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Injectivity Radius of Lorentzian Manifolds

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Abstract: Motivated by the application to general relativity we study the geometry and regularity of Lorentzian manifolds under natural curvature and volume bounds, and we establish several injectivity radius estimates at a point or on the past null cone of a point. Our estimates are entirely local and geometric, and are formulated via a reference Riemannian metric that we canonically associate with a given observer (p, T) –where p is a point of the manifold and T is a future-oriented time-like unit vector prescribed at p only. The proofs are based on a generalization of arguments from Riemannian geometry. We first establish estimates on the reference Riemannian metric, and then express them in terms of the Lorentzian metric. In the context of general relativity, our estimate on the injectivity radius of an observer should be useful to investigate the regularity of spacetimes satisfying Einstein field equations.

1. Introduction

Aims of this paper. The regularity and compactness of Riemannian manifolds under a priori bounds on geometric quantities such as curvature, volume, or diameter represent a central theme in Riemannian geometry. In particular, the derivation of lower bounds on the injectivity radius of a Riemannian manifold, and the construction of local coordinate charts in which the metric has optimal regularity are now well-understood. Moreover, Cheeger-Gromov's theory provides geometric conditions for the strong compactness of sequences of manifolds and has become a central tool in Riemannian geometry. See, for instance, [1,4,5,7,8,15,20,21].

Our objective in this paper is to present an extension of these classical techniques and results to Lorentzian manifolds. Recall that a Lorentzian metric is not positive definite, but has signature $(-, +, \dots, +)$. Motivated by recent work by Anderson [2] and Klainerman and Rodnianski [18], we derive here several injectivity radius estimates for Lorentzian manifolds satisfying certain curvature and volume bounds. That is, we provide lower bounds on the size of the (geodesic) ball around one point within which the exponential map is a global diffeomorphism and, therefore, we obtain a sharp control of the manifold geometry. Our proofs rely on arguments that are known to be flexible and efficient in Riemannian geometry, and are here extended to the Lorentzian setting: we analyze the properties of Jacobi fields and rely on volume comparison and homotopy arguments.

In our presentation (see for instance our main result stated in Theorem 1.1 at the end of this introduction) we emphasize the importance of having assumptions and estimates that are stated locally and geometrically, and avoid direct use of coordinates. When necessary, coordinates can be constructed a posteriori, once uniform bounds on the injectivity radius have been established.

Our motivation comes from general relativity, where one of the most challenging problems is the formation and the structure of singularities in solutions to the Einstein field equations. Relating curvature and volume bounds to the regularity of the manifold, as we do in this paper, is necessary before tackling an investigation of the geometric properties of singular spacetimes satisfying Einstein equations. (See, for instance, [2,3] for some background on this subject.)

Two preliminary observations should be made. First, since the Lorentzian norm of a non-zero tensor may vanish it is clear that only limited information would be gained from an assumption on the Lorentzian norm of the curvature tensor. This justifies that we endow the Lorentzian manifold with a "reference" Riemannian metric (denoted below by g_T); this metric is defined at a point p once we prescribe a future-oriented time-like unit vector T in the tangent space at p. We refer to the pair (p, T) as an observer located at the point p. This reference vector is necessary in order to define appropriate notions of conjugate and injectivity radii. (See Sect. 2, below, for details.)

Secondly, we rely here on the elementary but essential observation that, in the flat Riemannian and Lorentzian spaces, geodesics (are straight lines and therefore) coincide. More generally, under our assumptions, we will see that geodesics associated with the given Lorentzian metric are comparable to geodesics associated with the reference Riemannian metric. On the other hand, it must be emphasized that the curvature bound assumed on the Lorentzian metric implies, in general, no information on the curvature of the reference metric. In fact, as we show below, one of the main technical difficulties is to construct a sufficiently regular foliation of the manifold by spacelike hypersurfaces.

Earlier work. Let us briefly review some classical results from Riemannian geometry. Let (M, g) be a differentiable n-manifold (possibly with boundary) endowed with a Riemannian metric g. (Throughout the present paper, the manifolds and metrics under consideration are always assumed to be smooth.) Denote by $\mathcal{B}(p, r)$ the corresponding geodesic ball centered at $p \in M$ and with radius $r > 0$. Suppose that at some point $p \in M$ the unit ball $\mathcal{B}(p, 1)$ is compactly contained in M and that the Riemann curvature bound and the lower volume bound

$$\|\mathbf{Rm}_g\|_{\mathbf{L}^\infty(\mathcal{B}(p,1))} \le K, \qquad \mathbf{Vol}_g(\mathcal{B}(p, 1)) \ge v_0, \tag{1.1}$$

hold for some constants $K, v_0 > 0$. (We use the standard notation $\mathbf{L}^m$, $1 \le m \le \infty$, for the spaces of Lebesgue measurable functions.) Then, according to Cheeger, Gromov, and Taylor [9] the injectivity radius $\mathbf{Inj}_g(M, p)$ at the point p is bounded below by a positive constant $i_1 = i_1(K, v_0, n)$,

$$\mathbf{Inj}_g(M, p) \ge i_1. \tag{1.2}$$

It should be noticed that this is a local statement; for earlier (global) results on the injectivity radius see [5,10,14].

Moreover, Jost and Karcher in [15] rely on the regularity theory for elliptic operators and establish the existence of coordinates in which the metric has optimal regularity and which are defined in a ball with radius $i_2 = i_2(K, v_0, n)$. Precisely, given $\varepsilon > 0$ and $0 < \gamma < 1$ there exist a positive constant $C(\varepsilon, \gamma)$ (depending also upon (K, v_0, n) and a system of harmonic coordinates defined in the geodesic ball $\mathcal{B}(p, i_2)$ in which the metric g is close to the Euclidian metric g_E in these coordinates and has optimal regularity, in the following sense:

$$\begin{aligned} &e^{-\varepsilon}\, g_E \le g \le e^{\varepsilon}\, g_E, \\ &r^{1+\gamma}\, \|\partial g\|_{\mathbf{C}^{\gamma}(\mathcal{B}(p,r))} \le C(\varepsilon, \gamma), \qquad r \in (0, i_2]. \end{aligned} \tag{1.3}$$

Here, $\mathbf{C}^0$ and $\mathbf{C}^{\gamma} = \mathbf{C}^{0,\gamma}$ are the spaces of continuous and Hölder continuous functions, respectively. Harmonic coordinates are optimal [11] in the sense that if the metric is of class $\mathbf{C}^{k,\gamma}$ in certain coordinates then it has at least the same regularity in harmonic coordinates.

The above results were later generalized by Anderson [1] and Petersen [21] who replaced the $\mathbf{L}^{\infty}$ curvature bound by an $\mathbf{L}^{m}$ curvature bound with $m > n/2$. For instance, one can take $m = 2$ in dimension $n = 3$ which is the situation met in the application to general relativity (since time-slices of Lorentzian 4-manifolds are Riemannian 3-manifolds).

It is only more recently that the same questions were tackled for Lorentzian $(n+1)$-manifolds (M, g). Anderson [2,3] studied the long-time evolution of solutions to the Einstein field equations and formulated several conjectures. In particular, assuming the Riemann curvature bound in some domain Ω,

$$\|\mathbf{Rm}_g\|_{\mathbf{L}^{\infty}(\Omega)} \le K, \tag{1.4}$$

and other regularity conditions, he investigated the existence of coordinates that are harmonic in each spacelike slice of a time foliation of M. Anderson's pioneering work motivated us throughout the present paper.

On the other hand, for applications to general relativity and nonlinear wave equations using harmonic analysis tools, Klainerman and Rodnianski [18] considered asymptotically flat spacetimes endowed with a time foliation and satisfying the $\mathbf{L}^2$ curvature bound

$$\|\mathbf{Rm}_g\|_{\mathbf{L}^{2}(\Sigma)} \le K \tag{1.5}$$

for every spacelike hypersurface Σ. They established an injectivity radius estimate for past null cones, by relying on their earlier work [16,17] on the conjugate radius of null cones in terms of Bell-Robinson's energy and energy flux, and by deriving in [18] a new estimate on the null cut locus of such manifolds. We refer to these papers for further details and references on the Einstein equations.

Outline of this paper. We establish here four estimates on the radius of injectivity of Lorentzian manifolds, which hold either in a neighborhood of a point or in the past null cone of a point. Our assumptions are formulated within a geodesic ball (or within a null cone) and possibly apply to a large ball with arbitrary size, as long as our curvature and volume assumptions hold. All assumptions and statements are local and geometric.

An outline of the paper follows. In Sect. 2, we begin with basic material from Lorentzian geometry and we introduce the notions of reference metric and injectivity radius associated with Lorentzian manifolds.

In Sect. 3, we state our first estimate (Theorem 3.1 below) for a class of manifolds that have bounded curvature and admit a time foliation by slices with bounded extrinsic curvature. In Sect. 4, we provide a proof of this first estimate and we introduce a technique that will be used (in variants) throughout this paper; we combine two main ingredients: sharp estimates for Jacobi fields along geodesics, and a homotopy argument based on contracting a possible loop to two linear segments.

In Sect. 5, our second main result (Theorem 5.1) shows, under the same assumptions, the existence of convex functions (distance functions) and convex neighborhoods; this result leads us to a lower bound for the convexity radius.

In Sect. 6, our third estimate (Theorem 6.1) covers the generalization to null cones, and we show that weaker assumptions are sufficient to control the geometry of null cones. This result is directly relevant for the application to general relativity.

Next, in Sect. 7, we establish our principal and fourth result (stated in Theorem 1.1 below) which provides an injectivity radius bound under the mild assumption that the exponential map $\mathbf{exp}_p$ is defined in some ball and the curvature $\mathbf{Rm}$ is bounded. Most importantly, this is a general result that does not require a time foliation of the manifold but solely a single reference (future-oriented time-like unit) vector T at the base point p. This is very natural in the context of general relativity and (p, T) can be interpreted as an observer at the point p.

For the convenience of the reader we state here our main result and refer to Sect. 7 for further details. Given such an observer (p, T), we consider the geodesic ball $B_T(0, r) \subset T_pM$ with radius r, determined by the reference Riemannian inner product at p, and we can also define the geodesic ball $\mathcal{B}_T(p, r) := \mathbf{exp}_p(B_T(0, r))$. In turn, the radius of injectivity $\mathbf{Inj}_g(M, p, T)$ is defined as the largest radius r such that the exponential map is a diffeomorphism from $B_T(0, r)$ onto $\mathcal{B}_T(p, r)$. Let us then consider an arbitrary geodesic $\gamma = \gamma(s)$ initiating at p and let us g-parallel transport the vector T along this geodesic, defining therefore a vector field T_γ along this geodesic, only. At every point of γ we introduce a reference inner product g_{T_γ} and compute the curvature norm $|\mathbf{Rm}_g|_{T_\gamma}$. This construction allows us to express our curvature assumption below.

Theorem 1.1. (Injectivity radius of Lorentzian manifolds). *Let M be a time-orientable, Lorentzian, differentiable $(n+1)$-manifold. Consider an observer (p, T) consisting of a point $p \in M$ and a reference (future-oriented time-like unit) vector $T \in T_pM$. Assume that the exponential map $\mathbf{exp}_p$ is defined in a ball $B_T(0, r) \subset T_pM$ and the Riemann curvature satisfies*

$$\sup_\gamma |\mathbf{Rm}_g|_{T_\gamma} \le \frac{1}{r^2}, \tag{1.6}$$

where the supremum is over the domain of definition of γ and over every g-geodesic γ initiating from a vector in the Riemannian ball $B_T(0, r) \subset T_pM$. Then, there exists a constant $c(n)$ depending only on the dimension of the manifold such that

$$\frac{\mathbf{Inj}_g(M, p, T)}{r} \ge c(n) \, \frac{\mathbf{Vol}_g\,(\mathcal{B}_T(p, c(n)r))}{r^{n+1}}. \tag{1.7}$$

Observe that the curvature assumption (1.6) is not a genuine restriction since it can always be satisfied by suitably rescaling the metric. This result should be compared with the injectivity radius estimate established by Cheeger, Gromov, and Taylor [9] in Riemannian geometry. We also point out that the importance of analyzing the geometry of radial geodesics, as we will do in the proof of Theorem 1.1, was emphasized by Anderson [3] in his pioneering work on the optimal regularity of Lorentzian metrics.

It would be interesting to refine our arguments and replace the volume term in the right-hand side of (1.7) by $\mathbf{Vol}_g\,(\mathcal{B}(p,r))$. In a related direction, in Sect. 8, we establish a volume comparison theorem for future cones which allows us to generalize our main theorem and use the volume of a future cone in the right-hand side of (1.7).

Finally, in Sect. 9, we briefly discuss the regularity of Lorentzian metrics in harmonic-like coordinates, and we provide a direct generalization to pseudo-riemannian manifolds.

2. Preliminaries on Lorentzian Geometry

Basic definitions. It is useful to discuss first some basic definitions from Lorentzian geometry, for which we can refer to the textbook by Penrose [19]. Throughout this paper, (M, g) is a connected and differentiable $(n+1)$-manifold, endowed with a Lorentzian metric g with signature $(-, +, \dots, +)$. To emphasize the role of the metric g or the point p we use any of the following notations:

$$g_p(X, Y) = \langle X, Y\rangle_{g_p} = \langle X, Y\rangle_g = \langle X, Y\rangle_p$$

for the inner product of two vectors X, Y at a point $p \in M$; we sometimes also write $|X|^2_{g_p}$ instead of $g_p(X, X)$. Recall that the tangent vectors $X \in T_pM$ are called time-like, null, or spacelike depending whether the norm $g_p(X, X)$ is negative, zero, or positive, respectively. Vectors that are time-like or null are called causal.

Time-like vectors form a cone with two connected components. The manifold (M, g) is said to be time-orientable if we can select in a continuous way a half-cone of time-like vectors at every point p. The choice of a specific orientation allows us to decompose the cone of time-like vectors into future-oriented and past-oriented ones. The set of all future-oriented, time-like vectors at p and the corresponding bundle on M are denoted by T_p^+M and T^+M, respectively. We also introduce the bundle T_1^+M consisting of elements of T^+M with unit length.

By definition, a trip is a continuous time-like curve $\gamma : (a, b) \to M$. We write $p << q$ if there exists a trip from p to q. A causal trip is defined similarly except that the geodesics may be causal instead of time-like, and we write $p < q$ if there exists a causal trip from p to q.

The set $\mathcal{I}^+(p) := \{q \in M \,/\, p << q\}$ is called the chronological future of the point p, and $\mathcal{I}^-(p) := \{q \in M \,/\, q << p\}$ is called the chronological past. The causal future and past $\mathcal{J}^-(p)$ are defined similarly by replacing $<<$ by $<$. The future or past sets of a set $S \subset M$ are defined by

$$\mathcal{I}^\pm(S) := \bigcup_{p\in S} \mathcal{I}^\pm(p), \qquad \mathcal{J}^\pm(S) := \bigcup_{p\in S} \mathcal{J}^\pm(p),$$

and one easily checks that $\mathcal{I}^\pm(S)$ are open, but that $\mathcal{J}^\pm(S)$ need not be closed in general.

A future set $F \subset M$ by definition has the form $F = \mathcal{I}^+(S)$ for some set $S \subset M$. Similarly, a past set satisfies $F = \mathcal{I}^-(S)$ for some S. A set is called achronal if no two points are connected by a time-like trip. Observe that a set can be spacelike at every point without being achronal and that an achronal set can be null at some (or even at every) point. A set $B \subset M$ is called an achronal boundary if it is the boundary of a future set, that is, $B = \partial \mathcal{I}^+(S) = \overline{\mathcal{I}^+(S)} \setminus \mathcal{I}^+(S)$ for some $S \subset M$. One can check that given a non-empty achronal boundary the manifold can be partitioned as $M = P \cup B \cup F$, where B is the boundary of both F and P and, moreover, any trip from $p \in P$ to $q \in F$ meets B at a unique point. Observe also that any achronal boundary is a Lipschitz continuous n-manifold. For instance, in Sect. 6 below, we will be interested in the geometry of past null cones, that is, the sets $\partial \mathcal{J}^-(p)$ for $p \in M$.

Given an arbitrary achronal and closed set $S \subset M$, we define the (future or past) domains of dependence of S in M by

$$\mathcal{D}^{\pm}(S) := \{p \in M \,/\, \text{every future (resp. past) endless trip containing } p \text{ meets } S\},$$
$$\mathcal{D}(S) := \mathcal{D}^-(S) \cup \mathcal{D}^+(S).$$

Observe that domains of dependence are closed sets. Next, define the (future or past) Cauchy horizons

$$\mathcal{H}^{\pm}(S) := \left\{p \in \mathcal{D}^{\pm}(S) \,/\, \mathcal{I}^{\pm}(p) \cap \mathcal{D}^{\pm}(S) = \emptyset\right\} = \mathcal{D}^{\pm}(S) \setminus \mathcal{I}^{\mp}(\mathcal{D}^{\pm}(S)),$$
$$\mathcal{H}(S) := \mathcal{H}^-(S) \cup \mathcal{H}^+(S).$$

For instance, the future Cauchy horizon is the future boundary of the future domain of dependence of S. One can check that the Cauchy horizons are closed and achronal sets, with $\partial \mathcal{D}^+(S) = \mathcal{H}^+(S) \cup S$ and $\partial \mathcal{D}(S) = \mathcal{H}(S)$.

Finally, a (future) Cauchy hypersurface for M is defined as an achronal (but not necessarily closed) set S satisfying $\mathcal{D}^+(S) = M$. For instance, it is sufficient for $\overline{S}$ to be smooth, achronal, spacelike and such that every endless null geodesic meets M.

Reference metric. As explained in the introduction one should not use the Lorentzian metric to compute the norm of a tensor since its Lorentzian norm may vanish even when the tensor does not. This motivates the introduction of a "reference" Riemannian metric associated with a time-like vector field, as follows.

Let T be a future-oriented, time-like, unit vector field, satisfying therefore $g_p(T, T) = -1$ at every point p (in an open subset of M, at least). We refer to T as the *reference vector field* prescribed on M (or on an open subset). Introduce a moving frame E_α ($\alpha = 0, 1, \ldots, n$) defined in M, that is, E_α is an orthonormal basis of the tangent space at every point and consists of the vector $E_0 = T$ supplemented with n spacelike unit vectors E_j ($j = 1, \ldots, n$). Denoting by E^α the corresponding dual frame, the Lorentzian metric takes the form

$$g = \eta_{\alpha\beta}\, E^\alpha \otimes E^\beta,$$

where $\eta_{\alpha\beta}$ is the Minkowski "metric". This decomposition suggests to consider the Riemannian version obtained by switching the minus sign in $\eta_{00} = -1$ into a plus sign, that is,

$$g_T := \delta_{\alpha\beta}\, E^\alpha \otimes E^\beta,$$

where $\delta_{\alpha\beta}$ is the Euclidian "metric". Clearly, g_T is a positive definite metric; it is referred to as the *reference Riemannian metric* associated with the frame E_α.

For every $p \in M$, since T_p is time-like, the restriction of the metric g_p to the orthogonal complement $\{T_p\}^\perp \subset T_P M$ is positive definite, and the reference metric can be computed as follows: if $V = a\, T_p + V'$ and $W = b\, T_p + W'$ with $V', W' \in \{T_p\}^\perp$, then

$$g_{T,p}(V, W) = a\, b + g_p(V', W').$$

In the following, we use the notation

$$\langle V, W\rangle_{T,p} := g_{T,p}(V, W), \qquad |V|^2_{T,p} := g_{T,p}(V, V)$$

for vectors; the norm of tensors is defined and denoted similarly.

In contrast with the Lorentzian norm, the Riemannian norm $|A|_{T,p}$ of a tensor A at a point $p \in M$ vanishes if and only if the tensor vanishes at p. Moreover, as long as T remains in a compact subset of the bundle of half-cones T^+M, the norms associated with different reference vectors are equivalent.

The reference Riemannian metric also allows one to define the functional norms associated with Lebesgue or Sobolev spaces of tensors defined on M (as well as on submanifolds of M). For instance, $\mathbf{L}^2(M, g_T)$ can now be viewed as a Banach space. In particular, we will use later the $\mathbf{L}^2$ norm of a tensor field A defined on M and restricted to a hypersurface Σ:

$$\|\nabla A\|_{\mathbf{L}^2(\Sigma, g_T)} := \int_\Sigma |A|^2_T \, dV_{\Sigma, g_T},$$

where dV_{Σ, g_T} is the volume form induced on Σ by the reference Riemannian metric. The functional norm above depends upon the choice of the vector field T, but another choice of T would give rise to an equivalent norm (provided T remains in a fixed compact subset). Observe in passing that the volume forms associated with the metrics g and g_T coincide, so that the spacetime integrals of *functions* in (M, g) or (M, g_T) coincide; for instance, the volumes $\mathbf{Vol}_g(A)$ and $\mathbf{Vol}_{g_T}(A)$ of a set $A \subset M$ coincide.

Finally, we observe that in order to define the reference inner-product g_T at a single point p, it suffices to prescribe a future-oriented time-like unit *vector* T at that point p only; it is not necessary to prescribe a vector field. In the situation where the reference metric need only be defined at the base point p, we refer to T as the *reference vector* (rather than vector field) and we refer to $(p, T) \in T_1^+ M$ as the *observer at the point p*. This will be the standpoint adopted for our main result in Sect. 7 below.

Exponential map. On a complete Riemannian manifold the exponential map $\mathbf{exp}_p : T_p M \to M$ at some point $p \in M$ is defined on the whole tangent space $T_p M$ and is smooth. For sufficiently small radius r the restriction of $\mathbf{exp}_p$ to the ball $B_{g_p}(0, r) \subset T_p M$ (determined by the metric g_p at the point p) is a diffeomorphism on its image. The radius of injectivity at the point p is defined as the largest value r such that the restriction $\mathbf{exp}_p|_{B_{g_p}(0,r)}$ is a global diffeomorphism.

In the Lorentzian case, the exponential map is defined similarly but some care is needed in defining the notion of radius of injectivity. First of all, if the manifold is not geodesically complete (which is a rather generic situation, as illustrated by Penrose and Hawking's incompleteness theorems [13]), the map $\mathbf{exp}_p$ need not be defined on the whole tangent space $T_p M$ but only on a neighborhood of the origin in $T_p M$. More importantly, the Lorentzian norm of a non-zero vector may well vanish; consequently, the radius of injectivity should not be defined directly from the Lorentzian metric g. The definition given now depends on the prescribed Riemannian metric $g_{T,p}$ at the point p, only.

Definition 2.1. *The conjugate radius* $\mathbf{Conj}_g(M, p, T)$ *of an observer* $(p, T) \in T_1^+M$ *is the largest radius* r *such that the exponential map* $\mathbf{exp}_p$ *is a local diffeomorphism from the Riemannian ball* $B_T(0, r) = B_{g_{T,p}}(0, r) \subset T_pM$ *to a neighborhood of* p *in the manifold* M*. Similarly, the* **injectivity radius** $\mathbf{Inj}_g(M, p, T)$ *of an observer* $(p, T) \in T_1^+M$ *is the largest radius* r *such that the exponential map is a global diffeomorphism in the ball* $B_T(0, r)$*.*

When a vector field T is prescribed on the manifold (rather than a vector at the point p), we use the notation $\mathbf{Inj}_g(M, p, T)$ instead of $\mathbf{Inj}_g(M, p, T_p)$. Note that the radii $\mathbf{Conj}_g(M, p, T)$ and $\mathbf{Inj}_g(M, p, T)$ are essentially independent of the choice of the reference vector, as long as it remains in a fixed compact subset of T_p^+M.

We will also need the notion of injectivity radius for null cones. Given a point $p \in M$ and a reference vector $T \in T_pM$, we consider the *past null cone* at p,

$$N_p^- := \left\{X \in T_pM \,\big/\, g_p(X, X) = 0,\ g_p(T, X) \geq 0\right\},$$

which defines a subset of the tangent space at p. Denote by

$$B_T^N(0, r) = B_{g_{T,p}}^N(0, r) := B_{g_{T,p}}(0, r) \cap N_p^-$$

the intersection of the Riemannian $g_{T,p}$-ball with radius r and the past null cone, and by

$$\mathcal{N}^-(p) := \partial \mathcal{I}^-(p)$$

the past null cone at p.

Consider now the restriction of $\mathbf{exp}_p$ to the past null cone, denoted by

$$\mathbf{exp}_p^N : B_T^N(0, r) \subset N_p^- \to \mathcal{N}^-(p) \subset M,$$

which we refer to as the *null exponential map.*

Definition 2.2. *The* **(past) null conjugate radius Null Conj**$_g(M, p, T)$ *of an observer* $(p, T) \in T_1^+M$ *is the largest radius* r *such that the null exponential map* $\mathbf{exp}_p^N$ *is a local diffeomorphism from the punctured Riemannian ball* $B_T^N(0, r)\backslash\{0\} \subset T_pM$ *to a neighborhood of* p *in the past null cone. The* **null injectivity radius Null Inj**$_g(M, p, T)$ *of an observer* $(p, T) \in T_1^+M$ *is defined similarly by requiring the map* $\mathbf{exp}_p^N$ *to be a global diffeomorphism.*

3. Lorentzian Manifold Endowed with a Reference Vector Field

A first injectivity radius estimate. From now on, we fix a reference vector field T which allows us to define the Riemannian metric g_T and compute the norms of tensors. We begin with a set of assumptions encompassing a large class of foliated Lorentzian manifolds with $\mathbf{L}^\infty$ bounded curvature and we state our first injectivity estimate, in Theorem 3.1 below. The forthcoming sections will be devoted to further generalizations and variants of this result.

We fix a point $p \in M$ and assume that a domain $\Omega \subset M$ containing p is foliated by spacelike hypersurfaces Σ_t with future-oriented time-like unit normal T,

$$\Omega = \bigcup_{t\in[-1,1]} \Sigma_t. \tag{3.1}$$

A positive scalar n is defined by the relation $\frac{\partial}{\partial t} = n\, T$, or

$$n^2 := -g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).$$

In the context of general relativity, n is the proper time of an observer moving orthogonally to the hypersurfaces, and is called the *lapse function*. The geometry of the foliation is determined by this function n together with the Lie derivative $\mathcal{L}_T g$. The latter is nothing but the second fundamental form, or extrinsic curvature, of the slices Σ_t embedded in the manifold M.

We always assume that the geodesic ball $\mathcal{B}_{\Sigma_0}(p, 1) \subset \Sigma_0$ (determined by the induced metric $g|_{\Sigma_0}$) is compactly contained in Σ_0. We introduce the following assumptions:

$$e^{-K_0} \leq n \leq e^{K_0} \quad \text{in } \Omega, \tag{A1}$$

$$|\mathcal{L}_T g|_T \leq K_1 \quad \text{in } \Omega, \tag{A2}$$

$$|\mathbf{Rm}_g|_T \leq K_2 \quad \text{in } \Omega, \tag{A3}$$

$$\mathbf{Vol}_{g|_{\Sigma_0}}(\mathcal{B}_{\Sigma_0}(p, 1)) \geq v_0, \tag{A4}$$

where K_0, K_1, K_2 and v_0 are positive constants. Observe that Assumption $(A4)$ is a condition on the initial slice only; together with the other assumptions it actually implies a lower volume bound for every slice of the foliation.

We will prove:

Theorem 3.1. (Injectivity radius of foliated manifolds). *Let M be a differentiable manifold endowed with a Lorentzian metric g satisfying the regularity assumptions* $(A1)$–$(A4)$ *at some point p and for some foliation* (3.1). *Then, there exists a positive constant i_0 depending only upon the foliation bounds K_0, K_1, the curvature bound K_2, the volume bound v_0, and the dimension of the manifold such that the injectivity radius at p satisfies*

$$\mathbf{Inj}_g(M, p, T) \geq i_0.$$

The following section is devoted to the proof of this theorem. Observe that conditions $(A1)$–$(A4)$ are *local* about one point of the manifold and are stated in purely geometric terms, requiring no particular choice of coordinates. Of course, the conclusion of Theorem 3.1 holds globally in M if the assumptions $(A1)$–$(A4)$ hold at every point of the manifold. Our assumptions do depend on the choice of the time-like vector field T, but the dependence of the constants arising in $(A1)$–$(A4)$ should not be essential in the applications. It is conceivable that a quantitatively sharper estimate would be obtained with a choice of an "almost Killing" field, that is, a field T corresponding to a "small" Lie derivative $\mathcal{L}_T g$. Later in Sect. 7, a more general approach is presented in which the vector field T is constructed from a *single* vector prescribed at the point p.

Basic estimates on the reference metric. To establish Theorem 3.1 it is convenient to introduce coordinates on Ω, chosen as follows. Fix arbitrarily some coordinates (x^i) on the initial slice Σ_0. Then, transport these coordinates to the whole of Ω along the integral curves of the vector field T. This construction generates coordinates (x^α) on Ω such that $x^0 = t$ and the vector $\partial/\partial t$ is orthogonal to each vector $\partial/\partial x^j$ $(1 \leq j \leq n)$, so that the Lorentzian metric takes the form

$$g = -n^2\, dt^2 + g_{ij}\, dx^i dx^j, \tag{3.2}$$

where n is the lapse function and g_{ij} is the Riemannian metric induced on the slices Σ_t. The reference Riemannian metric in the domain Ω then takes the form

$$g_T = n^2 dt^2 + g_{ij}\, dx^i dx^j, \tag{3.3}$$

and the Riemannian norm of a vector X equals $g_T(X, X) := n^2\, X^0 X^0 + X^j X_j$.

We want to control the discrepancy between the reference Riemannian metric g_T and the original Lorentzian metric g, as measured in the connections ∇ and ∇_{g_T} and the curvature tensors **Rm** and $\mathbf{Rm}_{g_T}$. Clearly, these estimates should involve the constants arising in $(A1)$–$(A4)$. Consider the general class of metrics

$$\widetilde{g} := f\, dt^2 + g_{ij}\, dx^i dx^j, \tag{3.4}$$

which allows us to recover both the Lorentzian ($f = -n^2$) and the Riemannian ($f = n^2$) metrics.

In view of the general expressions of the Christoffel and Riemann curvature coefficients

$$\begin{aligned}
\widetilde{\Gamma}^{\gamma}_{\alpha\beta} &:= \frac{1}{2}\widetilde{g}^{\gamma\delta}\left(\frac{\partial \widetilde{g}_{\delta\beta}}{\partial x^{\alpha}} + \frac{\partial \widetilde{g}_{\delta\alpha}}{\partial x^{\beta}} - \frac{\partial \widetilde{g}_{\alpha\beta}}{\partial x^{\delta}}\right),\\
\widetilde{R}^{\zeta}_{\alpha\beta\delta} &:= \frac{\partial \widetilde{\Gamma}^{\zeta}_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial \widetilde{\Gamma}^{\zeta}_{\alpha\delta}}{\partial x^{\beta}} + \widetilde{\Gamma}^{\zeta}_{\alpha\eta}\widetilde{\Gamma}^{\eta}_{\beta\delta} - \widetilde{\Gamma}^{\zeta}_{\beta\eta}\widetilde{\Gamma}^{\eta}_{\alpha\delta},\\
\widetilde{R}_{\alpha\beta\gamma\delta} &:= \widetilde{g}_{\gamma\zeta}\widetilde{R}^{\zeta}_{\alpha\beta\delta}, \qquad \widetilde{R}_{\alpha\beta} := \widetilde{R}_{\alpha\gamma\beta\delta}\widetilde{g}^{\gamma\delta},
\end{aligned}$$

we can compute explicitly the Christoffel symbols associated with the metric $\widetilde{g}$,

$$\begin{aligned}
&\widetilde{\Gamma}^0_{00} = \frac{1}{2f}\frac{\partial f}{\partial t}, \qquad \widetilde{\Gamma}^0_{0i} = \frac{1}{2f}\frac{\partial f}{\partial x^i}, \qquad \widetilde{\Gamma}^0_{ij} = -\frac{1}{2f}\frac{\partial g_{ij}}{\partial t},\\
&\widetilde{\Gamma}^k_{00} = -\frac{1}{2}g^{kl}\frac{\partial f}{\partial x^l}, \qquad \widetilde{\Gamma}^k_{i0} = \frac{1}{2}g^{kl}\frac{\partial g_{li}}{\partial t}, \qquad \widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij},
\end{aligned} \tag{3.5}$$

as well as the (non-trivial) curvature terms

$$\begin{aligned}
\widetilde{R}_{ijkl} &= R_{ijkl} - \frac{1}{4f}\left(\frac{\partial g_{ik}}{\partial t}\frac{\partial g_{jl}}{\partial t} - \frac{\partial g_{il}}{\partial t}\frac{\partial g_{jk}}{\partial t}\right),\\
\widetilde{R}^p_{0jl} &= \frac{\partial \Gamma^p_{jl}}{\partial t} - \frac{\partial}{\partial x^j}\left(\frac{1}{2}g^{pq}\frac{\partial g_{ql}}{\partial t}\right) + \frac{1}{2}g^{pq}\frac{\partial}{\partial t}g_{qk}\Gamma^k_{jl} - \Gamma^p_{jk}\left(\frac{1}{2}g^{kq}\frac{\partial g_{lq}}{\partial t}\right)\\
&\quad + \left(\frac{1}{4f}g^{pq}\frac{\partial f}{\partial x^q}\right)\frac{\partial g_{jl}}{\partial t} - \frac{1}{2}g^{pq}\frac{\partial g_{qj}}{\partial t}\frac{1}{2f}\frac{\partial f}{\partial x^l},\\
\widetilde{R}_{0jil} &= \frac{1}{2}\left(\nabla_l\left(\frac{\partial}{\partial t}g_{ij}\right) - \nabla_i\left(\frac{\partial}{\partial t}g_{lj}\right)\right) + \frac{1}{4f}\left(\frac{\partial f}{\partial x^i}\frac{\partial g_{jl}}{\partial t} - \frac{\partial f}{\partial x^l}\frac{\partial g_{ij}}{\partial t}\right),\\
\widetilde{R}^p_{i00} &= \frac{\partial}{\partial x^i}\left(-\frac{1}{2}g^{pq}\frac{\partial f}{\partial x^q}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}g^{pq}\frac{\partial g_{qi}}{\partial t}\right) + \Gamma^p_{il}\left(-\frac{1}{2}g^{lq}\frac{\partial f}{\partial x^q}\right)\\
&\quad - \left(\frac{1}{2}g^{pq}\frac{\partial g_{ql}}{\partial t}\right)\left(\frac{1}{2}g^{lr}\frac{\partial g_{ri}}{\partial t}\right) + \frac{1}{2f}\frac{\partial f}{\partial t}\left(\frac{1}{2}g^{pq}\frac{\partial g_{qi}}{\partial t}\right) + \frac{1}{2}g^{pq}\frac{\partial f}{\partial x^q}\frac{1}{2f}\frac{\partial f}{\partial x^i},
\end{aligned}$$

and

$$\widetilde{R}_{i0j0} = -\frac{1}{2}\left(\nabla_i\nabla_j f + \frac{\partial^2 g_{ij}}{\partial t^2}\right) + \frac{1}{4}g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + \frac{1}{4f}\frac{\partial f}{\partial t}\frac{\partial g_{ij}}{\partial t} + \frac{1}{4f}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

By applying the formulas above to the metrics g and g_T we can estimate the Christoffel symbols, as follows. Recall that the difference $\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{g_T,\alpha\beta}$ can be regarded as a tensor field on M, so that the following (Riemannian) norm squared is a scalar field on the manifold M:

$$|\nabla_{g_T} - \nabla|_T^2 := |\Gamma_{g_T} - \Gamma|_T^2 = (\Gamma^{\alpha}_{g_T,\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma})\,(\Gamma^{\alpha'}_{g_T,\beta'\gamma'} - \Gamma^{\alpha'}_{\beta'\gamma'})\, g_{T,\alpha\alpha'}\, g_T^{\beta\beta'}\, g_T^{\gamma\gamma'}.$$

Lemma 3.2. (Levi-Cevita connection of the reference metric). *Suppose that g satisfies Assumptions $(A1)$–$(A2)$. Then, the covariant derivative of the Lorentzian and Riemannian metrics are comparable, precisely at every point of Ω,*

$$|\nabla_{g_T} - \nabla|_T = n^2\,|\mathcal{L}_T g|_T^2 \le e^{2K_0} K_1^2 =: K_3.$$

Proof. In view of (3.5) the difference $\Gamma_{g_T} - \Gamma$ depends essentially upon the terms $\frac{\partial n}{\partial x^i}$ and $\frac{\partial g_{ij}}{\partial t}$ which precisely appear in the expression of the Lie derivative of g along the vector field T (as follows by a direct computation from (3.2))

$$(\mathcal{L}_T g)_{00} = 0, \qquad (\mathcal{L}_T g)_{0i} = \frac{1}{n}\frac{\partial n}{\partial x^i}, \qquad (\mathcal{L}_T g)_{ij} = \frac{1}{n}\frac{\partial g_{ij}}{\partial t}. \tag{3.6}$$

We omit the details. □

Next, observe that the difference between the curvature tensors can not be similarly estimated, and that this is one of the main difficulties to deal with in the present work. For future reference we provide here the expressions of certain curvature coefficients of g and g_T in terms of (first-order derivatives of) the lapse function n and the induced metric g_{jk}:

$$\begin{aligned}
R_{ijkl} &= R^{\Sigma}_{ijkl} + \frac{1}{4n^2}\left(\frac{\partial g_{ik}}{\partial t}\frac{\partial g_{jl}}{\partial t} - \frac{\partial g_{il}}{\partial t}\frac{\partial g_{jk}}{\partial t}\right),\\
R_{0jil} &= \frac{1}{2}\left(\nabla_l(\frac{\partial}{\partial t} g_{ij}\right) - \nabla_i\left(\frac{\partial}{\partial t} g_{lj})\right) + \frac{1}{2n}\left(\frac{\partial n}{\partial x^i}\frac{\partial g_{jl}}{\partial t} - \frac{\partial n}{\partial x^l}\frac{\partial g_{ij}}{\partial t}\right),\\
R_{i0j0} &= \frac{1}{2}\left(\nabla_i\nabla_j(n^2) - \frac{\partial^2 g_{ij}}{\partial t^2}\right) + \frac{1}{4} g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + \frac{1}{2n}\frac{\partial n}{\partial t}\frac{\partial g_{ij}}{\partial t} - \frac{\partial n}{\partial x^i}\frac{\partial n}{\partial x^j},
\end{aligned}$$

and

$$\begin{aligned}
R_{T,ijkl} &= R^{\Sigma}_{ijkl} - \frac{1}{4n^2}\left(\frac{\partial g_{ik}}{\partial t}\frac{\partial g_{jl}}{\partial t} - \frac{\partial g_{il}}{\partial t}\frac{\partial g_{jk}}{\partial t}\right),\\
R_{T,0jil} &= \frac{1}{2}\left(\nabla_l\left(\frac{\partial}{\partial t} g_{ij}\right) - \nabla_i\left(\frac{\partial}{\partial t} g_{lj}\right)\right) + \frac{1}{2n}\left(\frac{\partial n}{\partial x^i}\frac{\partial g_{jl}}{\partial t} - \frac{\partial n}{\partial x^l}\frac{\partial g_{ij}}{\partial t}\right),\\
R_{T,i0j0} &= \frac{1}{2}\left(\nabla_i\nabla_j(-n^2) - \frac{\partial^2 g_{ij}}{\partial t^2}\right) + \frac{1}{4} g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + \frac{1}{2n}\frac{\partial n}{\partial t}\frac{\partial g_{ij}}{\partial t} + \frac{\partial n}{\partial x^i}\frac{\partial n}{\partial x^j},
\end{aligned}$$

where R^{Σ}_{ijkl} denotes the induced curvature tensor on the time slices $\Sigma = \Sigma_t$.

4. Derivation of the First Injectivity Radius Estimate

In this section we provide a proof of Theorem 3.1.

Step 1. Radius of definition of the exponential map. First of all, the injectivity radius of the Riemannian metric $g|_{\Sigma_0}$ induced on the initial hypersurface $\Sigma_0 = t^{-1}(0)$ is controled as follows. In view of Assumptions $(A3)$ and $(A4)$, the Riemann curvature of the metric $g|_{\Sigma_0}$ is bounded and the volume of the unit geodesic ball $\mathbf{Vol}_{g|_{\Sigma_0}}(\mathcal{B}_{\Sigma_0}(p, 1))$ is bounded below. Therefore, according to [9], there exists a constant $i_1 = i_1(K_2, v_0)$ such that the injectivity radius of $g|_{\Sigma_0}$ at the point p is i_1 at least:

$$\mathbf{Inj}_{g|_{\Sigma_0}}(\Sigma_0, p) \geq i_1.$$

Moreover, according to [15] we can also assume that i_1 is sufficiently small so that, given any $\varepsilon > 0$ there exist coordinates (x^α) defined in a ball with definite size near p, with $x^\alpha(p) = 0$, such that the metric $g|_{\Sigma_0}$ is close to the n-dimensional Euclidian metric $g_{E'} = \delta_{ij}$ (in these coordinates). More precisely, on the initial slice Σ_0 we have

$$e^{-\varepsilon}\,\delta_{ij} \leq g_{ij}(0, x^1, \dots, x^n) \leq e^{\varepsilon}\,\delta_{ij}, \quad (x^1, \dots, x^n) \in B_{E'}(0, i_1),$$

where we have set $B_{E'}(0, r) := \left\{(x^1)^2 + \dots + (x^n)^2 < r^2\right\} \subset \mathbb{R}^n$. The latter can be regarded as a subset of Σ_0 by identifying a point with its coordinates; we will also use the notation $\mathcal{B}_{E'}(p, r)$ for this Euclidean ball.

We can next introduce some coordinates $(x^\alpha) = (t, x^j)$ on the manifold, by propagating the coordinates (x^j) chosen on Σ_0 along the integral curve of the vector field T. This construction allows us to cover the domain Ω. From Assumption $(A2)$ (together with $(A1)$ and (3.6)) we deduce that the induced metric on *each slice* of the foliation is comparable with the n-dimensional Euclidean metric in some time interval $[-i_2, i_2]$, that is,

$$\begin{aligned}(e^{-\varepsilon} - K\,i_2)\,\delta_{ij} &\leq g_{ij}(x) \leq (e^{\varepsilon} + K\,i_2)\,\delta_{ij},\\ x = (t, x^1, \dots, x^n) &\in [-i_2, i_2] \times B_{E'}(0, i_1),\end{aligned}$$

for some $K > 0$ depending only on K_0, K_1, K_2.

We then restrict attention to a smaller radius $i_2 = i_2(K_0, K_1, K_2) \leq i_1$ chosen such that $e^{-\varepsilon} - K\,i_2 > 0$, and we pick up $c_1 \geq 0$ sufficiently large so that $e^{-c_1} \leq e^{-\varepsilon} - K\,i_2 \leq e^{\varepsilon} + K\,i_2 \leq e^{c_1}$. In turn, in view of Assumption $(A1)$ on the lapse function n and of the expression (3.3) of the reference Riemannian metric g_T, the above inequalities imply that g_T is comparable to the $(n+1)$-dimensional Euclidean metric:

$$e^{-c_2}\,\delta_{\alpha\beta} \leq g_{T,\alpha\beta} \leq e^{c_2}\,\delta_{\alpha\beta}, \quad x = (t, x^1, \dots, x^n) \in [-i_2, i_2] \times B_{E'}(0, i_2)$$

for some constant $c_2 \geq c_1$ depending upon c_1 and K_0.

Introducing on the manifold the $(n+1)$-dimensional Euclidian metric E (which we define in the constructed coordinates (x^α) and is, of course, independent of the point on the manifold) and introducing also the corresponding Euclidian metric ball $\mathcal{B}_E(p, i_2)$, we have established

$$e^{-c_2}\,g_E \leq g_{T,q} \leq e^{c_2}\,g_E, \quad q \in \mathcal{B}_E(p, i_2). \tag{4.1}$$

In the following we use the notation $|X|_E$ for the Euclidean norm of a vector X.

Our first task is to determine the radius of a ball on which the exponential map is well-defined. This radius depends upon the reference vector field T. Let $\gamma : [0, s_0] \to M$ be a geodesic associated with the Lorentzian metric g and satisfying $\gamma(0) = p$. Assume that this geodesic is included in the Euclidean ball $\mathcal{B}_E(p, i_2)$ (in which we already control the metric g_T). Obviously, the Lorentzian norm

$$\langle \gamma'(s), \gamma'(s)\rangle_g = \langle \gamma'(0), \gamma'(0)\rangle_g, \qquad s \in [0, s_0],$$

is constant. On the other hand, to determine the length of $\gamma'(s)$ with respect to the *reference* metric g_T, we proceed as follows:

$$\begin{aligned}
\left| \frac{d}{ds} \langle \gamma'(s), \gamma'(s)\rangle_T \right| &= \left| \nabla_{T,\gamma'(s)} \left(g_T(\gamma'(s), \gamma'(s)) \right) \right| \\
&= 2 \left| \langle \nabla_{g_T,\gamma'(s)} \gamma'(s), \gamma'(s)\rangle_T \right| \\
&= 2 \left| \langle (\nabla_{g_T} - \nabla_g)_{\gamma'(s)}\, \gamma'(s), \gamma'(s)\rangle_T \right| \\
&\le 2\, |\nabla_{g_T} - \nabla_g|_T\, |\gamma'(s)|_T^3.
\end{aligned}$$

So, by Lemma 3.2, $\left| \frac{d}{ds} |\gamma'(s)|_T^2 \right| \le 2\, K_3\, |\gamma'(s)|_T^3$, and, in consequence,

$$\left| \frac{d}{ds} |\gamma'(s)|_T^{-1} \right| \le K_3.$$

By integration of the above inequality and *provided* s is small enough so that $2s\, K_3\, |\gamma'(0)|_T < 1$, we see that

$$\frac{1}{2}\, |\gamma'(0)|_T \le |\gamma'(s)|_T \le 2\, |\gamma'(0)|_T. \tag{4.2}$$

In view of (4.1) this implies

$$\frac{e^{-c_2}}{2}\, |\gamma'(0)|_E \le |\gamma'(s)|_E \le 2\, e^{c_2}\, |\gamma'(0)|_E. \tag{4.3}$$

These inequalities hold for all $s \in [0, 1/(2K_3\, |\gamma'(0)|_T)]$ as long as $\gamma(s) \in \mathcal{B}_E(p, i_2)$. In particular, by restricting attention to geodesics whose initial vector has unit Euclidean length, $|\gamma'(0)|_E = 1$, we see that $\gamma([0, r_2]) \subset \mathcal{B}_E(p, i_2)$ where $r_2 := i_2 e^{-c_2}/2$. Hence, for such curves the geodesic equation is well-defined, and this establishes that the exponential map at the point p is well-defined on the ball $B_E(0, r_2)$ with a range included in the geodesic ball $\mathcal{B}_E(p, i_2)$.

Step 2. Conjugate radius estimate. Our second task is to determine a ball on which the exponential map is a local diffeomorphism and we, therefore, need to control the length of Jacobi fields along a geodesic. Let $\gamma : [0, r_2] \to M$ be a g-geodesic satisfying $\gamma(0) = p$ and $|\gamma'(0)|_E = 1$. By the discussion in Step 1 we already know that the curve γ lies in $\mathcal{B}_E(p, i_2)$ and that $\max_{s\in[0,r_2]} |\gamma'(s)|_T \le 2\, e^{2c_2}$. Given an arbitrary Jacobi field along γ, $J = J(s)$, satisfying

$$\begin{aligned}
&J''(s) = -\mathbf{Rm}(J(s), \gamma'(s))\gamma'(s), \\
&J(0) = 0, \qquad |J'(0)|_T = 1,
\end{aligned}$$

we need to control its Riemannian length $F(s) := |J|_T(s)$ (as stated in (4.7), below).

Let $[0, s_0]$ be the largest subinterval of $[0, r_2/4]$ in which the inequality $|J|_T \leq 1$ holds. Using the equation satisfied by the Jacobi field and taking into account the curvature bound $(A3)$, we deduce that, in the interval $[0, s_0]$,

$$\begin{aligned}\left|\frac{d}{ds}\langle\nabla_{\gamma'}J, \nabla_{\gamma'}J\rangle_T\right| &= 2\,\left|\langle\nabla_{g_T,\gamma'}\nabla_{\gamma'}J, \nabla_{\gamma'}J\rangle_T\right| \\ &\leq 2\,|\nabla_{g_T} - \nabla_g|_T|\gamma'|_T\,|\nabla_{\gamma'}J|_T^2 + 2\,K_2\,|\gamma'|_T^2|J|_T\,|\nabla_{\gamma'}J|_T.\end{aligned}$$

With (4.2) and the covariant derivative estimate in Lemma 3.2, we obtain

$$\left|\frac{d}{ds}|\nabla_{\gamma'}J|_T\right| \leq 4\,K_3\,|\nabla_{\gamma'}J|_T + 8\,K_2. \tag{4.4}$$

Next, we integrate (4.4) over an arbitrary interval $[0, s] \subset [0, s_0]$, use the initial condition on the Jacobi field, and obtain

$$1 + \frac{2K_2}{K_3}\,(1 - e^{-4K_3 s}) \leq |\nabla_{\gamma'}J|_T \leq 1 + \frac{2K_2}{K_3}\,(e^{4K_3 s} - 1).$$

Assuming that r_2 is small enough so that $2K_2\,(1-e^{-4K_3 s})/K_3 \geq -1/2$ and $2K_2\,(e^{4K_3 s} - 1)/K_3 \leq 1$, we infer that

$$\frac{1}{2} \leq |\nabla_{\gamma'}J|_T \leq 2. \tag{4.5}$$

Hence, using this inequality and Lemma 3.2 we find that $\frac{d}{ds}|J|_T \leq 2 + 2K_3 \leq 1$ and

$$F(s) = |J|_T(s) \leq (2 + 2K_3)\,s \leq (2 + 2K_3)\,r_2. \tag{4.6}$$

Further assuming that $(2 + 2K_3)\,r_2 \leq 1$, we conclude that $s_0 = r_2$.

Next, we want to improve the rough estimate (4.6). Since

$$\frac{d}{ds}\langle\nabla_{\gamma'}J, J\rangle_T = \langle\nabla_{g_T,\gamma'}\nabla_{\gamma'}J, J\rangle_T + \langle\nabla_{g_T,\gamma'}J, \nabla_{\gamma'}J\rangle_T,$$

then by substituting the previous estimates of $|J|_T(s)$ and $|\nabla_{\gamma'}J|_T(s)$ and performing similar calculations as above, we get

$$e^{-c_3} \leq \frac{d}{ds}\langle\nabla_{\gamma'}J, J\rangle_T \leq e^{c_3}$$

for some constant $c_3 > 0$. By integration this implies

$$e^{-c_3}\,s \leq \langle\nabla_{\gamma'}J, J\rangle_T \leq e^{c_3}\,s$$

and, for some $c_4 > 0$, we arrive at the following lower bound for the norm of the Jacobi field:

$$F(s) \geq \frac{\left|\langle\nabla_{\gamma'}J, J\rangle_T\right|}{|\nabla_{\gamma'}J|_T} \geq \frac{e^{-c_3}\,s}{2} \geq e^{-c_4}\,s.$$

On the other hand, using again the above estimates we have

$$\begin{aligned}\frac{d}{ds}F &\le \frac{1}{F}(\langle \nabla_{g_T,\gamma'} J, J\rangle_T + K_3\, F^2)\\ &\le \frac{e^{c_4}}{s}\left(e^{c_3}\, s + K_3(2+2K_3)^2\, s^2\right) \le e^{c_5}\end{aligned}$$

for some constant $c_5 > 0$, which now yields the upper bound

$$F(s) \le e^{c_5}\, s.$$

In summary, we have established that the norm of the Jacobi field is comparable with s:

$$e^{-c_4}\, s \le F(s) \le e^{c_5}\, s, \qquad s \in [0, r_2]. \tag{4.7}$$

By the definition of Jacobi fields these inequalities are equivalent to a uniform control of the differential of the exponential map, that is, for $s \in [0, r_2]$,

$$e^{-c_4}\, |W|_T \le |D\mathbf{exp}_{p,s\gamma'(0)}(W)|_T \le e^{c_5}\, |W|_T.$$

By the inverse mapping theorem, $D\mathbf{exp}_{p,s\gamma'(0)}$ is a local diffeomorphism.

We also conclude that the pull back of the reference metric to the tangent space at p satisfies

$$\begin{aligned}&e^{-c_4}\, g_{T,p} \le \left(\mathbf{exp}_p\right)^\star g_T \le e^{c_5}\, g_{T,p}\\ &\text{in the ball } B_T(0, r_2) \subset T_pM.\end{aligned} \tag{4.8}$$

Since the conjugate radius of the Lorentzian metric is precisely defined from the reference Riemannian metric, these inequalities show that the conjugate radius of the exponential map is r_2, at least.

Step 3. Injectivity radius estimate. We are now in a position to establish that $\mathbf{Inj}_g(M, p, T) \ge r_3 := r_2 e^{-3c_2}/4$. We argue by contradiction and assume that $\gamma_1 : [0, s_1] \to M$ and $\gamma_2 : [0, s_2] \to M$ are two distinct g-geodesics satisfying $\max(s_1, s_2) \le r_3$ and

$$\begin{aligned}&\gamma_1(0) = \gamma_2(0) = p, \quad |\gamma_1'(0)|_T = |\gamma_2'(0)|_T = 1,\\ &\gamma_1(s_1) = \gamma_2(s_2) =: q.\end{aligned}$$

We will reach a contradiction and this will establish that the injectivity radius is greater or equal to r_3 (as can be checked by using the fact that the exponential map is at least a local diffeomorphism).

By Step 1 (since $r_3 \le r_2 \le i_2$) we know that $\gamma_1, \gamma_2 \subset \mathcal{B}_E(p, 2e^{2c_2} r_3)$. By concatenating these two curves, we construct a geodesic loop containing p,

$$\gamma = \gamma_2^{-1} \cup \gamma_1 : [0, s_1 + s_2] \to \mathcal{B}_E(p, 2e^{2c_2} r_3),$$

which need not be smooth at p nor q. Since γ is contained in the image of the ball $B_T(p, r_2)$ under the exponential map, we can define a homotopy of γ with the origin ($x = 0$), by setting (in the coordinates constructed earlier)

$$\Gamma_\varepsilon(s) = \varepsilon\gamma(s), \quad \varepsilon \in [0, 1].$$

The curves $\Gamma_\varepsilon : [0, s_1 + s_2] \to \mathcal{B}_E(p, 2e^{2c_2} r_3)$ satisfy

$$\Gamma_\varepsilon(0) = \Gamma_\varepsilon(s_1 + s_2) = p, \quad \Gamma_0([0, 1]) = p, \quad \Gamma_1 = \gamma.$$

Moreover, we have $|\Gamma'_\varepsilon(s)|_E \le \varepsilon 2e^{2c_2} \le 2e^{2c_2}$ and thus (in view of (4.1)) $|\Gamma'_\varepsilon(s)|_T \le 2e^{3c_2}$. In particular, the g_T-lengths (computed with the reference metric) of the loops Γ_ε are less than

$$L(\Gamma_\varepsilon, g_T) \le 2e^{3c_2} r_3 = \frac{r_2}{2},$$

due to the choice made for r_3.

Since the exponential map is a local diffeomorphism from the ball $B_T(0, r_2) \subset T_pM$ to the manifold, and in view of the estimate (4.8) on the exponential map, it follows that all the loops Γ_ε can be lifted to the ball $B_T(0, r_2)$ in the tangent space with the *same* origin 0. Consequently, we obtain a *continuous* family of curves $\widetilde{\Gamma}_\varepsilon : [0, s_1+s_2] \to T_pM$ satisfying

$$\widetilde{\Gamma}_\varepsilon(0) = 0, \quad \varepsilon \in [0, 1].$$

At this juncture we observe that, since $\widetilde{\Gamma}_\varepsilon(s_1 + s_2)$ (for $\varepsilon \in [0, 1]$) all cover the same point p and since the curve $\widetilde{\Gamma}_0$ is trivial and the family is continuous,

$$\widetilde{\Gamma}_\varepsilon(s_1 + s_2) = 0, \quad \varepsilon \in [0, 1].$$

It remains to consider the lift of the original geodesic loop γ: under the lifting the geodesics γ_1, γ_2 are sent to two distinct *line segments* (with respect to the vector space structure) originating at the origin 0 which obviously do not intersect. This is a contradiction and we conclude that, in fact, $\mathbf{Inj}_g(M, p, T) \ge r_3$ as announced. This completes the proof of Theorem 3.1.

5. Convex Functions and Convex Neighborhoods

We establish now the existence of convex functions and convex neighborhoods in M. Let us recall first some basic definitions. A function u is said to be *geodesically convex* if the composition of u with any geodesic is a convex function (of one variable). A set $\Omega' \subset \Omega''$ is said to be *relatively geodesically convex in* Ω'' if, given any points $p, q \in \Omega'$ and any geodesic (segment) γ from p to q contained in Ω'', one has $\gamma \subset \Omega'$. A set Ω' is said to be *geodesically convex in* Ω'' if Ω' is relatively geodesically convex in Ω'' and, in addition, for any $p, q \in \Omega'$ there exists a unique geodesic γ connecting p and q and lying in Ω'.

We denote by d_T the distance function associated with the reference Riemannian metric g_T.

Theorem 5.1. (Existence of geodesically convex functions). *Let (M, g) be a differentiable $(n+1)$-manifold endowed with a Lorentzian metric g, satisfying the regularity assumptions $(A1)$–$(A4)$ for some point $p \in M$ and some future-oriented, unit, time-like vector field T, and let g_T be the associated Riemannian metric. Then, for any $\varepsilon \in (0, 1)$ there exists a positive constant r_0 depending only upon ε, the foliation bounds K_0, K_1, the curvature bound K_2, the volume bound v_0, and the dimension of the manifold and there exists a smooth function u defined on $\mathcal{B}_T(p, r_0)$ such that*

$$(1-\varepsilon)\, d_T(p, \cdot)^2 \le u \le (1+\varepsilon)\, d_T(p, \cdot)^2,$$
$$(2-\varepsilon)\, g_T \le \nabla^2 u \le (2+\varepsilon)\, g_T.$$

Hence, the function u above is equivalent to the Riemannian distance function from p and is geodesically convex for the Lorentzian metric. In the proof given below, the function u is the Riemannian distance function associated with a new time-like vector field (denoted by N in the proof below). The following corollary is immediate and provides us with a control of the radius of convexity, which generalizes the Whitehead theorem from Riemannian geometry [22,6].

Corollary 5.2. (Existence of geodesically convex neighborhoods). *Under the assumptions of Theorem 5.1, for any $0 < r < r_0$ there exists a set $\Omega_r \subset \Omega$ which is geodesically convex in $\mathcal{B}_T(p, 2r_0)$ and satisfies*

$$\mathbf{exp}_p(B_T(0, r)) \subset \Omega_r \subset \mathbf{exp}_p(B_T(0, (1 + \delta)r)).$$

Moreover, one can choose Ω_r so that

$$\mathcal{B}_T(p, r) \subset \Omega_r \subset \mathcal{B}_T(p, (1 + \delta)r)$$

also holds, where $\mathcal{B}_T(p, r)$ is the geodesic ball determined by the reference Riemannian metric.

Proof of Theorem 5.1. Step 1. Synchronous coordinate system. Given $\varepsilon > 0$, by applying the injectivity radius estimate in Theorem 3.1 to all points near p, we can find a constant r_0 depending on $K_0, K_1, K_2, v_0, \varepsilon, n$ such that for any $q \in \mathcal{B}_T(p, 2r_0)$ the injectivity radius at q is $2r_0$ at least, and we can assume that

$$e^{-\varepsilon}\, g_{T,q} \le (\mathbf{exp}_q)^\star g_T \le e^{\varepsilon}\, g_{T,q}, \qquad B_T(0, r_0) \subset T_qM,\ \ q \in \mathcal{B}_T(p, 2r_0).$$

Let $\gamma = \gamma(s)$ be the past time-like geodesic from p and satisfying $\gamma(0) = p$ and $\gamma'(0) = -T_p$, and consider the (past) point $q := \gamma(r_0/2)$. The future null cone at q with radius r_0 (the orientation being determined by the vector field T) is defined by

$$C_q(r_0) := \Big\{ V \in T_qM \,\big/\, |V|_{g_{T,q}} < r_0,\ |V|^2_{g_q} < 0,\ \langle V, T\rangle > 0 \Big\}.$$

Observe that the g_T-length of γ between p and q is approximatively $r_0/2$ and that the norm $|\gamma'|_T$ is almost 1, while $|\gamma'(q)|^2_{g_q} = 1$ and $\langle -\gamma', T\rangle_g > 0$. By the injectivity radius estimate in Theorem 3.1 the exponential map at q is a diffeomorphism from $C_q(r_0)$ onto its image which, moreover, contains the original point p.

Next, introduce the set of vectors that are "almost" parallel to T:

$$C_q(r_0, \varepsilon) := \left\{ V \in T_qM \,\big/\, |V|_{T,q} < r_0,\ \langle V, T\rangle_{g_q} > 0,\ \frac{\langle V, V\rangle_{g_q}}{\langle V, V\rangle_{T,q}} > 1 - \varepsilon \right\}.$$

The notation $c(\varepsilon) > 0$ is used for constants that depend only on $K_0, K_1, K_2, v_0, n, \varepsilon$ and satisfy $\lim_{\varepsilon\to 0} c(\varepsilon) = 0$. We claim that there is constant $c(\varepsilon) > 0$ such that

$$\mathcal{B}_T(p, c(\varepsilon)r_0) \subset \mathbf{exp}_q(C_q(r_0, \varepsilon)). \tag{5.1}$$

Actually, we have $\mathcal{B}_T(p, c(\varepsilon)r_0) \subset \mathcal{B}_T(q, (\frac{1}{2} + c(\varepsilon))r_0)$, hence

$$\mathcal{B}_T(p, c(\varepsilon)r_0) \subset \mathbf{exp}_q\left(B_T(0, (\frac{1}{2} + c(\varepsilon))r_0) \right).$$

Since the metrics $g_{T,0}$ and $g_{T,q}$ are comparable (under the exponential map at q) we see that geodesics σ connecting q and points of $\mathcal{B}_T(p, c(\varepsilon)r_0)$ make an angle $\leq c(\varepsilon)$ with $-\gamma'(q)$ at the point q (as measured by the metric $g_{T,q}$). By reducing the constant $c(\varepsilon)$ if necessary, the claim is proved.

Let τ be the Lorentzian distance from q: it is defined on $\mathbf{exp}_q(C_q(r_0))$ and is a smooth function on $\mathbf{exp}_q(C_q(r_0)) \setminus \{p\}$. From (5.1) we deduce that τ is smooth and satisfies

$$\left(\frac{1}{2} - c(\varepsilon)\right) r_0 < \tau < \left(\frac{1}{2} + c(\varepsilon)\right) r_0 \quad \text{in the ball } \mathcal{B}_T(p, c(\varepsilon)r_0). \tag{5.2}$$

It is clear also that

$$|\nabla\tau|_g^2 = -1, \qquad \nabla^2\tau(\nabla\tau, \cdot) = 0.$$

We now introduce a new foliation based on this Lorentzian distance function. Let (z^j) be coordinates on the level set hypersurface $\tau = \tau(p)$. By following the integral curves of the (unit, time-like) vector field

$$N := \nabla\tau$$

construct coordinates (z^α) in which $z_0 := \tau$ and the Lorentzian metric takes the form

$$g = -(dz^0)^2 + g_{ij}\, dz^i dz^j.$$

Let g_N be the reference Riemannian metric based on this (new) vector field N.

In view of Lemma 3.2 and the geodesic equation we see that all (future) g-geodesics σ satisfy the uniform bound

$$\left|\frac{d}{d\tau} \log|\sigma'(\tau)|\right| \leq K_3\, r_0.$$

(Recall that we allow r_0 to depend upon ε.) This inequality shows that the vector field N makes an angle $\leq c(\varepsilon)$ with T, everywhere on $\mathbf{exp}_q(C_q(r_0, \varepsilon))$. From this, we conclude that the two metrics are comparable:

$$(1 - c(\varepsilon))\, g_T \leq g_N \leq (1 + c(\varepsilon))\, g_T \quad \text{in the cone } \mathbf{exp}_q(C_q(r_0, \varepsilon)).$$

Step 2. Hessian comparison theorem and curvature bound for the reference metric g_N. Since $p \in \mathbf{exp}_q(C_q(r_0))$, let $\sigma : [0, \tau(p)] \to M$ be the future time-like geodesic connecting q to p, and let J be the Jacobi field defined along σ such that

$$J(0) = 0, \quad J(\tau(p)) = V,$$

where $V \in T_pM$ is given and satisfies the orthogonality condition $\langle\nabla\tau, V\rangle = 0$. Then, we have

$$\begin{aligned} -\nabla^2\tau(V, V) &= -\langle J, \nabla_{\nabla\tau} J\rangle = \langle J, \nabla_{\frac{\partial}{\partial\tau}} J\rangle \\ &= \int_0^{\tau(p)} \langle\nabla_{\frac{\partial}{\partial\tau}} J, \nabla_{\frac{\partial}{\partial\tau}} J\rangle_g - \mathbf{Rm}(\sigma', J, \sigma', J) =: I(J, J). \end{aligned}$$

Recall that, in the absence of conjugate points along the geodesic, Jacobi fields minimize the index form $I(V, V)$ among all vector fields with fixed boundary values. By applying a standard comparison technique from Riemannian geometry on the orthogonal space

$(\nabla\tau)^\perp$ (on which the Lorentzian metric induces a Riemaniann metric) we can control the Hessian of τ in terms of the curvature bound K_2, as follows:

$$\frac{\sqrt{K_2(1+c(\varepsilon))}}{\tan\sqrt{K_2(1+c(\varepsilon))}\tau}\, g\,|_{(\nabla\tau)^\perp} \le (-\nabla^2\tau)|_{(\nabla\tau)^\perp} \le \frac{\sqrt{K_2(1+c(\varepsilon))}}{\tanh\sqrt{K_2(1+c(\varepsilon))}\tau}\, g\,|_{(\nabla\tau)^\perp}.$$

Next, since $-\nabla^2_{ij}\tau = \frac{1}{2}\frac{\partial g_{ij}}{\partial\tau}$, we deduce from the above inequalities that

$$\frac{g_{ij}}{\tau} \le \frac{\partial g_{ij}}{\partial\tau} \le \frac{3g_{ij}}{\tau} \quad \text{in the cone } \mathbf{exp}_q(C_q(r_0)). \tag{5.3}$$

In view of the curvature expressions given at the end of Sect. 3, i.e. since the lapse function is now constant

$$\begin{aligned} R_{ijkl} &= R^\Sigma_{ijkl} + \frac{1}{4}\left(\frac{\partial g_{ik}}{\partial\tau}\frac{\partial g_{jl}}{\partial\tau} - \frac{\partial g_{il}}{\partial\tau}\frac{\partial g_{jk}}{\partial\tau}\right), \\ R_{0jil} &= \frac{1}{2}\left(\nabla_l(\frac{\partial}{\partial\tau}g_{ij}) - \nabla_i(\frac{\partial}{\partial\tau}g_{lj})\right), \\ R_{i0j0} &= -\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial\tau^2} + \frac{1}{4}g^{pq}\frac{\partial g_{ip}}{\partial\tau}\frac{\partial g_{jq}}{\partial\tau}, \end{aligned}$$

we conclude that

$$\left|\frac{\partial^2 g_{ij}}{\partial\tau^2}\right| \le \frac{C}{\tau^2} \quad \text{on } \mathbf{exp}_q(C_q(r_0)). \tag{5.4}$$

Finally, relying on the formulas for the curvature of the reference Riemannian metric g_N, we obtain

$$|\mathbf{Rm}_{g_N}|_N \le \frac{C}{\tau^2} \quad \text{on } \mathbf{exp}_q(C_q(r_0)). \tag{5.5}$$

Observe that, as could have been expected, the upper bounds in (5.4) and (5.5) blow-up as one approaches the point q which is the base point in our definition of the distance.

In particular, (5.5) shows the desired curvature estimate near the point p:

$$|\mathbf{Rm}_{g_N}|_N \le C r_0^{-2} \quad \text{on the ball } \mathcal{B}_T(p, c(\varepsilon)r_0).$$

Step 3. Constructing geodesically convex functions. Since the metrics g_T and g_N are comparable, the volume ratio $\mathbf{Vol}_{g_N}(\mathcal{B}_N(p, c(\varepsilon)r_0))/r_0^{n+1}$ is uniformly bounded (above and) below. Thanks to the theory for Riemannian metrics [9], the injectivity radius of the metric g_N is bounded from below by $c(\varepsilon)r_0$. Let

$$u(x) := d_{g_N}(p, x)^2$$

be the (square) of the distance function associated with the Riemannian metric g_N, which is a smooth function defined on the geodesic ball $\mathcal{B}_N(p, c(\varepsilon)r_0)$. By the standard Hessian comparison theorem for Riemannian manifold we have

$$(2-\varepsilon)\, g_{N,\alpha\beta} \le \nabla_{g_N,\alpha}\nabla_{g_N,\beta}u \le (2+\varepsilon)\, g_{N,\alpha\beta} \qquad \text{on the ball } \mathcal{B}_N(p, c(\varepsilon)r_0).$$

Next, in terms of the original Lorentzian metric g, the Hessian of the function u is

$$\nabla_\alpha \nabla_\beta u = \nabla_{g_N,\alpha} \nabla_{g_N,\beta} u + (\Gamma^\gamma_{g_N,\alpha\beta} - \Gamma^\gamma_{\alpha\beta}) \frac{\partial u}{\partial x^\alpha}.$$

Since $|\Gamma_{g_N} - \Gamma|_N \le C \sup |\frac{\partial g_{ij}}{\partial \tau}| \le C'$ by the estimate (5.3) and since also $|\nabla u|_N \le 2\, d_{g_N}$ on $\mathcal{B}_N(p, r_0)$, we conclude that

$$(2-\varepsilon)\, g_{N,\alpha\beta} \ge \nabla_\alpha \nabla_\beta u \ge (2+\varepsilon)\, g_{N,\alpha\beta} \qquad \text{in the ball } \mathcal{B}_N(p, c(\varepsilon) r_0).$$

This completes the proof of Theorem 5.1. □

6. Injectivity Radius of Null Cones

We now turn our attention to null cones within foliated Lorentzian manifolds. Our main result (Theorem 6.1 below) provides a lower bound for the null injectivity radius under the main assumption that the exponential map is defined in some ball and the null conjugate radius is already controled. Hence, contrary to the presentation in Sect. 3 our main assumption (see $(A3')$) below) is not directly stated as a curvature bound. However, under additional assumptions, it is known that the conjugate radius estimate can be deduced from an $\mathbf{L}^p$ curvature bound, so that our result is entirely relevant for the applications.

Indeed, in a series of fundamental papers [16–18], Klainerman and Rodnianski assumed an $\mathbf{L}^2$ curvature bound and estimated the null conjugate and injectivity radii for Ricci-flat Lorentzian $(3+1)$-manifolds. Our result in the present section is a continuation of the recent work [18] and covers a general class of Lorentzian manifolds with arbitrary dimension, while our proof is local and geometric and so conceptually simple.

We use the terminology and notation introduced in Sect. 2. In particular, a point $p \in M$ and a reference vector field T are given, and N_p^- denotes the past null cone in the tangent space at p. Recall from Sect. 2 that we defined the null exponential map $\mathbf{exp}_p^N : B_T^N(0, r) \to M$ over a subset of this cone,

$$B_T^N(0, r) := B_T(0, r) \cap N_p^-,$$

and we introduced the (past) null injectivity radius $\textbf{Null Inj}_g(M, p, T)$. We also set

$$\mathcal{B}_T^N(p, r) := \mathbf{exp}_p^N(B_T^N(0, r)).$$

We consider a domain $\Omega \subset M$ containing the point p on a final slice Σ_0 and foliated as

$$\Omega = \bigcup_{t\in[-1,0]} \Sigma_t, \qquad p \in \Sigma_0. \tag{6.1}$$

We assume that there exist positive constants K_0, K_1, K_2 such that

$$e^{-K_0} \le n \le e^{K_0} \quad \text{in } \Omega, \tag{A1}$$

$$|\mathcal{L}_T g|_T \le K_1 \quad \text{in } \Omega, \tag{A2}$$

the null conjugate radius at p is r_0 (at least) and the null exponential map satisfies

$$e^{-K_2}\, g_{T,p}\, |_{B_T^N(0,r_0)} \le \left(\mathbf{exp}_p^N\right)^\star (g_T\, |_{\mathcal{B}_T^N(0,r_0)}) \le e^{K_2}\, g_{T,p}\, |_{B_T^N(0,r_0)} \tag{A3'}$$

and, finally, there exists a coordinate system on the initial slice Σ_{-1} such that the metric $g\,|_{\Sigma_{-1}}$ is comparable to the n-dimensional Euclidean metric $g_{E'}$ in these coordinates:

$$e^{-K_0}\, g_{E'} \leq g\,|_{\Sigma_{-1}} \leq e^{K_0}\, g_{E'} \quad \text{in } \mathcal{B}_{\Sigma_{-1},E'}(p, r_0). \tag{A4$'$}$$

We refer to K_2 as the *effective conjugate radius constant.*

Theorem 6.1. (Injectivity radius of null cones). *Let M be a differentiable $(n+1)$-manifold, endowed with a Lorentzian metric g satisfying the regularity assumptions $(A1)$, $(A2)$, $(A3')$, and $(A4')$ at some point p and for some foliation (6.1). Then, there exists a positive constant i_0 depending only upon the foliation bounds K_0, K_1, the null conjugate radius r_0, the effective conjugate radius constant K_2, and the dimension n such that the null injectivity radius of the metric g at p satisfies*

$$\textbf{Null Inj}_g(M, p, T) \geq i_0.$$

It is interesting to compare the assumptions above with the ones made in Sect. 3. Assumptions $(A1)$ and $(A2)$ are concerned with the property of the foliation and were already required in Sect. 3.

Assumption $(A3')$ should be viewed as a weaker version of the L^∞ curvature condition $(A3)$. Recall that, under the assumptions of Theorem 3.1 which included a curvature bound, an analogue of $(A3')$ valid in the whole of Ω was already established in (4.8). It is expected that $(A3')$ is still valid when the curvature in every spacelike slice is solely bounded in some L^m space.

Indeed, at least when the spatial dimension is $n = 3$ and the manifold is Ricci-flat, Assumption $(A3')$ is a consequence of the following $\mathbf{L}^2$ curvature bound (for some constant $K_2' > 0$)

$$\|\mathbf{Rm}_g\|_{\mathbf{L}^2(\Sigma_{-1}, g_T)} \leq K_2', \tag{6.2}$$

as was established by Klainerman and Rodnianski [16,17].

Assumption $(A4')$ concerns the metric on the initial hypersurface and is only slightly stronger than the volume bound $(A4)$. Furthermore, according to Anderson [1] and Petersen [21] the property $(A4')$ is a consequence of the curvature bound (for $m > n/2$ and some constant $K_2' > 0$)

$$\|\mathbf{Rm}_g\|_{\mathbf{L}^m(\Sigma_{-1}, g_T)} \leq K_2', \tag{6.3}$$

and a volume lower bound at every scale

$$r^{-n}\, \mathbf{Vol}_{g|_{\Sigma_0}}(\mathcal{B}_{\Sigma_0}(p, r)) \geq v_0, \qquad r \in (0, r_0]. \tag{6.4}$$

Proof of Theorem 6.1. Step 1. Localization of the past null cone $\mathcal{N}^-(p)$ between two flat null cones. Assumption $(A3')$ provides us with a bound on the null conjugate radius, we need to control the injectivity radius. We proceed as in Sect. 4 and introduce coordinates near the point p such that $x^\alpha(p) = 0$. Precisely, relying on Assumptions $(A1)$, $(A2)$, and $(A4')$, we determine the coordinates $x = (x^\alpha)$ by setting $x^0 = t$ and transported (via the gradient of the function t) spatial coordinates (x^j) initially given on Σ_{-1}.

The Lorentzian metric reads $g = -n^2\, dt^2 + g_{ij}\, dx^i dx^j$ and satisfies for some $C_0, C_1 > 0$,

$$\frac{1}{C_0} \leq n^2 \leq C_0, \qquad \frac{1}{C_1}\, \delta_{ij} \leq g_{ij} \leq C_1\, \delta_{ij}, \tag{6.5}$$

for all $-r_0 < t \le 0$ and $(x^1)^2 + \ldots + (x^n)^2 \le (r_0)^2$, and in these coordinates the reference Riemannian metric g_T is comparable to the $(n+1)$-dimensional Euclidean metric $g_E := dt^2 + (dx^1)^2 + \ldots (dx^n)^2$:

$$\frac{1}{C_1}\, g_E \le g_T \le C_1\, g_E. \tag{6.6}$$

Denote by $\mathcal{B}_E(q, r)$ the Euclidean ball with center q and radius r. Note that these inequalities hold within a neighborhood of p in Ω. The forthcoming bounds will hold in a neighborhood of the past null cone, only. To simplify the notation, we set

$$c_0 := \frac{1}{C_0}, \qquad c_1 := \frac{1}{C_1}.$$

In each time slice of parameter value $t = a$ we introduce the n-dimensional Euclidean ball with radius b,

$$\mathcal{A}^a_{<b} := \Big\{ t = a, \quad (x^1)^2 + \ldots + (x^n)^2 < b^2 \Big\} \subset \Sigma_a,$$

which is centered around the point p' with coordinates $(a, 0, \cdots, 0)$. We also define $\mathcal{A}^a_{>b}$, $\mathcal{A}^a_{[c,d]}, \ldots$ in a similar way.

For any point q in a slice Σ_{t_0} satisfying $-r_0 \le t_0 < 0$ and $x^1(q)^2 + \cdots + x^n(q)^2 < c_1^2\, t_0^2$ we consider the line (for the Euclidean metric) connecting q to p:

$$\gamma(\tau) = \left(\tau, \frac{\tau}{t_0} x^1(q), \cdots, \frac{\tau}{t_0} x^n(q) \right), \qquad \tau \in [t_0, 0].$$

This is a time-like curve for the Lorentzian metric g, since

$$|\gamma'(\tau)|^2 = -n^2 + g_{ij}\, \frac{x^i(q)}{t_0}\, \frac{x^j(q)}{t_0} < -c_0 + c_1 < 0,$$

which shows that

$$\mathcal{A}^t_{<c_1 |t|} \subset \mathcal{I}^-(p), \qquad t \in (-r_0, 0).$$

On the other hand, we claim that the larger Euclidean cone $\mathcal{A}^t_{<C_1\, |t|}$ contains the null cone, in other words

$$\mathcal{A}^t_{\ge C_1\, |t|} \subset (\mathcal{N}^-(p) \cup \mathcal{I}^-(p))^c, \qquad t \in (-c_1\, r_0, 0).$$

Indeed, arguing by contradiction we suppose there exist a time $t_0 \in (-c_1\, r_0, 0)$ and a point $q \in \mathcal{A}^{t_0}_{\ge C_1\, t_0}$ connected to p by a causal curve $\gamma = \gamma(s)$ with $\gamma(0) = p$. After reparametrizing (in time) the curve if necessary, we can assume that $\gamma(\tau) = (\tau, x^j(\tau))$ for some $t_0' \le \tau \le 0$, as long as the point $\gamma(\tau)$ lies in the coordinate system under consideration. For this part of the curve at least we have

$$0 \ge |\gamma'|^2 = -n^2 + g_{ij}\, \frac{dx^i}{d\tau}\, \frac{dx^j}{d\tau},$$

which by (6.5) implies that $(\frac{dx^1}{d\tau})^2 + \ldots + (\frac{dx^n}{d\tau})^2 < C_1\, C_0$. Therefore, after integration we find

$$\left(x^1(q)^2 + \cdots + x^n(q)^2 \right)^{1/2} (t_0') \le \sqrt{C_0\, C_1}\, t_0' \le \sqrt{C_0 c_1}\, r_0 < r_0.$$

Hence, we can choose $t_0' = t_0$, the whole curve lies in our coordinate system, and is parametrized in the form $\gamma(\tau) = (\tau, x(\tau))$, $(\tau \in [t_0, 0])$. Moreover, we have $|x(t_0)| \le \sqrt{C_1\, C_0}\, |t_0| < C_1\, |t_0|$, which contradicts our assumption $q \in \mathcal{A}^{t_0}_{\ge C_1\, t_0}$.

In conclusion, we have localized the slices of the past null cone within "annulus" regions:

$$\mathcal{N}^-(p) \cap \Sigma_t \subset \mathcal{A}^t_{[c_1\, |t|, C_1 |t|]}, \qquad t \in [-c_1\, r_0, 0].$$

Step 2. The past null cone $\mathcal{N}^-(p)$ can be viewed as a graph with bounded slope. We now obtain a Lipschitz continuous parametrization of the null cone. For any fixed $q \in \mathcal{A}^{-c_1 r_0}_{\le c_1^2 r_0}$ we consider the vertical curve passing through q:

$$\gamma_q(\tau) = (\tau, x^1(q), \cdots, x^n(q)), \qquad \tau \in [-c_1 r_0, 0].$$

By Step 1 we know that there exists τ_q such that $\gamma_q(\tau_q) \in \mathcal{N}^-(p)$. Moreover, τ_q is unique since $\mathcal{N}^-(p)$ is achronal, and this defines a map

$$F : \mathcal{A}^{-c_1 r_0}_{\le c_1^2 r_0} \to \mathcal{N}^-(p)$$

such that $F(q) = \gamma_q(\tau_q)$. It is obvious $F(-c_1 r_0, 0) = p$.

We claim that the map F is Lipschitz continuous with Lipschitz constant less than C_1, as computed with the Euclidean metric E. Namely, by contradiction, suppose that $|F(q_1) - F(q_2)|_E > C_1\, |q_1 - q_2|_E$ for some $q_1, q_2 \in \mathcal{A}^{-c_1 r_0}_{\le c_1^2 r_0}$, then by (6.6) in Step 1, $F(q_1)$ would be chronologically related to $F(q_2)$ and this would contradict the fact that $\mathcal{N}^-(p)$ is achronal. Moreover, from Step 1 it follows that

$$F(\mathcal{A}^{-c_1 r_0}_{\le c_1^2 r_0}) \supset \mathcal{N}^-(p) \cap \mathcal{B}_E(p, c_1^3 r_0).$$

Step 3. Constructing a homotopy of curves on the null cone. Suppose that γ_1, γ_2 are two (past) null geodesics from p satisfying

$$\begin{aligned} &\gamma_1(0) = \gamma_2(0), \qquad |\gamma_1'(0)|_T = |\gamma_2'(0)|_T = 1, \\ &\gamma_1(s_1) = \gamma_2(s_2). \end{aligned}$$

We claim that $\max(s_1, s_2) > c_1^6 r_0$, which will establish the desired injectivity bound by setting $i_0 = c_1^6 r_0$.

We argue by contradiction and assume that $\max(s_1, s_2) < c_1^6 r_0$. Taking into account Assumption $(A2)$ and applying exactly the same arguments as in Step 1 of Sect. 4 we see that the g_T-lengths of the curves γ_1, γ_2 satisfy

$$L(\gamma_j, g_T) \le s_j\, e^{C C_1 s_j} \le c_1^{5+3/4}\, r_0 \qquad (j = 1, 2).$$

By Step 1 of the present proof we know that the Euclidean lengths of γ_1, γ_2 satisfy

$$L(\gamma_j, g_E) \le c_1^{5+1/4}\, r_0 \qquad (j = 1, 2).$$

In particular, $\gamma_1, \gamma_2 \subset \mathcal{N}^-(p) \cap \mathcal{B}_E(p, c_1^5 r_0)$ and we can thus concatenate the curve γ_1, γ_2 and obtain

$$\gamma := \gamma_2^{-1} \cup \gamma_1 : [0, s_1 + s_2] \to \mathcal{N}^-(p) \cap \mathcal{B}_E(p, c_1^5 r_0).$$

Since $F(\mathcal{A}^{-c_1 r_0}_{\leq c_1^2 r_0}) \supset \mathcal{N}^-(p) \cap \mathcal{B}_E(p, c_1^3 r_0)$, there exists a smooth family of curves $\sigma_\varepsilon : [0, s_1 + s_2] \to \mathcal{N}^-(p)$ such that

$$\sigma_1 = \gamma, \quad \sigma_0 = p,$$
$$\sigma_\varepsilon(0) = \sigma_\varepsilon(s_1 + s_2) = p, \qquad \varepsilon \in [0, 1].$$

Specifically, we choose

$$\sigma_\varepsilon(s) := F(\varepsilon F^{-1} \gamma(s)),$$

where the multiplication by ε is defined by relying on the linear structure of $\mathcal{A}^{-c_1 r_0}_{\leq c_1^2 r_0} \approx B_{\mathbb{R}^n}(0, c_1^2 r_0)$. Equivalently, by setting $x^i(s) = x^i(\gamma(s))$ we have the explicit formula

$$\sigma_\varepsilon(s) = F\left(-c_1 r_0, \varepsilon x^1(s), \cdots, \varepsilon x^n(s)\right).$$

It is clear that the Euclidean and g_T-lengths of σ_ε satisfy

$$L(\sigma_\varepsilon, g_E) \leq \varepsilon(1 + C_1)\, L(\gamma, g_E) \leq c_1^{4+1/8}\, r_0,$$
$$L(\sigma_\varepsilon, g_T) \leq c_1^{3+5/8}\, r_0.$$

By Assumption $(A3')$ on the null conjugate radius, we can lift to the null cone of the tangent space $T_p M$ the continuous family of loops σ_ε, and we obtain a continuous family of curves $\widetilde{\sigma}_\varepsilon$ defined on the interval $[0, s_1 + s_2]$ and such that

$$\widetilde{\sigma}_\varepsilon(0) = 0, \quad L(\widetilde{\sigma}_\varepsilon, g_{T,p}) \leq c_1^3\, r_0.$$

Observe that the property $L(\widetilde{\sigma}_\varepsilon, g_{T,p}) \leq c_1^3\, r_0 \preceq r_0$ guarantees the existence of this continuous lift. By continuity, all of the curves $\widetilde{\sigma}_\varepsilon$ are loops containing 0. As observed earlier (in the proof for the case of bounded curvature), $\widetilde{\sigma}_1$ consists of two distinct segments which, clearly, can not form a closed loop. We have reached a contradiction and the proof of Theorem 6.1 is completed. □

7. Injectivity Radius of an Observer in a Lorentzian Manifold

Main result. We are now in a position to discuss and prove Theorem 1.1 stated in the introduction. As we have seen in the proof given in the previous section, once the injectivity radius is controled, one can construct a foliation satisfying certain "good" properties. On the other hand, the concept of injectivity radius is clearly independent of any prescribed foliation. As this is more natural, we will now present a general result which avoids assuming a priori the existence of a foliation. This will be achieved by relying on purely geometric and intrinsic quantities and constructing coordinates adapted to the geometry. Such a result is conceptually very important in the applications. The result and proof in this section should be viewed as a Lorentzian generalization of Cheeger, Gromov, and Taylor's technique [9], originally developed for Riemannian manifolds.

Let (M, g) be a differentiable $(n+1)$-manifold endowed with a Lorentzian metric g, and consider a point $p \in M$ and a vector $T \in T_p M$ with $g_p(T, T) = -1$. That is, we now fix a single observer located at the point p. As explained in Sect. 2 the vector T induces an inner product $g_T = \langle\, ,\, \rangle_T$ on the tangent space $T_p M$. We assume that the exponential map $\mathbf{exp}_p$ is defined in some ball $B_T(0, r) \subset T_p M$ determined by this inner product, which is of course always true in a sufficiently small ball. Controling the geometry at the point p precisely amounts to estimating the size of this radius r where the exponential map is defined and has some good property. We restrict attention to the geodesic ball $\mathcal{B}_T(p, r) := \mathbf{exp}_p(B_T(0, r))$; recall that these sets depend upon the vector T given at p.

As explained in the introduction, by g-parallel translating the vector T at p along a geodesic γ from p, we can get a future-oriented unit time-like vector field T_γ defined along this geodesic. To this vector field and the Lorentzian metric g we can associate a reference Riemannian metric g_{T_γ} along the geodesic. In turn, this allows us to compute the norm $|\mathbf{Rm}_g|_{T_\gamma}$ of the Riemann tensor along the geodesic.

Of course, whenever two such geodesics γ, γ' meet away from p, the corresponding vectors T_γ and $T_{\gamma'}$ are generally *distinct*. If we consider the family of all such geodesics we therefore obtain a (generally) multi-valued vector field defined in the geodesic ball $\mathcal{B}_T(p, r)$. We use the same letter T to denote this vector field. In turn, we can compute the Riemann curvature norm $|\mathbf{Rm}_g|_T$ by taking into account every value of T.

The key objective of the present section is the study of the geometry of the local covering map $\mathbf{exp}_p : B_T(0, r) \to \mathcal{B}_T(p, r)$ by comparing the Lorentzian metric g defined on the manifold M with the reference Riemannian metrics g_T. As we will see in the proof below, it will be convenient to pull the metric "upstairs" on the tangent space at p, using the exponential map. This will be possible once we will have estimated the conjugate radius (in Step 1 of the proof below) and will know that the exponential map is non-degenerate on $B_T(0, r)$. By pulling back the Lorentzian metric g on M by the exponential map we get a Lorentzian metric $g = \mathbf{exp}_p^\star g$ defined in the tangent space, on the ball $B_T(0, r)$. We use the same letter g to denote this metric. The geometry in the tangent space is particularly simple, since the g-geodesics on M passing through p are radial straightlines in $B_T(0, r)$.

Note that another (equivalent) standpoint could be adopted here by restricting attention to the domain within the cut-locus from the point p, and by imposing the curvature assumption within the cut-locus only. We now prove our main result stated in Theorem 1.1.

Proof of Theorem 1.1. After scaling we may assume that $r = 1$, and so we need to show

$$\mathbf{Inj}_g(M, p, T) \geq c(n)\, \mathbf{Vol}_g(\mathcal{B}_T(p, c(n))). \tag{7.1}$$

Step 1. Estimates for the metric g_T and its covariant derivative. Let $E_0 = T, E_1, \cdots, E_n$ be an orthonormal frame in T_pM for the Lorentzian inner product g_p, where E_j are space-like vectors. By g-parallel transporting this basis along a radial geodesic $\gamma = \gamma(r)$, satisfying $\gamma(0) = 0$, $|\gamma'(0)|_T = 1$, we get an orthonormal frame defined along the geodesic. We use the same letters E_α to denote these vector fields. Since

$$\frac{d}{dr}\langle E_\alpha, E_\beta\rangle_g = 0,$$

we infer that $|E_i|_T^2 = |E_i|_g^2 = 1$ along the geodesic. The same argument also implies

$$|\gamma'(r)|_T^2 = |\gamma'(0)|_T^2 = 1, \tag{7.2}$$

and $\gamma'(r) = c^\alpha E_\alpha(r)$ with constant (in r) scalars c^α and $\sum |c^\alpha|^2 = |\gamma'(0)|_T = 1$. We used here that, by definition, γ' is g-parallel transported along γ.

Let $V = a^\alpha(r)\, E_\alpha(r)$ be a Jacobi field along a radial geodesic $\gamma = \gamma(r)$, with $V(0) = 0$ and $|V'(0)|_T = 1$. Then, the Jacobi equation takes the form

$$a''_\alpha(r) = -\langle E_\alpha, R(E_\beta, E_\gamma)\, E_\delta \rangle_T\, c^\beta c^\delta\, a^\gamma(r),$$

and since, by our curvature assumption (1.6),

$$-2 \sum_\alpha \left({a'_\alpha}^2 + a_\alpha^2 \right) \le \frac{d}{dr} \left(\sum_\alpha {a'_\alpha}^2 + a_\alpha^2 \right) \le 2 \sum_\alpha \left({a'_\alpha}^2 + a_\alpha^2 \right),$$

we obtain $|V'(r)|_T \le e^r$ and thus $|V(r)|_T \le (e^r - 1)$.

By substituting this result into the above formulas, the estimate can be improved again. Indeed, by computing and estimating the second-order derivative $\frac{d}{dr} \sum_\alpha a'_\alpha a_\alpha$ as we did for the Jacobi field estimate of Sect. 4, we can check that

$$r - C(n)\, r^2 \le \left(\sum |a_\alpha(r)|^2 \right)^{1/2} \le (e^r - 1) \quad \text{along the geodesic,}$$

where $C(n)$ depends only on the dimension.

Denote by g_0 and $g_{T,0}$ the Lorentzian and the Riemannian metrics at the origin 0 (which are nothing but the metrics at the point p), and let $y^0, \dots, y^n$ be Cartesian coordinates on $B_T(0,1)$, with $\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \rangle_{g_0}(0) = \eta_{\alpha\beta}$ (where $\eta_{\alpha\beta}$ is the Minkowski metric). Assuming that the radius under consideration is sufficiently small so that $(1 - C(n)\, |y|) < 1$, we conclude from the Jacobi field estimate that the exponential map is non-degenerate and that the inner products along the geodesic are comparable. In turn, since this is true for every radial geodesic, we can define the pull back of the metric to the tangent space and the conclusion holds in the whole ball $B_T(0,1)$, that is,

$$(1 - C(n)\, |y|)\, g_{T,0} \le g_{T,y} \le (1 + C(n)\, |y|)\, g_{T,0}, \qquad y \in B_T(0,1). \tag{7.3}$$

We next try to compare the covariant derivative operators. By construction of the metric g_T we have $\nabla_{g_T} - \nabla = \nabla T * T$ (schematically) with $\nabla T(0) = 0$, and so we need to control ∇T. We write the radial vector field as

$$\frac{\partial}{\partial r} = \frac{y^\alpha}{r} \frac{\partial}{\partial y^\alpha}, \qquad r^2 := \sum |y^\alpha|^2,$$

with $|\frac{\partial}{\partial r}|_T^2 \equiv 1$ (as stated already in (7.2)). Using that $|\nabla T|_T^2 = \nabla_\alpha T^\xi \nabla_\beta T^\eta g_{T,\xi\eta}\, g_T^{\alpha\beta}$ and computing the derivative of $|\nabla T|_T^2$ along any radial geodesic, we find

$$\frac{d}{dr} |\nabla T|_T^2 \le C(n)\, |\nabla T|_T^3 + 2\, \langle \nabla_{\frac{\partial}{\partial r}} \nabla T, \nabla T \rangle_T.$$

By using that

$$\nabla_{\frac{\partial}{\partial r}} T = 0, \qquad \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial y^\alpha} \right] = -\frac{1}{r} \frac{\partial}{\partial y^\alpha} + \frac{y^\alpha}{r^2} \frac{\partial}{\partial r},$$

we obtain

$$\nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial y^\alpha}} T^\gamma = -\frac{1}{r} \nabla_{\frac{\partial}{\partial y^\alpha}} T^\gamma + R(\frac{\partial}{\partial r}, \frac{\partial}{\partial y^\alpha}) T^\gamma$$

and, therefore, thanks to the curvature assumption we find

$$\frac{d}{dr}|\nabla T|_T^2 \le -\frac{2}{r}|\nabla T|_T^2 + C(n)\,|\nabla T|_T^3 + C(n)\,|\nabla T|_T.$$

This implies a uniform bound for the covariant derivative of T

$$|\nabla T|_T(y) \le C(n)\,|y|, \qquad |y| \le 1/C(n), \tag{7.4}$$

which also provides a bound for the difference $\nabla_{g_T} - \nabla$.

Step 2. Estimate of the injectivity radius of g on $B_T(0, c(n))$. Since the curvature on $B_T(0, 1)$ is bounded and $|\nabla_{g_T} - \nabla|_T^2 \le C(n) = 1/c(n)$ on the ball $B_T(0, c(n))$ we can follow the argument in Sect. 4 and obtain a uniform lower bound the conjugate radius at any point of the ball $B_T(0, 3c(n)/4)$.

Next, given any point $y \in B_T(0, c(n)/2)$, let γ_1 and γ_2 be two geodesics which meet at their end points and have "short" length with respect to the metric g_T (or, equivalently, $g_{T,0}$). By using the linear structure on $B_T(0, 1)$ (as a subset of the vector space T_pM) we can construct a homotopy of the loop $\gamma_1 \cup \gamma_2^{-1}$ to the origin, such that each curve has also "short" length for the metric g_T. By lifting the homotopy to the tangent space $T_y B_T(0, 1)$ and by relying on the conjugate radius bound, we reach a contradiction as was done in Sect. 4.

In summary, there exists a universal constant $C(n) = 1/c(n)$ (depending only on the manifold dimension) such that the injectivity radius at each point y of $B_T(0, c(n))$ is bounded from below by $4c(n)$. Moreover, using again a Jacobi field estimate we can check that the ball $B_{T,p}(0, c(n)) \subset T_pM$ defined by the Euclidean metric $g_{T,p}$ is covered by $\exp_y(B_{T,y}(0, 3c(n)))$, where $B_{T,y}(0, 3c(n)) \subset T_yT_pM$ is a ball of radius $3c(n)$ defined by metric $g_{T,y}$ and, moreover, any two points in $B_{T,p}(0, c(n))$ can be connected by a g-geodesic totally contained in $B_{T,p}(0, 2c(n))$. Further arguments are now required to arrive at the desired bound (7.1).

Step 3. New Riemannian metric g_N induced on $B_T(0, 2c(n))$. Consider a geodesic γ satisfying $\gamma(0) = 0$ and $\gamma'(0) = -T$, and define

$$\gamma(c(n)/2) =: q, \qquad \tau := d_g(\cdot, q) - d_g(q, 0).$$

Then, by following exactly the same arguments as in the main proof of Sect. 5, we construct a normal coordinate system (of definite size) such that

$$g = -d\tau^2 + g_{ij}\,dx^i dx^j, \qquad g_N = d\tau^2 + g_{ij}dx^i dx^j,$$

and the Riemannian metric satisfies the following properties:

(i) $(1 - c(n))\,g_N \le g_T \le (1 + c(n))\,g_N$,
(ii) g_N has bounded curvature ($\le C(n)$) (see (5.4)), and
(iii) for any fixed $y_0 \in B_T(0, c(n))$ the distance function $d_{g_N}(y_0,)^2$ is strictly g-convex on the ball $B_T(0, 2c(n))$ and, more precisely, for any $y_0 \in B_T(0, c(n))$,

$$(2 + c(n))\,g_N \ge \nabla_g^2 d_{\bar g}^2(y_0, \cdot) \ge (2 - c(n))\,g_N \qquad \text{on } B_T(0, 2c(n)).$$

Recall that the Hessian of the distance function (defined by the Riemannian metric g_N) is computed with the covariant derivative operator associated with the Lorentzian metric.

Step 4. Suppose that $p_1, \cdots, p_N$ are distinct pre-images of p in the ball $B_T(0, c(n))$. We claim that any $p' \in \mathcal{B}_T(p, c(n))$ has at least N distinct pre-images in $B_T(0, 1)$, and refer to this property as a "lower semi-continuity" property.

Generalizing the terminology in [9], we use the notation $a \underset{(g_{T,0}, A)}{\sim} b$ when two curves a, b defined on M and with the same endpoints are homotopic through a family of curves whose lift have $g_{T,0}$-lengths $\leq A$. Relying on the lift and the linear structure, we see that, for any curve ξ starting from p with (after lifting through 0) $g_{T,0}$-length $A \leq 1$, there exists a unique g-geodesic γ_ξ (with the same end points as ξ) defined on M such that $\xi \underset{(g_{T,0}, A)}{\sim} \gamma_\xi$. This fact establishes a one-to-one correspondence between the following three concepts:

(i) equivalence class of curves through p with $g_{T,0}$-lengths $\leq 3c(n)$,
(ii) radial geodesic segments of $g_{T,0}$-lengths $\leq 3c(n)$, and
(iii) points in the ball $B_T(0, 3c(n)) \subset T_pM$.

Let σ be a g-geodesic connecting p to p' in $\mathcal{B}_T(p, c(n))$. Observe that the images of the lines $\overline{Op_i}$ by the exponential map, $\sigma_i = \mathbf{exp}_p(\overline{Op_i})$, are distinct geodesic loops through p. Denote by $\widetilde{\sigma}_i$ the lift of $\sigma_i \cup \sigma$ through 0, and denote by p_i' the end point of $\widetilde{\sigma}_i$. Then, it is clear that all the points p_i' $(i = 1, \cdots, N)$ are the pre-images of p' in $B_T(0, 1/2)$. We claim that they are distinct.

Indeed, assuming that $p_i' = p_j'$ for some $i \neq j$, we would find $\sigma \cup \sigma_i \underset{g_{T,0}, 2c(n)}{\sim} \sigma \cup \sigma_j$, which gives

$$\sigma_i \underset{g_{T,0}, 3c(n)}{\sim} \sigma^{-1} \cup \sigma \cup \sigma_i \underset{g_{T,0}, 3c(n)}{\sim} \sigma^{-1} \cup \sigma \cup \sigma_j \underset{g_{T,0}, 3c(n)}{\sim} \sigma_j.$$

This would imply $\sigma_i \underset{g_{T,0}, 3c(n)}{\sim} \sigma_j$ and, therefore, $p_i = p_j$, which is a contradiction. In short, this argument shows that the "cancellation law" holds for the homotopy class of "not too long" curves.

Step 5. Suppose that there exist two distinct g-geodesics $\gamma_1 : [0, l_1] \to M$ and $\gamma_2 : [0, l_2] \to M$ satisfying

$$\gamma_1(0) = \gamma_2(0) = p, \qquad |\gamma_1'(0)|_T^2 = |\gamma_2'(0)|_T^2 = 1,$$

and meeting at their endpoints, that is, $\gamma_1(l_1) = \gamma_2(l_2)$. Then, let $l := l_1 + l_2$ and $\gamma := \gamma_2^{-1} \cup \gamma_1 : [0, l] \to M$. Our aim is to prove that

$$l \geq c(n) \, \mathbf{Vol}_g \left(\mathcal{B}_T(p, c(n)) \right),$$

which will give us the desired injectivity radius.

From the loop γ we define a map $\pi_\gamma : B_T(0, c(n)) \to B_T(0, 2c(n))$ as follows: for any $y \in B_T(0, c(n))$, the point $\pi_\gamma(y)$ is the end point of the lift $\mathbf{exp}_p(\overline{Oy}) \cup \gamma$ (through the origin). If one would have $\pi_\gamma(y) = y$ then by the cancellation law established in Step 4, we would have $\gamma \underset{g_{T,0}, 2c(n)}{\sim} 0$, which is a contradiction. So, the map π_γ has no fixed point.

Without loss of generality, we assume that $l \leq c(n)^5$. Let $N = [c(n)^3/l]$ be the largest integer less than $c(n)^3/l$, and let us use the notation $2\gamma = \gamma \circ \gamma$, etc.

Claim. The classes $[\gamma], [2\gamma], \cdots, [N\gamma]$ are distinct homotopy classes for the relation $\underset{g_{T,0}, c(n)^2}{\sim}$.

If this were not true, then by the cancellation law we would have $[j\gamma] \underset{g_{T,0,c(n)^2}}{\sim} 0$ for some $1 \le j \le N$. We already know that all π_γ^i is defined from $B_T(0, c(n)^2)$ to $B_T(0, c(n))$ for $i \le j$. Since for any $y \in B_T(0, c(n)^2)$ we have

$$\mathbf{exp}_p(\overline{Oy}) \cup j\gamma \underset{g_{T,0,c(n)}}{\sim} \mathbf{exp}_p(\overline{Oy}),$$

which implies that $\pi_\gamma^j = id$. We use here the notation $\pi_\gamma^2 := \pi_\gamma \circ \pi_\gamma$, etc.

Then, we define a function $u : B_T(0, c(n)) \to \mathbb{R}$ by

$$u(y) = d_{\widetilde{g}}^2(0, y) + d_{\widetilde{g}}^2(0, \pi_\gamma y) + \cdots + d_{\widetilde{g}}^2(0, \pi_\gamma^{j-1} y).$$

Since $\pi_\gamma^j = id$, it is easy to see $u(\pi_\gamma y) = u(y)$ for any $y \in B_T(0, c(n))$. That is to say, u is π_γ-invariant. By Step 3, u is strictly g-geodesically convex on $B_T(0, c(n))$. More precisely, since for any g-geodesic $\xi : [0, s_0] \to B_T(0, c(n))$, $\pi_\gamma^i \xi$ are still g-geodesics in $B_T(0, c(n))$, and

$$\begin{aligned}
&\frac{d^2}{ds^2} u(\xi(s)) \\
&\quad = \nabla^2 d_{\widetilde{g}}^2(0, \cdot)(\xi'(s), \xi'(s)) + \ldots + \nabla^2 d_{\widetilde{g}}^2(0, \cdot) \left(d\pi_\gamma^{j-1}{}_{\xi(s)}(\xi'(s)), d\pi_\gamma^{j-1}{}_{\xi(s)}(\xi'(s)) \right) \\
&\quad \ge \widetilde{g}(\xi'(s), \xi'(s)) > 0.
\end{aligned}$$

Observe that

$$u \mid_{B_T(0,c(n))^c} \ge j\,(1 - c(n))^2 \left(c(n) - \frac{2l\, c(n)^3}{l} \right)^2 \ge \frac{j\, c(n)^2}{2},$$

and

$$u(0) \le j\,(jl)^2 \le j\, c(n)^5 < \frac{j\, c(n)^2}{2},$$

so the minimum of function u over $\overline{B_T(0, c(n))}$ is only achieved at an interior point, say $y_0 \in B_T(0, c(n))$. Then by π_γ invariance of u, we have $u(\pi_\gamma y_0) = u(y_0) < jc(n)^2/2$, and this implies $\pi_\gamma(y_0) \in B_T(0, c(n))$. By the injectivity radius estimate at $y_0 \in (T_pM, g)$, there exists a g-geodesic connecting y_0 to $\pi_\gamma(y_0)$, which is contained in $B_{T,p}(0, 2c(n))$. By using the strong g-geodesic convexity of u, we conclude that $\pi_\gamma y_0 = y_0$. This contradicts the fact that π_γ has no fixed point, and the claim is proved.

Step 6. The pull back of the volume element of g is the same as the one of g_T. By combining this observation with our results in Steps 4 and 5 we find

$$\mathbf{Vol}_{g_T}(B_T(0, 1)) \ge \frac{c(n)^3}{l} \mathbf{Vol}_g(\mathcal{B}_T(p, c(n))),$$

which implies

$$l \ge c(n) \frac{\mathbf{Vol}_g(\mathcal{B}_T(p, c(n)))}{\mathbf{Vol}_{g_T}(B_T(0, 1))} \ge c(n)\, \mathbf{Vol}_g(\mathcal{B}_T(p, c(n)))$$

and completes the proof of Theorem 1.1. □

8. Volume Comparison for Future or Past Cones

In Riemannian geometry, Bishop-Gromov's volume comparison theorem assumes a lower bound on the Ricci curvature and compares the volume of small and large balls in a sharp and qualitative manner. Our aim in this section is to provide an extension to cones in a Lorentzian manifold, and to use this result to refine our main injectivity radius estimate. For definiteness we state the result for future cones.

Theorem 8.1. (Volume comparison theorem for cones). *Let (M, g) be a globally hyperbolic, Lorentzian $(n+1)$-manifold. Fix $p \in M$ and a vector $T \in T_pM$ with $g_p(T,T) = -1$, and suppose that the exponential map $\mathbf{exp}_p$ is defined on the ball $B_T(0, r_0) \subset T_pM$ (determined by the reference inner product g_T at p). Suppose also that the Ricci curvature on $\mathcal{B}_T(p, r_0)$ satisfies for some K the inequality*

$$\mathbf{Ric}_g(V, V) \geq -n\, K\, |g(V, V)| \quad \textit{for all time-like vector fields } V.$$

Then, by setting $\mathcal{F}C(p, r) := \mathbf{exp}_p(FC(p, r))$ and

$$FC(p, r) := \Big\{0 < g_{T,p}(V, V) < r_0^2,\ g_p(V, V) < 0,\ g_p(T, V) < 0\Big\},$$

for any $0 < r < s < r_0$ the following inequality holds

$$\frac{\mathbf{Vol}_g(\mathcal{F}C(p, r))}{\mathbf{Vol}_g(\mathcal{F}C(p, s)))} \geq \frac{\mathbf{Vol}_K(\widetilde{B}_K(r))}{\mathbf{Vol}_K(\widetilde{B}_K(s)))},$$

where $\mathbf{Vol}_K(\widetilde{B}_K(r))$ is the volume of the ball $\widetilde{B}_K(r) \subset \widetilde{M}_K$ in the simply-connected Lorentzian $(n+1)$-manifold $(\widetilde{M}_K, \widetilde{g}_K)$ with constant curvature K (that is, with curvature tensor $\widetilde{R}_{K,\alpha\beta\gamma\delta} = -K\,(\widetilde{g}_{K,\alpha\gamma}\,\widetilde{g}_{K,\beta\delta} - \widetilde{g}_{K,\alpha\delta}\,\widetilde{g}_{K,\beta\gamma})$).

For the proof of this result we return to the technique in Step 2 of Sect. 5, where we introduced the index form associated with a synchronous coordinate system based on time-like geodesics. By observing that the index form is symmetric and that Jacobi fields "minimize" the index form, we can extend the method of proof of the index comparison theorem. However, in a general Lorentzian manifold, since the index form we needed (without imposing a restriction on the geodesics) is non-symmetric, we need to adapt the method of the index comparison theorem.

More generally, the same proof allows to establish that, if Σ is a subset of the unit sphere S^n such that $g(V, V) < 0$ and $g(T, V) < 0$ for all $V \in \Sigma$, then by setting with $\mathcal{F}C_\Sigma(p, r) := \mathbf{exp}_p(FC_\Sigma(p, r))$ and

$$FC_\Sigma(p, r) := \left\{V \in FC(p, r)\, /\, \frac{V}{|V|_{g_T}} \in \Sigma\right\},$$

the following inequality holds

$$\frac{\mathbf{Vol}_g(\mathcal{F}C_\Sigma(p, r))}{\mathbf{Vol}_g(\mathcal{F}C_\Sigma(p, s)))} \geq \frac{\mathbf{Vol}_K(\widetilde{B}_K(r))}{\mathbf{Vol}_K(\widetilde{B}_K(s)))}.$$

From Theorem 8.1 it follows:

Corollary 8.2. (Injectivity radius based on the volume of a future cone). *Let M be a manifold satisfying the assumptions in Theorem 1.1 and assumed to be globally hyperbolic, and let $T \in T_pM$ be a reference vector. Let Σ be a subset in the unit sphere S^n included in the future cone N_p^+. If $\mathbf{Vol}_g(\mathcal{F}C_\Sigma(p,r)) \geq v_0 > 0$, then the inequality*

$$\frac{\mathbf{Inj}_g(M,p,T)}{r} \geq c(\Sigma)\,\frac{v_0}{r^{n+1}}$$

holds, where $\mathcal{F}C_\Sigma(p,r) := \mathbf{exp}_p(FC_p(r))$ with

$$FC_p(r) := \left\{0 < |V|_T < r,\ \langle T, V\rangle_T < 0,\ |V|_g^2 < 0,\ \frac{V}{|V|_T} \in \Sigma\right\},$$

and the constant $c(\Sigma)$ depends only on the distance (measured by T) of Σ to the null cone.

Proof of Theorem 8.1. Given a future-oriented time-like geodesic $\gamma : [0, s_0] \to M$ satisfying $\gamma(0) = p$ and $|\gamma'(0)|_{g_T} = -1$, let us compute the rate of change of the volume element along γ. For simplicity in the derivation, we fix $s_1 \in (0, s_0)$ sufficiently small so that every point in the interval $(0, s_1]$ is neither a conjugate point nor a cut point with respect to the base point p. Let $v_0 = \gamma'(s_1), v_1, v_2, \cdots, v_n$ be an arbitrary orthonormal basis prescribed at the final point $\gamma(s_1)$ (with respect to the Lorentzian metric $g_{\gamma(s_1)}$). Let J_α be the Jacobi field defined on the interval $[0, s_1]$ and satisfying the two conditions $J_\alpha(0) = 0$ and $J_\alpha(s_1) = v_\alpha$ at the point $\gamma(s_1)$.

Clearly, the vector fields J_0 coincides (up to rescaling) with the tangent vector along the geodesic:

$$J_0(s) = \frac{s}{s_1}\,\gamma'(s),$$

while the vectors J_i and $\nabla_{\gamma'} J_i$ are orthogonal to the geodesic. Recall that the Jacobian of the exponential map $\varphi(s) := Jac(D\mathbf{exp}_{\gamma(s)})$ along the geodesic is given by the formula

$$\varphi(s)^2 = \frac{|\gamma'(s) \wedge J_1(s) \wedge \cdots \wedge J_n(s)|_g^2}{s^{2n}\,|\gamma'(0) \wedge J_1'(0) \wedge \cdots J_n'(0)|_g^2}.$$

We will use also below the function $\widetilde{\varphi}_K(s)$ defines as the corresponding quantity in the simply connected Lorentzian $(n+1)$-manifold with constant curvature $-K$.

Along the geodesic we can consider the index form

$$I_s(X,Y) := \int_0^s \big(\langle \nabla_{\gamma'} X, \nabla_{\gamma'} Y\rangle_g - \mathbf{Rm}_g(\gamma', X, \gamma', Y)\big)\, ds,$$

where X, Y are arbitrary vector fields and $\mathbf{Rm}_g(\gamma', X, \gamma', Y) := -\langle \mathbf{Rm}_g(\gamma', X)\gamma', Y\rangle_g$. Observe that the index form is symmetric in its arguments X, Y; moreover, using the fact that J_i and $\nabla_{\gamma'} J_i$ are orthogonal to the geodesic we can express the derivative of the Jacobian of the exponential map in terms of the index form evaluated on Jacobi fields, as follows:

$$\begin{aligned}\frac{d}{ds}\left(\log \varphi^2\right)(s_1) &= \sum_i \langle J_i'(s_1), J_i(s_1)\rangle_g - \frac{2n}{s_1}\\ &= \sum_i I_{s_1}(J_i, J_i) - \frac{2n}{s_1}.\end{aligned}$$

Recall also that since, by assumption, there are no conjugate points along γ, the Jacobi field minimizes the index form among all vector fields with fixed boundary values. This is the same property as in Riemannian geometry, which follows from the fact that a time-like geodesic without conjugate points has a locally maximizing length among all nearby time-like curves with the same end points.

Finally, let $E_i(s)$ be the vector field obtained by parallel transporting v_i (prescribed at the end-point $\gamma(s_1)$) along γ. Setting

$$\widetilde{J}_i(s) = \frac{\sinh s}{\sinh s_1} E_i(s),$$

for which we already know that $I_{s_1}(J_i, J_i) \leq I_{s_1}(\widetilde{J}_i, \widetilde{J}_i)$, we can compute that

$$\begin{aligned} \frac{d}{ds}\left(\log \frac{\varphi^2}{\widetilde{\varphi}_K^2}\right)(s_1) &\leq -\sum_i \int_0^{s_1} \frac{(\sinh s)^2}{(\sinh s_1)^2} \big(\mathbf{Rm}_g(\gamma', E_i, \gamma', E_i) - K\big)\, ds \\ &= -\int_0^{s_1} \frac{(\sinh s)^2}{(\sinh s_1)^2} \big(\mathbf{Ric}_g(\gamma', \gamma') - n\, K\big)\, ds, \end{aligned} \tag{8.1}$$

which is non-positive. Hence, the function $\varphi(s)/\widetilde{\varphi}_K(s)$ is non-increasing.

To conclude we need an observation due to Gromov in the Riemannian setting, which we now extend to globally hyperbolic Lorentzian manifolds. Let A be the star-shaped domain (with respect to 0) in T_pM, such that $\mathbf{exp}_p : A \cap B_T(0, r_0)$ is a diffeomorphism on its image and the image of $\partial A \cap B_T(0, r_0)$ is the cut locus (in $\mathcal{B}_T(p, r_0)$). Let χ_A be the characteristic function of the set A. Since $\varphi/\widetilde{\varphi}_K$ is non-increasing, we see that $\chi_A\varphi/\widetilde{\varphi}_K$ is also non-increasing. Hence, we have two functions defined on the ball $B_T(0, r_0)$, whose quotient is non-increasing along any radial geodesics.

Since, by assumption, the manifold M is globally hyperbolic, any point in the set $\mathcal{FC}(p, r_0)$ can be connected to p by a maximizing time-like geodesic. This implies that the integration of the function $\chi_A\varphi$ over the ball $B_T(0, s)$ coincides with the volume $\mathbf{Vol}_g(\mathcal{B}_T(p, s))$ of the geodesic ball. In conclusion, by integrating $\chi_A\varphi$ and $\widetilde{\varphi}_K$ over the ball $B_T(0, s)$ and after a simple calculation we obtain that $\mathbf{Vol}_g(\mathcal{FC}(p, s))/\mathbf{Vol}_K(\widetilde{B}_K(s))$ is non-increasing in s. The proof of the theorem is completed. □

Proof of Corollary 8.2. Observe that there is a constant $C(\Sigma)$ depending only on the distance of Σ to the null cone such that $\mathbf{Ric}(\gamma', \gamma') \geq -C(\Sigma)\, g(\gamma', \gamma')$ for any time-like geodesic γ satisfying $\gamma'(0) \in \Sigma$. From the volume comparison theorem for future cones (Theorem 8.1) we deduce

$$\frac{\mathbf{Vol}_g(\mathcal{FC}_\Sigma(p, c(n)r))}{\mathbf{Vol}_g(\mathcal{FC}_\Sigma(p, r))} \geq C(\Sigma),$$

and, by combining this result with Theorem 1.1, the corollary follows. □

9. Final Remarks

Regularity of Lorentzian metrics. Following the strategy proposed in the present paper, we can also transfer to the Lorentzian metric the regularity available on any reference Riemannian metric. Clearly, the regularity obtained in this manner depends on the choice of the reference metric. The interest of the approach we now describe is to provide a simple proof of a regularity result: we use harmonic-like coordinates for the Riemannian metric that we constructed in the proof of our main theorem and we see immediately that the Lorentzian metric has uniformly bounded first-order derivatives. For a discussion of the optimal regularity achievable with Lorentzian metrics we refer to Anderson [3].

Proposition 9.1. (Regularity in harmonic-like coordinates). *Under the assumptions and notation of Theorem 1.1, define*

$$r_1 := c(n)\, \frac{\mathbf{Vol}_g(\mathcal{B}_T(p, c(n)\, r))}{r^{n+1}}\, r,$$

where $c(n)$ is the constant determined therein. Then, for any $\varepsilon > 0$ there exist a constant $c_1(n, \varepsilon)$ with $\lim_{\varepsilon\to 0} c_1(n, \varepsilon) = 0$ and a coordinate system (x^α) satisfying $x^\alpha(p) = 0$ and defined for all $(x^0)^2+(x^1)^2+\ldots+(x^n)^2 < (1-\varepsilon)^2\, r_1^2$, such that in the L^∞ sup-norm

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| + r_1\, |\partial g_{\alpha\beta}| \le c_1(n, \varepsilon), \tag{9.1}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric in these coordinates.

Proof. By scaling we may assume $r_1 = 1$. By Step 1 in the proof of Theorem 1.1, we know that the Riemannian metric g_T is equivalent to the Riemannian metric $g_{T,p}$ on the ball $B_T(0, 4c_1(n))$ for some $c_1(n) > 0$. By considering a lift and using again the results in Step 1 this implies

$$\mathcal{B}_T(p, c_1(n)) \subset \mathcal{B}_T(q, 3c_1(n)) \quad q \in \mathcal{B}_T(p, c_1(n)).$$

Applying the same argument as in Theorem 1.1, we deduce that the injectivity radius of any point in $\mathcal{B}_T(p, c_1(n))$ is bounded from below by $c_1(n)$. As in Step 3 in the proof of Theorem 1.1 (or in Step 2 of Sect. 5), we see that there exists a synchronous coordinate system $(y^\alpha) = (\tau, y^j)$ of definite size around p such that the metrics $g = -d\tau^2 + g_{ij}\, dy^i dy^j$ and $g_N = d\tau^2 + g_{ij}\, dy^i dy^j$ (the Riemannian metric constructed therein) satisfy the following properties on the geodesic ball $\mathcal{B}_T(p, c_1(n))$:

(a) $(1 - c_1(n))\, g_N \le g_T \le (1 + c_1(n))\, g_N$,
(b) g_N has bounded curvature ($\le 1/c_1(n)$),
(c) $|\tau| + \frac{1}{|\tau|} + |\nabla^2\tau|_N \le 1/c_1(n)$.

(In particular, this implies $|\nabla_{g_N} g|_N < 1/c_1(n)$.) Since the volume $\mathbf{Vol}_{g_N}(\mathcal{B}_T(p, c_1(n)))$ is bounded from below, it follows from [9] that the injectivity radius of g_N at p is bounded from below by $c_1(n)$. By the theorem in [15] on the existence of harmonic coordinates, for any small $\varepsilon > 0$ there exists a harmonic coordinate system (x^α) with respect to the Riemannian metric g_N such that $\sum_\alpha |x^\alpha|^2 < (1 - \varepsilon)^2$ and for every $0 < \gamma < 1$,

$$|g_{N,\alpha\beta} - \delta_{\alpha\beta}| < c_1(n, \varepsilon), \qquad |\partial g_N| < 1/c_1(n), \qquad |\partial g_N|_{\mathbf{C}^\gamma} < 1/c(n, \varepsilon, \gamma).$$

In the construction of harmonic coordinates, we may also assume that $|\frac{\partial}{\partial y_0} - \frac{\partial}{\partial \tau}|_{g_{T,p}} < c_1(n, \varepsilon)$.

Since $|\nabla_{g_N} g|_N < 1/c_1(n)$ and that, in these coordinates, $|\nabla_{g_N}| \le 1/c_1(n)$, we have $|\partial g| < 1/c_1(n)$. Finally, to estimate the metric we write $|g_{\alpha\beta} - \eta_{\alpha\beta}|_p < c_1(n, \varepsilon)$ and $|\partial g| < 1/c_1(n)$ and we conclude that $|g_{\alpha\beta} - \eta_{\alpha\beta}| < \frac{1}{C(n)}\varepsilon + c_1(n, \varepsilon)$. The proof is completed. □

Pseudo-Riemannian manifolds. Finally, we would like to discuss the more general situation of pseudo-Riemannian manifolds (M, g) (also referred to as semi-Riemannian manifolds). Consider a differentiable manifold M endowed with a symmetric, non-degenerate covariant 2-tensor g. We assume that the signature of g is (n_1, n_2), that is, n_1 negative signs and n_2 positive signs. Riemannian and Lorentzian manifolds are obviously special cases of pseudo-Riemannian manifolds. Fix $p \in M$ and an orthonormal family T consisting of n_1 vectors $E_1, E_2, \cdots, E_{n_1} \in T_pM$ such that $\langle E_i, E_j \rangle_g = -\delta_{ij}$. Based on this family, we can define a reference inner product g_T on T_pM by generalizing our construction in the Lorentzian case, and by using this inner product we can then define the ball $B_T(0, s) \subset T_pM$. By parallel translating $E_1, E_2, \cdots, E_{n_1}$ along radial geodesics from the origin in T_pM, we obtain vector fields $E_1, E_2, \cdots, E_{n_1}$ defined in the tangent space (or multi-valued vector fields on the manifold). This also induces a (multi-valued) Riemannian metric g_T as was explained before.

The following corollary immediately follows by repeating the proof of Theorem 1.1. We note that the curvature covariant derivative bound imposed below is in fact superfluous and could be removed by introducing a foliation based on certain synchronous-type coordinates, as we did in Sect. 5 for Lorentzian manifolds. On the other hand, to the best of our knowledge this is the first injectivity radius estimate for pseudo-Riemannian manifolds.

Corollary 9.2. (Injectivity radius of pseudo-Riemannian manifolds). *Let (M, g) be a differentiable pseudo-Riemannian n-manifold with signature (n_1, n_2), and let $p \in M$ and $T = (E_1, \cdots, E_{n_1})$ be a family of vectors in T_pM satisfying $g(E_i, E_j) = -\delta_{ij}$. Suppose that the exponential map $\mathbf{exp}_p$ is defined on $B_T(0, r) \subset T_pM$ and that*

$$|\mathbf{Rm}_g|_T \le r^{-2}, \quad |\nabla \mathbf{Rm}_g|_T \le r^{-3} \qquad \textit{on } B_T(0, r).$$

Then, there exists a positive constant $c(n)$ such that

$$\frac{\mathbf{Inj}_g(M, p, T)}{r} \ge c(n)\, \frac{\mathbf{Vol}_g(\mathcal{B}_T(p, c(n)\, r))}{r^n},$$

where $\mathcal{B}_T(p, r) = \mathbf{exp}_p(B_T(0, r))$ is the geodesic ball at p with radius r.

Proof. Without loss of generality we assume $r = 1$. In local coordinate system y^α, let

$$E_i =: E_i^\beta \frac{\partial}{\partial y^\beta}, \quad E_{i\alpha} = E_i^\beta g_{\alpha\beta}, \qquad i = 1, \ldots, n_1,$$

then $g_{T,\alpha\beta} = g_{\alpha\beta} + 2\sum_{i=1}^{n_1} E_{i\alpha} E_{i\beta}$. By the same computations as in the proof of Theorem 1.1 we obtain

$$|\nabla E_i|_T \le \frac{1}{c(n)},$$
$$|g_T - g_{T,p}| + |g - \eta| < c(n) \qquad \text{on the ball } B_T(0, c(n)),$$

where $\eta_{\alpha\beta} := \mp\delta_{\alpha\beta}$ (a minus sign for $\alpha \le n_1$, and a plus sign for $\alpha > n_1$). In view of the computations in [12] (Theorem 4.11 and Corollary 4.12) we deduce that $|\partial g| < r/c(n)$, where $r^2 = (y^1)^2 + \cdots + (y^n)^2$. Since $d^2_{g_{T,p}}(y_0, y) = |y - y_0|^2$, we have for any point $y_0 \in B_T(0, c(n))$,

$$\nabla^2_{\alpha\beta} d^2_{g_{T,p}}(y_0, \cdot) \ge \delta_{\alpha\beta} = g_{T,p} \qquad \text{on the ball } B_T(0, c(n)).$$

Since the metric $g_{T,p}$ plays the same role as g_N (cf. the proof of Theorem 1.1), all arguments can be carried out and this completes the proof of the corollary. □

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