

with the following constitutive relations for the chemical potential  $\mu(\varphi)$  and the non-monotone pressure  $p(\varphi)$ :

$$\mu(\varphi) = W'(\varphi) - \gamma \varepsilon^2 \Delta \varphi, \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

The outer system for  $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \mu_0, \mu_1, p_0, p_1)$ :

$$\begin{aligned}(m_j \Delta - m_r) \mu_0 &= 0, \\ \nabla p_0 &= 0, \\ \partial_t \varphi_0 + \operatorname{div}(\varphi_0 \mathbf{v}_0) &= c_+ (m_j \Delta - m_r) (c_+ \mu_1 + c_- \lambda_0), \\ \rho(\varphi_0) (\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0) + \nabla p_1 + \nabla \lambda_0 &= 0, \\ \operatorname{div} \mathbf{v}_0 &= c_- (m_j \Delta - m_r) (c_+ \mu_1 + c_- \lambda_0).\end{aligned}$$

The inner system is an ODE system w.r.t.  $z$ :

$$\mathcal{L}[\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0, M_0, M_1, P_0, P_1] = \mathbf{F}$$

**Note:** Equations for  $M_0$ , and  $P_0$  decouples from the system above and can be independently solved:

$$M_0 \equiv 0, \quad P_0 \equiv 0.$$

$M_0$  satisfies the following:

$$\partial_{zz} M_0 = 0, \quad M_0 = W'(\Phi_0) - \gamma \partial_{zz} \Phi_0 \quad \Rightarrow \quad M_0 = 0.$$

### Proposition

Let  $\mu_0 \in C([0, T]; C^2(\bar{\Omega}^\pm))$  satisfy the outer system, and the boundary conditions

$$\begin{aligned} \nabla \mu_0 \cdot \nu &= 0 && \text{on } \partial\Omega, \\ \mu_0|_\Gamma &= \lim_{z \rightarrow \pm\infty} M_0(\mathbf{s}, z) && \text{on } \Gamma, \end{aligned}$$

where  $M_0 \in C([0, T]; C^2(\bar{U} \times \mathbb{R}))$  satisfies inner equations with  $\partial_z M_0 \rightarrow 0$  for  $z \rightarrow \pm\infty$ . Then,

- $\mu_0$  vanishes in whole  $\Omega$ ,
- in particular, we have  $\varphi_0 \in \{-1, +1\}$ .

### Proof.

■ Inner equation + Maxwell construction  $\implies M_0 \equiv 0$ .

■ Then we observe that the system

$$\begin{aligned}(m_j \Delta - m_r) \mu_0 &= 0 && \text{in } \Omega^- \cup \Omega^+, \\ \nabla \mu_0 \cdot \nu &= 0 && \text{on } \partial\Omega, \\ \mu_0|_{\Gamma} &= 0 && \text{on } \Gamma.\end{aligned}$$

admits a unique solution,  $\mu_0 \equiv 0$ , due to the maximum principle.

■ Recall that  $\mu_0 = W'(\varphi_0)$ .

■ Then we have  $\varphi_0|_{\Omega^\pm} = \pm 1$ .



## Reduction of the outer system

Due to the proposition above, we have a reduced bulk system for the tuple  $(\varphi_1, \mathbf{v}_0, \lambda_0)$

$$\begin{cases} (m_j \Delta - m_r)(c_+ \mu_1 + c_- \lambda_0) = 0 \\ \tilde{\rho}_i(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0) \nabla p_1 + \nabla \lambda_0 = 0 \\ \operatorname{div} \mathbf{v}_0 = 0 \end{cases}$$

where  $i = 1, 2$  denotes two phases occurring in  $\Omega^\pm$ , respectively. Now we can identify  $\Phi_0$  uniquely as follows:

### Corollary

Let  $\Phi_0 \in C^1([0, T] \times U; C^3(\mathbb{R}))$  satisfy inner equations with  $\Phi_0 \rightarrow \pm 1$  as  $z \rightarrow \pm\infty$  and  $\Phi_0(t, \mathbf{s}, 0) = 0$ . Then the function  $\Phi_0$  is unique and independent of  $t$  and  $\mathbf{s}$ .

### Proof.

It directly follows from  $M_0 = 0$  such that the equation

$$\gamma \partial_{zz} \Phi_0 - W'(\Phi_0) = 0 \quad \text{with} \quad \Phi_0 \rightarrow \pm 1 \quad \text{for} \quad z \rightarrow \pm\infty,$$

admits a unique solution  $\Phi_0(t, \mathbf{s}, z) =: \Phi_0(z)$  which obeys  $\Phi_0(0) = 0$ . □

- Matching inner and outer settings yields a jump condition for the velocity field:

$$[\mathbf{v}_0 \cdot \boldsymbol{\nu}]_{-}^{+} = \frac{c_{-}}{c_{+}} (\langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle_{-}^{+} - w_{\boldsymbol{\nu}})$$

Here, the jump of a quantity  $f$  across the interface and the mean value:

$$[f]_{-}^{+} = f^{+} - f^{-} \quad \text{and} \quad \langle f \rangle_{-}^{+} = f^{+} + f^{-}.$$

- In the case of identical fluids, i.e.  $\tilde{\rho}_1 = \tilde{\rho}_2$ , we recover the continuity of the velocity at the interface, that is  $[\mathbf{v}_0 \cdot \boldsymbol{\nu}]_{-}^{+} = 0$
- The continuity of the mass flux through the interface as follows:

$$[\rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]_{-}^{+} = 0 \quad \text{with} \quad \rho_0^{+} = \tilde{\rho}_1, \rho_0^{-} = \tilde{\rho}_2$$

- The normal velocity of the interface is then given by:

$$2w_{\boldsymbol{\nu}} = \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle_{-}^{+} + \frac{m_j}{2} c_{+} [c_{+} \nabla \mu_1 \cdot \boldsymbol{\nu} + c_{-} \nabla \lambda_0 \cdot \boldsymbol{\nu}]_{-}^{+}$$

- Inner variables  $(P_1, M_1)$  are explicitly  $\Phi_1$  related. Furthermore we have the relation

$$c_+ \partial_z M_1 + c_- \partial_z \Lambda_0 = 0 \quad \implies \quad \tilde{L}\Phi_1 = \tilde{f}.$$

- We further perturb  $\Phi_1$  such that:  $\Psi = \Phi_1 - \tilde{\Phi}_1$  with  $\Psi \in W^{3,1}$ .

- Therefore we have a an operator  $L : W^{3,1} \rightarrow L^1$  defined as

$$L\Psi := -(c_+ - c_- \Phi_0) \partial_z (W''(\Phi_0)\Psi - \gamma \partial_{zz} \Psi)$$

satisfying the equation

$$L\Psi = f(\Phi_0, V_0, \tilde{\Phi}_1)$$

- where the right hand side is given by

$$\begin{aligned} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1) &= -c_- \partial_z (J_0 \mathbf{V}_0 \cdot \boldsymbol{\nu}) + c_- \partial_z ((\eta(\Phi_0) + 2\hat{\eta}(\Phi_0)) \partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu}) \\ &\quad + \kappa \gamma (c_+ - c_- \Phi_0) \partial_{zz} \Phi_0 + (c_+ - c_- \Phi_0) \partial_z (W''(\Phi_0) \tilde{\Phi}_1 - \gamma \partial_{zz} \tilde{\Phi}_1). \end{aligned}$$

### Proposition

The equation  $L\Psi = f$  admits a solution if and only if

$$\int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1) dz = 0, \quad \int_{-\infty}^{\infty} \frac{f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)}{c_+ - c_- \Phi_0} dz = 0$$

holds. This means, in particular,

$$\begin{aligned} [j_0 \mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]_-^+ &= C_1(\Phi_0) \kappa \gamma, \\ [\mu_1]_-^+ &= \frac{\tilde{p}_2 - \tilde{p}_1}{2} \left( \frac{1}{2} c_- c_+ j_0^2|_{\Gamma} + C_2(\Phi_0) j_0|_{\Gamma} \right). \end{aligned}$$

**Remark 1.** The coefficients are given by

$$C_1(\Phi_0) = \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz, \quad C_2(\Phi_0) = \int_{-\infty}^{\infty} (\eta(\Phi_0) + 2\hat{\eta}(\Phi_0)) \left( \frac{1}{\rho(\Phi_0)} \right)_z^2 dz.$$

**Remark 2.** We have no longer classical Gibbs-Thompson law valid, but a jump for the chemical potential.

The proof essentially follows from the Fredholm alternative theorem:

$$L\Psi = f \quad \text{is solvable if and only if} \quad \int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)\phi \, dz = 0$$

for all solutions  $\phi(z) \in L^\infty(\mathbb{R})$  of the homogeneous problem for the adjoint operator  $L^* : L^\infty(\mathbb{R}) \rightarrow W^{-3,1}(\mathbb{R})$  given by

$$L^* \phi = (W''(\Phi_0)\partial_z - \gamma\partial_{zzz})((c_+ - c_- \Phi_0)\phi).$$

Therefore, we need to find all linearly independent solutions to  $L^* \phi = 0$  which are in  $L^\infty(\mathbb{R})$ .

- First we observe that every constant is a solution, i.e.  $\phi_1 = \text{constant}$ .
- The other two solutions can be found by reducing the system

$$\phi_2(z) = \frac{c_+/c_-}{c_+ - c_- \Phi_0}, \quad \phi_3(z) = \int_0^z \phi_2'(\tilde{z}) \int_0^{\tilde{z}} \frac{c_+ - c_- \Phi_0}{(\partial_z \Phi_0)^2} d\tilde{z}$$

- Observe  $\phi_1, \phi_2 \in L^\infty(\mathbb{R})$  and  $\phi_3 \notin L^\infty(\mathbb{R})$

## The sharp interface limit in the Euler regime

In  $\Omega^\pm$ , respectively, for  $i = 1, 2$ ,

$$\begin{aligned}(m_j \Delta - m_r)(c_+ \mu_1 + c_- \lambda_0) &= 0, \\ \tilde{\rho}_i (\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0) + \nabla p_1 + \nabla \lambda_0 &= 0, \\ \operatorname{div} \mathbf{v}_0 &= 0,\end{aligned}$$

At the interface

$$\begin{aligned}[j_0 \mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]_-^+ &= \gamma C_1 \kappa, \\ [\mathbf{v}_0 \cdot \boldsymbol{\nu}]_-^+ &= \frac{c_-}{c_+} \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} - w_\nu \rangle_-^+, \\ c_- [\lambda_0]_-^+ &= -c_+ [\mu_1]_-^+, \\ [\mu_1]_-^+ &= \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \left( \frac{1}{2} c_- c_+ j_0^2 |_\Gamma + j_0 |_\Gamma C_2 \right), \\ 2w_\nu &= \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle_-^+ + \frac{m_j}{2} c_+ [c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}]_-^+, \end{aligned}$$

Implicitly given the continuity of mass flux:  $[\rho_0 (\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_\nu)]_-^+ = 0$ .

- During the phase separation (or transition) it might occur that  $\mathbf{v}_0 \cdot \boldsymbol{\nu} = w_\nu$ , i.e.  $j_0 = 0$

$$\begin{aligned} [p_1]_{-}^{+} &= \kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz, \\ [\mathbf{v}_0]_{-}^{+} &= [\mu_1]_{-}^{+} = [\lambda_0]_{-}^{+} = 0, \\ \mu_1|_{\Gamma} &= \frac{1}{2}\kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz, \\ w_\nu &= \mathbf{v}_0|_{\Gamma} \cdot \boldsymbol{\nu}, \end{aligned}$$

where we observe that due to the last relation the chemical potential  $\mu_1$  is expected to decouple from the system. However, we have the Lagrange multiplier coupled to the diffusion equation for  $\mu_1$ .

$$(m_j \Delta - m_r)(c_+ \mu_1 + c_- \lambda_0) = 0$$

- In the case of the identical fluids, i.e.  $\tilde{\rho}_2 = \tilde{\rho}_1$ , the terms with  $c_-$  drop out. The chemical potential decouples from the system. Therefore, the whole sharp interface model in the limit recovers the free boundary problem of Euler type.

## Diffuse model in Navier-Stokes (N-S) regime with similar densities

Similar densities restriction:  $\tilde{\rho}_2 - \tilde{\rho}_1$  is only  $\varepsilon$  comparable quantity.

Further, we compare the Mach number and Reynolds number as follows:

$$M \sim \varepsilon^{1/2}, \quad \frac{1}{\text{Re}} \sim 1.$$

which means no viscosity restriction in this case.

Scaled Navier-Stokes-Korteweg/Cahn-Hilliard system reads

$$\begin{aligned} \partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) &= \frac{1}{\varepsilon} c_+ (m_j \Delta - m_r) (c_+ \mu(\varphi) + \varepsilon c_- \lambda), \\ \rho(\varphi) (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) + \frac{1}{\varepsilon} (\nabla p(\varphi) + \nabla \lambda) &= \nabla(\eta(\varphi) \operatorname{div} \mathbf{v}) + \operatorname{div}(\hat{\eta}(\varphi) (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ &\quad + \gamma \varepsilon \varphi \nabla \Delta \varphi, \\ \operatorname{div} \mathbf{v} &= c_- (m_j \Delta - m_r) (c_+ \mu(\varphi) + \varepsilon c_- \lambda), \end{aligned}$$

with the constitutive relations for the chemical potential  $\mu$  and the non-monotone pressure  $p$ :

$$\mu(\varphi) = W'(\varphi) - \gamma \varepsilon^2 \Delta \varphi, \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

### Proposition

The leading order chemical potential  $\mu_0$  vanishes in whole  $\Omega$ . In particular, we have

$$\varphi_0 \in \{-1, +1\}.$$

This reduces the outer system:

$$\left\{ \begin{array}{l} (m_j \Delta - m_r)(c_+ \mu_1 + c_- \lambda_0) = 0, \\ \nabla \lambda_0 = 0, \\ \rho_0(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0) + \nabla p_1 + \nabla \lambda_1 = \operatorname{div}(\hat{\eta}(\varphi_0)(\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^T)), \\ \operatorname{div} \mathbf{v}_0 = 0, \end{array} \right.$$

for the tuple  $(\varphi_1, \mathbf{v}_0, \lambda_0, \lambda_1)$  where  $\rho_0 = \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2}$ . Furthermore, we can identify  $\Phi_0 = \Phi_0(z)$  uniquely as a corollary as before.

**Remark.**  $\nabla \lambda_0 = 0$  implies that  $\lambda_0(t, \mathbf{x}) = \lambda_0(t)$ .

- Continuity of the velocity field, leading order Lagrange multiplier and the chemical potential at the interface:

$$[\mathbf{v}_0]_{-}^{+} = 0, \quad [\lambda_0]_{-}^{+} = 0, \quad [\mu_1]_{-}^{+} = 0$$

- Interfacial stress balances satisfy the Young-Laplace law:

$$[p_1 + \lambda_1]_{-}^{+} \boldsymbol{\nu} - [\hat{\eta}_i (\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^T)]_{-}^{+} \boldsymbol{\nu} = \kappa \gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz \boldsymbol{\nu}$$

- Gibbs-Thompson law:

$$\mu_1|_{\Gamma} = \frac{1}{2} \gamma \kappa \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz$$

- Interfacial velocity:

$$w_{\boldsymbol{\nu}} = \mathbf{v}_0|_{\Gamma} \cdot \boldsymbol{\nu} + \frac{m_i}{2} c_+ [c_+ \nabla \mu_1 \cdot \boldsymbol{\nu}]_{-}^{+}$$

## The limiting model in the (N-S) regime

An incompressible sharp interface model where  $\lambda_0(t)$  appears only if  $m_r \neq 0$ :

$$\begin{aligned}c_+ (m_j \Delta - m_r) \mu_1 + c_- m_r \lambda_0(t) &= 0, \\ \rho_0 (\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0) + \nabla p_1 + \nabla \lambda_1 &= \operatorname{div}(\hat{\eta}(\varphi_0)(\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^T)), \\ \operatorname{div} \mathbf{v}_0 &= 0,\end{aligned}$$

equipped with the interface conditions

$$\begin{aligned}[p_1 + \lambda_1]_{-}^{+} \boldsymbol{\nu} - [\hat{\eta}_i(\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^T)]_{-}^{+} \boldsymbol{\nu} &= \kappa \gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz \boldsymbol{\nu}, \\ w_{\boldsymbol{\nu}} &= \mathbf{v}_0|_{\Gamma} \cdot \boldsymbol{\nu} + \frac{m_j}{2} c_+ [c_+ \nabla \mu_1 \cdot \boldsymbol{\nu}]_{-}^{+}, \\ \mu_1|_{\Gamma} &= \frac{1}{2} \gamma \kappa \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 dz\end{aligned}$$

with

$$[\mathbf{v}_0]_{-}^{+} = 0, \quad [\mu_1]_{-}^{+} = 0.$$

Further in formal asymptotic analysis:

- Temperature dependent model
- Semi-compressible model: Introducing the phase fraction  $\varphi$  for an incompressible constituent and keeping the density for the compressible one.

$$\rho_1 = \tilde{\rho}_L \frac{1 + \varphi}{2}, \quad \rho_2 = \tilde{\rho}_V \frac{1 - \varphi}{2} \quad \text{with} \quad \rho_L = \text{const} \quad \text{and} \quad \rho_V \neq \text{const}$$

Open analytical questions:

- Well-posedness of the diffuse model...
- Well-posedness of the limiting sharp models...
- A rigorous sharp interface limit...

Thank You!