A short review on « Massive Gravity »
(some of its open problems)

1. Pauli-Fierz (PF) theory
2. Non linear PF theory
3. DGP Model
4. Vainshtein mechanism for static spherically symmetric solutions of non linear PF.
Why being interested in « massive gravity »?

One way to modify gravity at « large distances » … and get rid of dark energy?

Changing the dynamics of gravity?

$H^2 = \frac{8\pi G}{3}\rho$

Dark matter or dark energy?

Historical example the success/failure of both approaches: Le Verrier and

• The discovery of Neptune
• The non discovery of Vulcan… but that of General Relativity
for this idea to work...

I.e. to « replace » the cosmological constant by a non vanishing graviton mass...

NB: It seems one of the Einstein’s motivations to introduce the cosmological constant was to try to « give a mass to the graviton »

1. Quadratic massive gravity: the Pauli-Fierz theory and the vDVZ discontinuity

Pauli-Fierz action: second order action for a massive spin two

\[
\int d^4x \sqrt{g} R_g + m^2 \int d^4 x h_{\mu\nu} h_{\alpha\beta} \left( \eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\nu} \eta^{\alpha\beta} \right)
\]

second order in \( h_{\mu\nu} \cdot g_{\mu\nu} - \eta_{\mu\nu} \)

Only Ghost-free (quadratic) action for a massive spin two

(Pauli, Fierz 1939)

Only Ghost-free (quadratic) action for a massive spin two (NB: breaks explicitly gauge invariance)

The propagators read

propagator for \( m = 0 \) \[ D_{0}^{\partial \cdot \bar{\epsilon}} (p) = \frac{\bar{n} \partial \bar{\epsilon} - i}{2p^2} + O(p) \]

propagator for \( m \neq 0 \) \[ D_{m}^{\partial \cdot \bar{\epsilon}} (p) = \frac{\bar{n} \partial \bar{\epsilon} + \bar{n} \partial \bar{\epsilon}}{2(p^2 m^2)} \]

\( + O(p) \)
Coupling the graviton with a conserved energy-momentum tensor

\[ S_{\text{int}} = \int d^4x \, P^- g h^{\dot{\alpha}\dot{\gamma}} T^{\dot{\alpha}\dot{\gamma}} \]

\[ h^{\dot{\alpha}\dot{\gamma}} = \int D^{\dot{\alpha}\dot{\gamma}i} (x \dot{\alpha} x^0) T_{\dot{\alpha}i} (x^0) d^4x^0 \]

The amplitude between two conserved sources T and S is given by

\[ A = \int d^4x S^{\dot{\alpha}\dot{\gamma}}(x) h^{\dot{\alpha}\dot{\gamma}}(x) \]

For a massless graviton: \( A_0 = \int \hat{\phi}^{\dot{\alpha}\dot{\gamma}} \dot{\alpha} \frac{1}{2} \hat{n}^{\dot{\alpha}\dot{\gamma}} \hat{\phi}^{\dot{\alpha}\dot{\gamma}} \)

For a massive graviton: \( A_m = \int \hat{\phi}^{\dot{\alpha}\dot{\gamma}} \dot{\alpha} \frac{1}{3} \hat{n}^{\dot{\alpha}\dot{\gamma}} \hat{\phi}^{\dot{\alpha}\dot{\gamma}} \)
e.g. amplitude between two non relativistic sources:

\[
\begin{align*}
\hat{\Phi}^\circ / \text{diag}(\hat{n}_1; 0; 0; 0) & \quad \text{and} \quad \hat{S}^\circ / \text{diag}(\hat{n}_2; 0; 0; 0) \\
\end{align*}
\]

\[
\left\{ \begin{array}{c}
A \varnothing \frac{2}{3}\hat{n}_1\hat{n}_2 \\
A \varnothing \frac{1}{2}\hat{n}_1\hat{n}_2 \\
\end{array} \right.
\]

Rescaling of Newton constant

\[G_{\text{Newton}} = \frac{4}{3}G^{(4)}\]

defined from Cavendish experiment

appearing in the action

but amplitude between an electromagnetic probe and a non-relativistic source is the same as in the massless case (the only difference between massive and massless case is in the trace part) ➔ wrong light bending! (factor \(\_\))
N.B., the PF mass term reads

\[ M^2_P m^2 \int d^4x \left( h_{ij} h_{ij} - 2h_{0i} h_{0i} - h_{ii} h_{jj} + 2h_{ii} h_{00} \right) \]

\( h_{00} \) enters linearly both in the kinetic part and the mass term, and is thus a Lagrange multiplier of the theory…

… which equation of motion enables to eliminate one of the a priori 6 dynamical d.o.f. \( h_{ij} \)

By contrast the \( h_{0i} \) are not Lagrange multipliers

5 propagating d.o.f. in the quadratic PF
\( (h_{\mu\nu} \) is transverse traceless in vacuum)
2. Non linear Pauli-Fierz theory and the Vainshtein Mechanism

Can be defined by an action of the form

\[ S = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R_g + L_g \right) + S_{\text{int}}[f, g], \]

The interaction term \( S_{\text{int}}[f, g] \), is chosen such that

- It is invariant under diffeomorphisms
- It has flat space-time as a vacuum
- When expanded around a flat metric

\[ (g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, f_{\mu \nu} = \eta_{\mu \nu}) \]

It gives the Pauli-Fierz mass term

leads to the e.o.m.

Matter energy-momentum tensor

Effective energy-momentum tensor ( \( f, g \) dependent)

Isham, Salam, Strathdee, 1971
Some working examples

Boulware Deser, 1972, BD in the following

Arkani-Hamed, Georgi, Schwarz, 2003
AGS in the following

(with)

(infinite number of models with similar properties)
(Damour, Kogan, 2003)

Look for static spherically symmetric solutions
With the ansatz (not the most general one!)

\[ g_{\alpha\beta} \, dx^\alpha \, dx^\beta = \sqrt{J(x)} \, dt^2 + K(x) \, dr^2 + L(x) \, r^2 \, d\theta^2 \]

Gauge transformation

\[ g_{\alpha\beta} \overset{\pi}{\mapsto} dx^\alpha \, dx^\beta = e^{\pi} \left( R \right) \, dt^2 + e^{\pi} \left( R \right) \, dr^2 + R^2 \, d\theta^2 \]

Then look for an expansion in \( G_N \) (or in \( R_S / G_N M \)) of the would be solution.
This coefficient equals +1 in Schwarzschild solution.

\[ ds^2 = e^{\mathbf{2}(\mathbf{u})} dt^2 + e^{\mathbf{2}(\mathbf{u})} d\mathbf{u}^2 + e^{\mathbf{2}(\mathbf{u})} \mathbf{u}^2 d\mathbf{O}_2^2 \]

\[ \mathbf{v}(\mathbf{r}) = \mathbf{a} \left( \frac{r_s}{\mathbf{r}} \right) (1 + \frac{7}{32} \mathbf{i} + \cdots) \]

\[ \mathbf{o}(\mathbf{r}) = \mathbf{a} \left( \frac{1}{2} \right) \frac{r_s}{\mathbf{r}} (1 + \frac{21}{8} \mathbf{i} + \cdots) \]

with \( \mathbf{i} = \frac{r_s}{m^4 r^5} \) 

Vainshtein 1972

In some kind of non linear PF

Wrong light bending!

Introduces a new length scale \( r_v \) in the problem below which the perturbation theory diverges!

For the sun: bigger than solar system! with \( r_v = (r_s m^4)^{1/5} \)
So, what is going on at smaller distances?

Vainshtein’s answer (1972):

There exists an other perturbative expansion at smaller distances, reading:

- This goes smoothly toward Schwarzschild as $m$ goes to zero
- This leads to corrections to Schwarzschild which are non analytic in the Newton constant

No warranty that this solution can be matched with the other for large $r$!

Boulware, Deser, 1972
This was investigated (by numerical integration) by Damour, Kogan and Papazoglou (2003)

No non-singular solution found matching the two behaviours (always singularities appearing at finite radius)

(see also Jun, Kang 1986)

In the 4th part of this talk:

A new look on this problem using in particular the « Goldstone picture » of massive gravity in the « Decoupling limit. »

(in collaboration with E. Babichev and R.Ziour)
The Goldstone picture

The theory considered has the usual diffeo invariance

\[
\pi(\mathbf{x}) = \pi(\mathbf{0}) + \mathbf{f}(\mathbf{x})
\]

This can be used to go from a « unitary gauge » where

To a « non unitary gauge » where some of the d.o.f. of the $g$ metric are put into $f$ thanks to a gauge transformation of the form

\[
f(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + \mathbf{gf}(\mathbf{x})
\]
One (trivial) example: our spherically symmetric ansatz

\[
\left\{ \begin{array}{l}
\text{Gauge transformation}
\end{array} \right. 
\]
Analogous to the Stuckelberg « trick » used to introduce gauge invariance into the Proca Lagrangian

\[ \text{Unitary gauge} \]

Do then the replacement with the new field

The obtained theory has the \( g_{\mu \nu} \) invariance

\[ \{ \]

The Proca action is just the same theory written in the \( f_{\mu \nu} \) gauge, while \( \partial_{\mu} \) gets a kinetic term via the Proca mass term \( ( \) \)
Back to massive gravity: Poincaré-covariant Goldstone

Expand the theory around the unitary gauge as

\[
A_\mu \text{ gets a kinetic term via the mass term}
\]

\[
\phi \text{ only gets one via a mixing term}
\]
One can demix $\phi$ from $h$ by defining

And the interaction term reads then at quadratic order

The canonically normalized $\phi$ is given by

Taking then the « Decoupling Limit »

One is left with …
\[ \frac{1}{2}(\nabla \tilde{\phi})^2 - \frac{1}{M_P} \tilde{\phi} T + \frac{1}{\Lambda^5} \left\{ (\nabla^2 \tilde{\phi})^3 + \ldots \right\} \]

With $\Lambda = (m^4 M_P)^{1/5}$

Other cubic terms omitted

« Strong coupling scale »
(hidden cutoff of the model?)

In the decoupling limit, the Vainshtein radius is kept fixed, and one can understand the Vainshtein mechanism as

E.g. around a heavy source: of mass M

Interaction $M/M_P$ of the external source with $\tilde{\phi}$

The cubic interaction above generates $O(1)$ correction at $R = R_v \, \tilde{n} \, (R_s m^{4})^{1=5}$
An other non trivial property of non-linear Pauli-Fierz: at non linear level, it propagates 6 instead of 5 degrees of freedom, the energy of the sixth d.o.f. having no lower bound!

Using the usual ADM decomposition of the metric, the non-linear PF Lagrangian reads (for $\eta_{\mu \nu}$ flat)

With

$$
\begin{align*}
N &\equiv (-g^{00})^{-1/2} \\
N_i &\equiv g_{0i}
\end{align*}
$$

Neither $N_i$, nor $N$ are Lagrange multipliers

The e.o.m. of $N_i$ and $N$ determine those as functions of the other variables

6 propagating d.o.f., corresponding to the $g_{ij}$

Boulware, Deser '72
Moreover, the reduced Lagrangian for those propagating d.o.f. read

\[ \left( \frac{1}{2} \left( \nabla \tilde{\phi} \right)^2 - \frac{1}{M_P} \tilde{\phi} T + \frac{1}{\Lambda^5} \left\{ (\nabla^2 \tilde{\phi})^3 + \ldots \right\} \right) \]

Leads to order 4 E.O.M. ), it describes two scalars fields, one being ghost-like
Some remarks and questions

• The « decoupling limit » and Goldstone picture we have just sketched gives a qualitative description of the Vainshtein mechanism… can one be more quantitative (see part 4) ?

• What about other models of « massive gravity ? » (see part 3)
3. DGP model (or brane-induced gravity).

Dvali, Gabadadze, Porrati, 2000

Usual 5D brane world action

\[ S = \underbrace{M_{(5)}^3 \int d^5 x \sqrt{g} \left( \tilde{R} + \cdots \right)}_{\text{Standard model}} + \int_{\text{brane}} d^4 x \sqrt{g} \mathcal{L}_{\text{matter}} + \underbrace{M_P^2 \int_{\text{brane}} d^4 x \sqrt{g} \left( R + \cdots \right)}_{\text{Peculiar to DGP model}} \]

A special hierarchy between \( M_{(5)} \) and \( M_P \) is required to make the model phenomenologically interesting.

Leads to the e.o.m.

\[ G^{(5)}_{AB} = \delta \left( \text{brane} \right) \frac{1}{M_{(5)}^3} \left[ G_{\mu\nu} - \frac{1}{M_P^2} T_{\mu\nu} \right] \]

- Brane localized kinetic term for the graviton
- Will generically be induced by quantum corrections
Phenomenological interest
A new way to modify gravity at large distance, with a new type of phenomenology …
(Important to have such models, if only to disentangle what does and does not depend on the large distance dynamics of gravity in what we know about the Universe)

Theoretical interest
Consistent (?) non linear massive gravity …
In the DGP model:

• Newtonian potential on the brane behaves as
  \[ V(r) / r \] 4D behavior at small distances
  \[ V(r) / r^2 \] 5D behavior at large distances

• The crossover distance between the two regimes is given by
  \[ r_c = \frac{M_P^2}{2M_5^3} \]

This enables to get a “4D looking” theory of gravity out of one which is not, without having to assume a compact (Kaluza-Klein) or “curved” (Randall-Sundrum) bulk.

• But the tensorial structure of the graviton propagator is that of a massive graviton (gravity is mediated by a continuum of massive modes)

 Leads to the van Dam-Veltman-Zakharov discontinuity on Minkowski background!
Homogeneous cosmology of DGP model

One obtains the following modified Friedmann equations (C.D. 2001)

\[ \sqrt{H^2 + \frac{k}{a^2}} = \frac{\epsilon}{2r_c} + \sqrt{\frac{\rho(M)}{3M_P^2}} + \frac{1}{4r_c^2} \]

with \( \epsilon = \pm 1 \)

\[ \dot{\rho} = -3H (P + \rho) \]

Analogous to standard (4D) Friedmann equations

\[ H^2 + \frac{k}{a^2} = \frac{\rho(M)}{3M_P^2} \]

In the early Universe
(small Hubble radii \( H^{-1} \ll r_c \) )
Late time cosmology

\[ H = \frac{\epsilon}{2r_c} + \sqrt{\frac{8\pi G \rho}{3}} + \frac{1}{4r_c^2} \]

Depending on the sign of \( \epsilon \):

- \( \epsilon = -1 \)
- \( \epsilon = +1 \)

One virtue of DGP model: can get accelerated universe by large distance modification of gravity (C.D (‘01); C.D., Dvali, Gabadaze (‘02)).

Brane cosmology in 5D Minkowski bulk with no R term on the brane (i.e.: solution to 5D Einstein-Hilbert Action)

Late time deviation from standard cosmology

Self accelerating solution (asymptotes de Sitter space even with zero matter energy density)
DGP self accelerating phase

The brane (first) Friedmann equation

\[ H^2 = -\frac{k}{a_{(b)}^2} + \left\{ \frac{\epsilon}{2r_c} + \sqrt{\frac{1}{4r_c^2} + \frac{8\pi G \rho}{3}} \right\}^2 \]

Can be rewritten as

with \( \Omega_{r_c} = \frac{1}{4r_c^2 H_0^2} \)

Phase diagram with \( \epsilon = +1 \)

Acts as a cosmological constant if \( \epsilon = +1 \)

Same number of parameter as \( \Lambda \)CDM

Maartens, Majerotto
Strictly speaking, only SN observations are depending solely on the background evolutions.

**DGP**

CMB and more importantly Baryon oscillations should be re-computed taking into account the peculiarities of DGP gravity.

**Vs. \( \Lambda \)CDM**

Maartens, Majerotto ‘06
Interesting (toy) model realizing the idea put forward in the beginning of this talk

However a ghost was found in the spectrum of the linearized theory in the self-accelerating phase!

Luty, Porrati, Rattazzi; Nicolis, Rattazzi, Koyama; Gorbunov, Koyama, Sibiryakov; Charmousis, Kaloper, Gregory, Padilla; C.D., Gabadadze, Iglesias

This ghost will be (briefly) discussed there-after, but irrespectively of its meaning for the self-accelerating phase, the DGP model remains interesting to study properties of (would-be existing) « massive gravity »
Van Dam-Veltman-Zakharov discontinuity in the DGP model...

- Exact cosmological solutions provide an explicit example of interpolation between theories with different tensor structure for the graviton propagator.

\[
H^2 = \frac{\rho}{3 M_P^2} \quad \text{large } r_c \\
S_0 = M_P^2 \int d^4 x \sqrt{-g} R
\]

\[
H^2 = \frac{\rho^2}{36 M_{(5)}^6} \quad \text{small } r_c \\
S_5 = M_{(5)}^3 \int d^5 x \sqrt{-g} R
\]

comes in support of a « Vainshtein mechanism » [non perturbative recovery of the « massless » solutions, originally (’72) proposed for « massive » gravity] at work in DGP…… More recently an other exact solution found by Kaloper for localized relativistic source showing the same recovery…..

C.D., Gabadadze, Dvali, Vainshtein (2002)
• Perturbative study of Schwarzschild type solutions of DGP model on a flat background space-time:

Gruzinov; Porrati; Lue; Lue & Starkman; Tanaka

Potential: 4D 4D 5D
Tensor structure: 4D 5D

No known exact solutions (similar to other « brane worlds »)

\[ r = \left( \frac{r_c^2 r_s}{2} \right)^{1/3} \]

Vainshtein radius for DGP model

Related to strong self interaction of the scalar sector

C.D., Gabadadze, Dvali, Vainshtein; Arkani-Hamed, Georgi Schwartz; Rubakov; Luty, Porrati, Rattazzi.
A good (at least qualitatively speaking) description of this is given by the action (obtained by taking a « decoupling limit ») 

\[
3\pi \square \pi - \frac{1}{\Lambda^3} (\partial_\mu \pi \partial^\mu \pi) \square \pi + \frac{1}{M_P} \pi T
\]

Scalar sector of the model

Energy scale \( \Lambda = (r_c^2/M_P)^{1/3} \)

This is in line with the appearance of the Vainshtein radius around a heavy source: \( r \) of mass \( M \)

Interaction \( M/M_P \) of the external source with \( \pi \)

The cubic interaction above generates \( O(1) \) correction at \( r = r_V \tilde{\eta} (r_c^2/r_S)^{1/3} \)
Note also that the cubic action

\[ 3\pi \Box \pi - \frac{1}{\Lambda^3} \left( \partial_\mu \pi \partial^\mu \pi \right) \Box \pi + \frac{1}{M_P} \pi T \]

Does not lead to equation of motion of order higher than 2 !: No ghost « à la Bouware-Deser » (revisited)  
(C.D., Rombouts)

Leads in vacuum to two branches of solutions, \( \pi \sim 0 \) and \( \Box \pi \sim \Lambda^3 \) representing the two branches of solutions of the original model…

(Nicolis, Rattazzi)
What about the ghost?

\[ 3\pi \Box \pi - \frac{1}{\Lambda^3} (\partial_\mu \pi \partial^\mu \pi ) \Box \pi + \frac{1}{M_P} \pi T \]

The kinetic terms for small perturbations of $\pi$ over the flat background and self-accelerating de Sitter background have opposite signs

Luty, Porrati, Rattazzi; Nicolis, Rattazzi
(in qualitative agreement with other refs cited above)

The one of the self accelerating being ghost-like…

**Problem:** this is only possible because the $\pi$ background for the self accelerating Universe is at the scale $\Lambda$ (i.e. $\Box \pi \gg \Lambda^3$) claimed to be also the UV cutoff of the model! If so quantum corrections can drastically change the conclusion about the existence of the ghost!
A crucial question for the sake of the DGP model: Find a proper UV completion of the model.

String theory?

Yes/ May be?
Antoniadis, Minasian, Vanhove; Kohlprath, Vanhove; Kiritsis, Tetrás, Tomaras; Corley, Lowe, Ramgoolam.

No/ May be not?
Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi.
4. Static spherically symmetric solutions of nonlinear Pauli-Fierz theory

\[ S = \int d^4x \sqrt{-g} \left( \frac{M^2}{2} R_g + L_g \right) + S_{\text{int}}[f, g], \]

The interaction term \( S_{\text{int}}[f, g] \), is chosen such that

- It is invariant under diffeomorphisms
- It has flat space-time as a vacuum
- When expanded around a flat metric
  \[ (g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, f_{\mu \nu} = \eta_{\mu \nu}) \]
  It gives the Pauli-Fierz mass term

Leads to the e.o.m.
In the following

Look for static spherically symmetric solutions with the ansatz (« λ, μ, ν » gauge)

4.1 A first look at the problem using a (the) decoupling limit (DL)

4.2 Full (non DL) case
In the « λ, μ, ν » gauge, the e.o.m (+ Bianchi) read

“$G_{tt}$”⇒

“$G_{RR}$”⇒

“Bianchi”⇒

With

$$f_t(\lambda, \nu, \mu', \mu, R) \equiv \frac{8\pi G}{m^2} T^{(g)}_{tt},$$

$$f_R(\lambda, \nu, \mu', \mu, R) \equiv \frac{8\pi G}{m^2} T^{(g)}_{RR},$$

$$- \frac{1}{R} f_g(\lambda, \lambda', \nu, \nu', \mu, \mu', \mu'', R) \equiv - \frac{1}{m^2 M_p^2} \frac{1}{R} \nabla^\mu T^g_{\mu R}.$$
e.g. for the « AGS » potential one finds

\[
\begin{align*}
    f_t^{(AGS)} &= \frac{1}{16} e^{-\lambda - 2\mu} \left[ (e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu}) R^2 \mu^2 - 4 \left( e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu} \right) R \mu' \\
    &\quad - 4 \left( -e^{\mu} - 2e^{\lambda+\mu} + 3e^{\lambda+2\mu} + 2e^{\nu} + e^{\lambda+\nu} - 3e^{\mu+\nu} - 6e^{\lambda+\mu+\nu} + 6e^{\lambda+2\mu+\nu} \right) \right], \\
    f_R^{(AGS)} &= \frac{1}{16} e^{-2\mu-\nu} \left[ (e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu}) R^2 \mu^2 - 4 \left( e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu} \right) R \mu' \\
    &\quad + 4 \left( -e^{\mu} + 2e^{\lambda+\mu} - 3e^{\lambda+2\mu} - 2e^{\nu} + e^{\lambda+\nu} + 3e^{\mu+\nu} - 6e^{\lambda+\mu+\nu} + 6e^{\lambda+2\mu+\nu} \right) \right], \\
    f_g^{(AGS)} &= \frac{1}{16 R} e^{-\lambda - 2\mu-\nu} \left( R \mu' - 2 \right) \left\{ \left[ (e^{\mu} + 4e^{\nu} - 3e^{\mu+\nu}) \mu'^2 - (e^{\mu} + 2e^{\nu} - 3e^{\mu+\nu}) \lambda' \mu' \\
    &\quad - (e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu}) \nu' \mu' - 2 \left( -e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu} \right) \mu'' \right] R^2 \\
    &\quad + 2 \left[ (e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu}) \lambda' - 4 \left( e^{\mu} + 2e^{\nu} - 3e^{\mu+\nu} \right) \mu' \\
    &\quad - (e^{\mu} - 2e^{\nu} + 3e^{\mu+\nu}) \nu' \right] R + 8 \left( -1 + e^{\lambda} \right) \left( -e^{\mu} - e^{\nu} + 3e^{\mu+\nu} \right) \right\}. 
\end{align*}
\]

The system can however be greatly simplified taking the « Decoupling Limit » (DL)
Indeed, doing the rescaling

\[
\begin{align*}
\tilde{\nu} & \equiv M_P \nu \\
\tilde{\lambda} & \equiv M_P \lambda \\
\tilde{\mu} & \equiv m^2 M_P \mu
\end{align*}
\]

And taking the « decoupling » limit

The full (non linear) system of e.o.m collapses to

\[
\begin{align*}
\frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} & = -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho} \\
\frac{\tilde{\nu}'}{R} - \frac{\tilde{\lambda}}{R^2} & = \tilde{\mu} \\
\frac{\tilde{\lambda}}{R^2} & = \frac{\tilde{\nu}'}{2R} + \frac{Q(\tilde{\mu})}{\Lambda^5}
\end{align*}
\]

System of equations to be solved in the DL
System of equations to be solved in the DL

\[ \begin{align*}
\frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} &= -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho} \\
\tilde{\nu}' - \frac{\tilde{\lambda}}{R^2} &= \tilde{\mu} \\
\frac{\tilde{\lambda}}{R^2} &= \frac{\tilde{\nu}'}{2R} + \frac{Q(\tilde{\mu})}{\Lambda^5} \\
\end{align*} \]

Where \( Q(\mu) \) is given by

\[
Q^{(\alpha,\beta)}(\mu) = -\frac{1}{2R} \left\{ 3\alpha \left( 6\mu\mu' + 2R\mu'' + \frac{3}{2}R\mu\mu'' + \frac{1}{2}R^2\mu'\mu'' \right) \right. \\
\left. + \beta \left( 10\mu\mu' + 5R\mu'' + \frac{5}{2}R\mu\mu'' + \frac{3}{2}R^2\mu'\mu'' \right) \right\},
\]

\( \alpha \) and \( \beta \) being constants depending on the potential

( For example, for the AGS potential, one finds: )
System of equations to be solved in the DL

\[
\begin{align*}
\frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} &= -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho} \\
\frac{\tilde{\nu}'}{R} - \frac{\tilde{\lambda}}{R^2} &= \tilde{\mu} \\
\frac{\tilde{\lambda}}{R^2} &= \frac{\tilde{\nu}'}{2R} + \frac{Q(\tilde{\mu})}{\Lambda^5}
\end{align*}
\]

Out of which \(\lambda\) and \(\nu\) can be eliminated to yield a single equation for \(\mu\) reading

\[
\frac{1}{\Lambda^5} \left[ 6Q(\tilde{\mu}) + 2RQ(\tilde{\mu}') \right] + \frac{9}{2} \tilde{\mu} + \frac{3}{2} R \tilde{\mu}' = \tilde{\rho}
\]

And can be integrated once to yield the first integral

\[
\frac{2}{\Lambda^5} Q(\tilde{\mu}) + \frac{3}{2} \tilde{\mu} = -\frac{K}{R^3}
\]
Recall that $\mu$ is encoding the gauge transformation upon the substitution

$$\tilde{\mu} = -\frac{2}{R} \tilde{\phi}'$$

This first integral

$$-\frac{3}{2} \tilde{\mu} - \frac{2}{\Lambda^5} Q(\tilde{\mu}) = \frac{K}{R^3}$$

Yields exactly one which is obtained in the Poincaré-Covariant DL (first part of this talk) for the Goldstone
\[ Q(\tilde{\mu}) \] corresponds exactly to the quadratic piece

\[
3\alpha \left( -4 \frac{\tilde{\phi}'}{R^4} + 2 \frac{\tilde{\phi}' \tilde{\phi}''}{R^3} + 2 \frac{\tilde{\phi}''}{R^2} + 2 \frac{\tilde{\phi}' \tilde{\phi}(3)}{R^2} + \frac{\tilde{\phi}'' \tilde{\phi}(3)}{R} \right)
\]

\[
+ \beta \left( -6 \frac{\tilde{\phi}''}{R^4} + 2 \frac{\tilde{\phi}' \tilde{\phi}''}{R^3} + 4 \frac{\tilde{\phi}''}{R^2} + 4 \frac{\tilde{\phi}' \tilde{\phi}(3)}{R^2} + 3 \frac{\tilde{\phi}'' \tilde{\phi}(3)}{R} \right)
\]

And depends only on two parameters \( \alpha \) and \( \beta \)

(depending on the exact form of the interaction term)

This can be understood easily in the Poincaré-covariant picture by the fact the \( \phi \) e.o.m. derive from a Poincaré-covariant action where (up to integration by part) the only cubic term which can be written are
To summarize, in the decoupling limit the full non linear system reduces to

\[
\begin{align*}
\frac{\ddot{\lambda}}{R} + \frac{\dot{\lambda}}{R^2} &= -\frac{1}{2}(3\mu + R\mu') + \rho \\
\frac{\ddot{\mu}}{R} - \frac{\dot{\mu}}{R^2} &= \mu \\
\frac{2}{\Lambda^5} Q(\mu) + \frac{3}{2} \dot{\mu} &= -\frac{K}{R^3}
\end{align*}
\]

Which can be shown to give the leading behaviour of the solution in the range \( R_v(R_v m) < R < m^{-1} \)

The Vainshtein radius \( R_v \) is in this range

hence a first way to investigate the success/failure of the Vainshtein mechanism is to look for solutions of the above system interpolating between the two limiting behaviour found by Vainshtein
In the DL, one only needs to solve the non linear ODE

\[ \frac{3}{2} \ddot{\mu} + \frac{2}{\Lambda^5} Q(\ddot{\mu}) = -\frac{K}{R^3} \]

Depending on the interaction term \( S_{int}[f, g] \)
E.g. in the Case of the two interaction terms \( (\alpha + \beta = 0) \)

\[ S(2)_{int} = \frac{1}{8} m^2 M^2 P Z d^4 x \]

\( \left( \frac{1}{2} \omega \pi \right)^{\omega} = \left( \frac{g}{\pi} + \omega \right)^{\omega} = \left( g + \omega \right)^{\omega} \)

\( (\text{Boulware Deser}) \)

\( (\text{Arkani-Hamed, Georgi, Schwarz}) \)

This equation boils down to the simple form

\[ 3w - s \left( \dot{w}^2 + 2w\ddot{w} + 8 \frac{w\ddot{w}}{\xi} \right) = \frac{2c_0}{\xi^3} \]

With \( s = \$ 1 \) and the dimensionless quantities

\[ \begin{align*}
    w &= (R_v m)^{-2} \mu \\
    \xi &= R/R_V \\
    c_0 &= \frac{K}{R_V^2 \Lambda^5}
\end{align*} \]
With $s = \S 1$ and the dimensionless quantities

\[
3w - s \left( \dot{w}^2 + 2w\ddot{w} + 8\frac{w\dot{w}}{\xi} \right) = \frac{2c_0}{\xi^3}
\]

\[
\begin{align*}
  w &= (R_v m)^{-2} \mu \\
  \xi &= R/R_v \\
  c_0 &= \frac{K}{R_v^2 \Lambda^5}
\end{align*}
\]

How to read the Vainshtein mechanism and scalings?

For $\xi \gg 1$ keep the linear part

\[
3w = \frac{2c_0}{\xi^3}
\]

However, numerical integration (and mathematical properties of the non linear ODE) shows that the situation is much more complicated!

Assume a power law scaling

\[
\Rightarrow w \propto \xi^{-1/2}
\]
Indeed ...

\[3w - s \left( \dot{w}^2 + 2w \ddot{w} + 8 \frac{w \dddot{w}}{\xi} \right) = \frac{2c_0}{\xi^3}\]

At large \(\xi\) (expect \(w / 1/ \xi^3\))

The change of variables

\[
\begin{align*}
\Xi_\infty &= \xi^{-3} \\
W_\infty &= (w(\xi))^{3/2}
\end{align*}
\]

Put the ODE in the form

\[sW''_{\infty} - \frac{W_{\infty}^{1/3}}{4\Xi_{\infty}^{8/3}} + \frac{c_0}{6\Xi_{\infty}^{5/3}W_{\infty}^{1/3}} = 0\]

And finding a solution with Vainshtein asymptotics translates into the (singular) initial value problem

\[
\begin{align*}
W_{\infty}(0) &= 0 \\
W'_{\infty}(0) &= 0
\end{align*}
\]

A power law expansion of the would-be solution to this problem can be found (here with \(c_0 = 1\) and \(s = +1\))

\[w(\xi) = \frac{2}{3\xi^3} - \frac{4}{3\xi^8} + \frac{1024}{27\xi^{13}} + \frac{712960}{243\xi^{18}} + \frac{104910848}{243\xi^{23}} - \frac{225030664192}{2187\xi^{28}} + \ldots\]
This serie appears to be divergent but seems to give a good asymptotic expansion of the numerical solution at large $\xi$. 

- Note that this can easily been checked numerically for $s=-1$ (Boulware Deser) (where the Vainshtein solution does not exist at small $\xi$, becoming complex!) 

- For $s=+1$ (Arkani-Hamed et al.) solution is numerically highly unstable, singularities are seemingly arising at finite $\xi$… 

However by using a combination of relaxation method / Runge-Kutta/ Asymptotic expansion, one can see that solutions with Vainshtein asymptotics at large $\xi$ do exist. 

We also have Mathematical existence proofs of those solutions (thanks to J. Ecalle; Cid, Pouso, Pouso)
Let us first discuss the $s = -1$ case (Boulware Deser)

In this case: no real Vainshtein solution with $w / \xi^{1/2}$

At small $\xi$ (expect $w / \xi^{1/2}$, when the solution is real)

Numerical solution

$w$ scales as $w / \xi^2$

$\xi^2 w(\xi)$

$w / \xi^3$

$w / \xi^{1/2}$ at small $\xi$ (Vainshtein)
E.g. the first terms of this expansion are given by

\[ w(\xi) = \frac{A_0}{\xi^2} + \frac{3A_0 B_0 + \ln \xi}{3A_0} \xi - \frac{3}{8} \xi^2 \]

\[ -1 + 6A_0 B_0 + 54A_0^2 B_0^2 + (2 + 36A_0 B_0) \ln \xi + 6 \ln^2 \xi \frac{\xi^4}{216A_0^3} \]

And matches well the numerics (fitting for \( A_0 \) and \( B_0 \))
Let us now discuss the $s=+1$ case (Arkani-Hamed et al.)

In this case the large distance behaviour

$$w(\xi) = \frac{2}{3\xi^3}$$

Does not lead to a unique small distance ($\xi < 1$) behaviour (and solution)
$w / \tilde{\xi}^3$ (large distance)

$w / \tilde{\xi}^2$ (new scaling)

$w / \tilde{\xi}^{1/2}$ (Vainshtein)
In our case, using the « resurgence theory » (J. Ecalle) extending Borel resummation

Formal (divergent) serie

Borel transform

Laplace transform or rather « convolution average» extension

Solution of the ODE

s = -1

s = 1

Unique solution with \( w / 1/ \xi^3 \) decay at infinity

The difference between any two solutions is given (asymptotically) by

\[ \xi^{3/2} \exp \left( -k \frac{3}{5} \xi^{5/2} \right) \]

(with integer \( k \) !)

Infinitely many solutions with \( w / 1/ \xi^3 \) decay at infinity

(proof provided to us by J. Ecalle)
4.2 Full non linear (non DL) problem

\[
\begin{align*}
\mathcal{f}_t(\lambda, \nu, \mu', \mu, R) & \equiv \frac{8\pi G}{m^2 T_{tt}^{(g)}} \\
\mathcal{f}_R(\lambda, \nu, \mu', \mu, R) & \equiv \frac{8\pi G}{m^2 T_{RR}^{(g)}} \\
- \frac{1}{R} \mathcal{f}_g(\lambda', \nu', \mu', \mu'', R) & \equiv - \frac{1}{m^2 M_P^2 R} \nabla^\nu T_{\mu R}^g
\end{align*}
\]

\[
\begin{align*}
\mathcal{f}_t^{(AGS)} & = \frac{1}{16} e^{-\lambda-2\mu} \left[ (e^\mu - 2e^\nu + 3e^{\mu+\nu}) R^2 \mu'^2 - 4 \left( e^\mu - 2e^\nu + 3e^{\mu+\nu} \right) R \mu' \\
& \quad - 4 \left( -e^\mu - 2e^{\lambda+\mu} + 3e^{\lambda+2\mu} + 2e^\nu + e^{\lambda+\nu} - 6e^{\lambda+\mu+\nu} + 6e^{\lambda+2\mu+\nu} \right) \right], \\
\mathcal{f}_R^{(AGS)} & = \frac{1}{16} e^{-2\mu-\nu} \left[ (-e^\mu - 2e^\nu + 3e^{\mu+\nu}) R^2 \mu'^2 - 4 \left( e^\mu - 2e^\nu + 3e^{\mu+\nu} \right) R \mu' \\
& \quad + 4 \left( -e^\mu + 2e^{\lambda+\mu} - 3e^{\lambda+2\mu} - 2e^\nu + e^{\lambda+\nu} + 3e^{\mu+\nu} - 6e^{\lambda+\mu+\nu} + 6e^{\lambda+2\mu+\nu} \right) \right], \\
\mathcal{f}_g^{(AGS)} & = \frac{1}{16 R} e^{-\lambda-2\mu-\nu} \left( R \mu' - 2 \right) \left\{ \left[ (e^\mu + 4e^\nu - 3e^{\mu+\nu}) \mu'^2 - (e^\mu + 2e^\nu - 3e^{\mu+\nu}) \lambda' \mu' \\
& \quad - (e^\mu - 2e^\nu + 3e^{\mu+\nu}) \nu' \mu' - 2 \left( -e^\mu - 2e^\nu + 3e^{\mu+\nu} \right) \mu'' \right] R^2 \\
& \quad + 2 \left[ (-e^\mu - 2e^\nu + 3e^{\mu+\nu}) \lambda' - 4 \left( e^\mu + 2e^\nu - 3e^{\mu+\nu} \right) \mu' \\
& \quad - (e^\mu - 2e^\nu + 3e^{\mu+\nu}) \nu' \right] R + 8 \left( -1 + e^\lambda \right) \left( -e^\mu - e^\nu + 3e^{\mu+\nu} \right) \right\}. 
\end{align*}
\]
To be solved with a source (perfect fluid)

And continuity equation

\[ P' = -\frac{\nu'}{2} (P + \rho) \]

And appropriate boundary and regularity conditions

\{
\{
\{

\[ \frac{R_{\odot}}{R_V} = 10^{-3} \text{ and } m \times R_V = 10^{-3} \]
So the Vainshtein mechanism seems to work!

Various left-over issues for non linear PF theory.

• Compact sources?
• Black holes?
• Spherical collapse (instabilities?)?

Is there a consistent theory of massive gravity (some UV completed DGP-like model?)?

How does the Vainshtein mechanism work there?