

Local foliations and optimal regularity of Einstein spacetimes

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Abstract

We investigate the local regularity of pointed spacetimes, that is, time-oriented Lorentzian manifolds in which a point and a future-oriented, unit timelike vector (an observer) are selected. Our main result covers the class of Einstein vacuum spacetimes. Under curvature and injectivity bounds only, we establish the existence of a *local coordinate* chart defined in a ball with definite size in which the metric coefficients have optimal regularity. The proof is based on *quantitative estimates*, on one hand, for a constant mean curvature (CMC) foliation by spacelike hypersurfaces defined locally near the observer and, on the other hand, for the metric in local coordinates that are spatially harmonic in each CMC slice. The results and techniques in this paper should be useful in the context of general relativity for investigating the long-time behavior of solutions to the Einstein equations.

1 Introduction

1.1 Quantitative estimates for CMC foliations

We denote by (\mathbf{M}, \mathbf{g}) a spacetime of general relativity, that is, a time-oriented, $(n + 1)$ -dimensional Lorentzian manifold whose metric \mathbf{g} , by definition, has signature $(-, +, \dots, +)$. Our main result in the present paper will concern vacuum spacetimes, that is, Ricci-flat manifolds, although this assumption will be made only later in the discussion. Building on our earlier work [12], we continue the investigation of the local geometry of Einstein spacetimes, using here techniques for partial differential equations. Our main objective will be,

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under natural geometric bounds on the curvature and the injectivity radius only, to establish the existence of local coordinate charts in which the metric coefficients have optimal regularity, that is, belong to the Sobolev space $W^{2,a}$ for all real $a \in (1, \infty)$. The construction proposed in the present paper is local in the neighborhood of a given “observer” and, in turn, our result provides a sharp control of the local geometry of the spacetime at every point. This optimal regularity theory should be useful for tackling the global regularity issue for Einstein spacetimes and investigating the long-time behavior of solutions to the Einstein equations.

As in [12], we consider a *pointed* Lorentzian manifold $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_{\mathbf{p}})$, that is, a time-oriented Lorentzian manifold supplemented with a point $\mathbf{p} \in \mathbf{M}$ and a future-oriented timelike vector $\mathbf{T}_{\mathbf{p}}$ at that point. The pair $(\mathbf{p}, \mathbf{T}_{\mathbf{p}})$ is called a (*local*) *observer* and is required for stating our curvature and injectivity radius bounds for some given curvature constant Λ and injectivity radius constant $\lambda > 0$; see (1.2) in Section 2.1 below. We will establish the existence of a neighborhood of the observer $(\mathbf{p}, \mathbf{T}_{\mathbf{p}})$ whose size depends on Λ, λ only and in which local coordinates exist in such a way that the regularity of the metric coefficients can be controlled by the same constants. Since the Riemann curvature involves up to two derivatives of the metric it is natural to search for an estimate of the metric in the $W^{2,a}$ norm, and this is precisely what we achieve in the present paper.

We will proceed as follows. Our first task is constructing a *constant mean curvature* (CMC) foliation by spacelike hypersurfaces, which is locally defined near the observer and satisfies quantitative bounds involving the constants Λ, λ , only; see Theorem 2.2 below. Our method can be viewed as a refinement of earlier works by Bartnik and Simon [9] (covering hypersurfaces in Minkowski space) and Gerhardt [14, 15] (global foliations of Lorentzian manifolds). If one would assume that the metric \mathbf{g} admits bounded covariant derivatives of sufficiently high order of regularity, then the techniques in [9, 14] would provide the existence of the CMC foliation and certain estimates. Hence, the construction of a CMC foliation on a *sufficiently smooth* manifold is standard at *small scales*.

In contrast, in the framework of the present paper only *limited differentiability* of the metric should be used and uniform bounds involving the curvature and injectivity radius bounds, only, be sought. We have to solve a boundary value problem for the prescribed mean curvature equation in a Lorentzian background and to establish that a CMC foliation exists in a neighborhood (of the observer) with *definite size* and to control the geometry of these slices in terms of Λ, λ , only. A technical difficulty in this analysis is ensuring that each hypersurface of the foliation is *uniformly* spacelike and can not approach a null hypersurface. Deriving a gradient estimate for prescribed curvature equations requires the use of barrier functions determined from (parts of) suitably constructed geodesic spheres.

1.2 Earlier works

An extensive study of (sufficiently regular) spacetimes admitting *global* foliations by spatially compact hypersurfaces with constant mean curvature is available in the literature. Andersson and Moncrief [7, 8] and Andersson [5, 6] have established global existence theorems for sufficiently small perturbations of a large family of spacetimes. For instance, their method allowed them to establish a global existence theorem for sufficiently small perturbation of Friedmann-Robertson-Walker type spacetimes. Their construction is based on constructing a global CMC foliation and uses harmonic coordinates on each slice. In these works, the authors derive (and strongly rely on) a priori estimates which are based on the so-called Bel-Robinson tensor and involve *up to third-order* derivatives of the metric. In contrast, we focus in the present paper on the local existence of such foliations but require only the sup norm of the curvature to be bounded. Our new approach leads to a construction of “good” local coordinates (see below) and allows us to explore the local optimal regularity of Lorentzian metrics. Another direction of research on CMC foliations is currently developed by Reiris [26, 27], who analyzes the CMC Einstein flow in connection with the Bel-Robinson energy and also imposes higher regularity of the metric.

We also refer the reader to an ambitious program (the L^2 curvature conjecture) initiated and developed by Klainerman and Rodnianski in a series of papers; see [19, 20, 21, 22]. In these works, the authors are interested in controlling the geometry of *null cones* which may become singular due to caustic formation. The regularity of null cones is needed in order to suitably extend the methods of harmonic analysis to the Einstein equations. In particular, the recent result [22] provides a breakdown criterion for solutions to the Einstein equations. In comparison with the present work, the objectives in [22] are different: these authors rely on *hyperbolic* techniques and investigate the geometry of *light cones*, while our approach in the present paper is purely *elliptic* in nature and addresses the geometry of *the spacetime itself*.

1.3 CMC–harmonic coordinates of an observer

Our second task is constructing local coordinates. In Riemannian geometry it is well-known that geodesic-based coordinates and distance-based coordinates fail to achieve the optimal regularity of the metric. The use of *harmonic* coordinates on Riemannian manifolds was first advocated by De Turck and Kazdan [13] and, later, a quantitative bound on the harmonic radius at a point was derived by Jost and Karcher [18] in terms of curvature and volume bounds, only. More recently, the issue of the optimal regularity of Lorentzian metrics was tackled by Anderson in the pioneering work [3, 4]. He proposed to use a combination of normal coordinates (based on geodesics) and spatially harmonic coordinates, and derived several uniform estimates for the metric coefficients. This construction based on geodesics does not lead to the desired optimal regularity, however. We also refer to earlier work by Anderson [1, 2] for further regularity results within the class of static and, more generally, stationary spacetimes.

Our main result covers Einstein vacuum spacetimes, that is, manifolds satisfying the Ricci-flat condition

$$\mathbf{Ric}_g = 0, \quad (1.1)$$

and the construction we propose is as follows. Relying on our quantitative estimates for CMC foliations near a given observer (Theorem 2.2) and then applying Jost and Karcher's theorem for Riemannian manifolds [18], we construct (spatially) harmonic coordinates on each spacelike CMC slice. We refer to such coordinates as *CMC-harmonic coordinates*, and we prove first that, on every slice, the spatial metric coefficients \mathbf{g}_{ij} belong to the Sobolev space $W^{2,a}$ and satisfy the *quantitative estimate*

$$\|\mathbf{g}_{ij}\|_{\mathbf{p}, \mathbf{T}_p, W^{2,a}} \leq C(a, \Lambda, \lambda)$$

for all $a < \infty$ and some constant $C(a, \Lambda, \lambda) > 0$ (depending also on the dimension n). In addition, we also control the lapse function and the shift vector associated with these local coordinates. The shift vector, denoted below by ξ , arises since coordinates are not simply transported from one slice to another but are chosen to be harmonic on each slice. The lapse function, denoted below by λ , is a measure of the distance between two nearby slices.

In turn, we arrive at the following main result of the present paper.

Theorem 1.1 (CMC-harmonic coordinates of an observer). *There exist constants $0 < \underline{c}(n) < c(n) < 1$ and $C(n), C_q(n) > 0$ depending upon the dimension n (and some exponent $q \in [1, \infty)$) such that the following properties hold. Let $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_p)$ be an $(n+1)$ -dimensional, pointed, Einstein vacuum spacetime satisfying the following curvature and injectivity radius bounds at the scale $r > 0$:*

$$\mathbf{R}_{\max}^r(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_p) \leq r^{-2}, \quad \mathbf{Inj}(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_p) \geq r. \quad (1.2)$$

Then, there exists a local coordinate system $\mathbf{x} = (t, x^1, \dots, x^n)$ having $p = (r_1, 0, \dots, 0)$ for some $r_1 \in [\underline{c}(n)r, c(n)r]$ and defined for all

$$|t - r_1| < c(n)^2 r, \quad ((x^1)^2 + \dots + (x^n)^2)^{1/2} < c(n)^2 r,$$

so that the following two properties hold:

- i) Each slice $\Sigma_t = \{(x^1)^2 + \dots + (x^n)^2 < c(n)^4 r^2\}$ on which t remains constant is a spacelike hypersurface with constant mean curvature $c(n)^{-1} r^{-2} t$ and the coordinates $x := (x^1, \dots, x^n)$ are harmonic for the metric induced on Σ_t .*
- ii) The Lorentzian metric in the spacetime coordinates $\mathbf{x} = (t, x^1, \dots, x^n)$ has the form*

$$\mathbf{g} = -\lambda(\mathbf{x})^2 (dt)^2 + g_{ij}(\mathbf{x})(dx^i + \xi^i(\mathbf{x}) dt)(dx^j + \xi^j(\mathbf{x}) dt) \quad (1.3)$$

and is close to the Minkowski metric in these local coordinates, in the sense that

$$e^{-C(n)} \leq \lambda \leq e^{C(n)},$$

$$e^{-C(n)} \delta_{ij} \leq g_{ij} \leq e^{C(n)} \delta_{ij}, \quad |\xi|_g^2 := g_{ij} \xi^i \xi^j \leq e^{-C(n)},$$

and for each $q \in [1, \infty)$

$$\frac{1}{r^{n-q}} \int_{\Sigma_t} |\partial_{\mathbf{x}} \mathbf{g}|^q dv_{\Sigma_t} + \frac{1}{r^{n-2q}} \int_{\Sigma_t} |\partial_{\mathbf{x}\mathbf{x}}^2 \mathbf{g}|^q dv_{\Sigma_t} \leq C_q(n).$$

The theorem above establishes the existence of *locally defined* CMC–harmonic coordinates near any observer. The coordinates cover a neighborhood of the base point, whose size is of order r in the timelike and in the spacelike directions. In the statement above, $\partial_{\mathbf{x}} \mathbf{g}$ and $\partial_{\mathbf{x}\mathbf{x}}^2 \mathbf{g}$ denote any *spacetime* first- and second-order derivatives of the metric coefficients in the local coordinates, respectively, while dv_{Σ_t} denotes the volume form induced on Σ_t by the spacetime metric and can be computed in terms of the spatial coordinates x .

Finally, let us put our results in a larger perspective. The proposed framework relies on constructing purely *local* CMC–harmonic coordinates and, therefore, applies to spacetimes which need not admit a global CMC foliation. However, based on our local regularity theory we can also control the *global* geometry of the spacetime, as follows.

Given a pointed Lorentzian manifold $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_{\mathbf{p}})$ satisfying a global version of the curvature and injectivity radius estimates (1.2), we can find a *global* atlas of local charts covering the whole of \mathbf{M} and in which the metric coefficients have the optimal regularity. Such a conclusion is achieved by introducing a notion of *global CMC–harmonic radius viewed by the observer* $(\mathbf{p}, \mathbf{T}_{\mathbf{p}})$: it is the “largest” radius $r > 0$ such that *local* CMC–harmonic coordinates exist in a ball of radius r about *each* point (and satisfy the uniform estimates stated in Theorem 1.1 for some fixed constants $\underline{c}(n), c(n), C(n), C_q(n)$). By establishing a lower bound on the radius of balls in which local CMC–harmonic coordinates exist at every point, we obtain the desired global optimal regularity. Again, this is a purely geometric result that involves the curvature and injectivity radius bounds, only. This development is a work in progress.

Throughout this paper, we use the notation C, C', C_1, \dots for constants that only depend on the dimension n and may change at each occurrence.

2 CMC foliation of an observer

2.1 Main statement in this section

In this section, we derive quantitative bounds on local CMC foliations for a general class of Lorentzian manifolds which need not satisfy the Einstein equations. For background on Riemannian or Lorentzian geometry we refer to [10, 17, 23]. Let (\mathbf{M}, \mathbf{g}) be a time-oriented, $(n + 1)$ -dimensional Lorentzian manifold, and

let ∇ be the Levi-Civita connection associated with \mathbf{g} . Given a point $\mathbf{p} \in \mathbf{M}$, we want to construct a constant curvature foliation that is defined near \mathbf{p} and whose geometry is uniformly controlled in terms of the curvature and injectivity radius, only. The inner product of two vectors X, Y is also written $\langle X, Y \rangle_g = \langle X, Y \rangle = g(X, Y)$.

In fact, rather than a single point on the manifold we must prescribe an *observer*, that is, a pair $(\mathbf{p}, \mathbf{T}_\mathbf{p})$ where $\mathbf{T}_\mathbf{p}$ is a unit, future-oriented, timelike vector at \mathbf{p} (also called a *reference vector*). We use the notation $(\mathbf{p}, \mathbf{T}_\mathbf{p}) \in T_1^+ \mathbf{M}$ for the bundle of such pairs, and we refer to $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_\mathbf{p})$ as a *pointed Lorentzian manifold*. The vector $\mathbf{T}_\mathbf{p}$ naturally induces a (positive-definite) inner product on the tangent space at \mathbf{p} , which we denote by $\mathbf{g}_{\mathbf{T}_\mathbf{p}} = \langle \cdot, \cdot \rangle_{\mathbf{T}_\mathbf{p}}$. We sometimes write $|X|_{g_{\mathbf{T}}}$ for the Riemannian norm of a vector X . To simplify the notation, we often write \mathbf{T} instead of $\mathbf{T}_\mathbf{p}$.

On a Lorentzian manifold the notion of injectivity radius is defined as follows. Consider the *exponential map* $\mathbf{exp}_\mathbf{p}$ at the point \mathbf{p} , as a map defined on the Riemannian ball $B_{\mathbf{g}_\mathbf{T}}(\mathbf{p}, r)$ (a subset of the tangent space at \mathbf{p}) and taking values in \mathbf{M} ; this map is well-defined for all sufficiently small radius r , at least.

Definition 2.1. *Given a Lorentzian manifold (\mathbf{M}, \mathbf{g}) , the injectivity radius $\mathbf{Inj}(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_\mathbf{p})$ of an observer $(\mathbf{p}, \mathbf{T}_\mathbf{p}) \in T_1^+ \mathbf{M}$ is the supremum over all radii $r > 0$ such that the exponential map $\mathbf{exp}_\mathbf{p}$ is well-defined and is a global diffeomorphism from the subset $B_{\mathbf{g}_\mathbf{T}}(\mathbf{p}, r)$ of the tangent space at \mathbf{p} to a neighborhood of \mathbf{p} in the manifold denoted by $\mathcal{B}_{\mathbf{g}_\mathbf{T}}(\mathbf{p}, r) := \mathbf{exp}_\mathbf{p}(B_{\mathbf{g}_\mathbf{T}}(\mathbf{p}, r)) \subset \mathbf{M}$.*

To simplify the notation, we will also use the notation $B_\mathbf{T}(\mathbf{p}, r)$ and $\mathcal{B}_\mathbf{T}(\mathbf{p}, r)$ for the above Riemannian balls. To state our assumption on the curvature we need a Riemannian metric defined in a neighborhood of the point \mathbf{p} . This reference metric is also denoted by $\mathbf{g}_\mathbf{T}$ and is defined as follows.

By parallel transporting the vector $\mathbf{T}_\mathbf{p}$, with respect to the Lorentzian connection ∇ and along radial geodesics leaving from \mathbf{p} , we construct a vector field \mathbf{T} which, however, may be multi-valued since two distinct geodesics leaving from \mathbf{p} , in general, may eventually intersect. We use the notation \mathbf{T}_γ for the vector field defined along a radial geodesic γ lying in the set $\mathcal{B}_\mathbf{T}(\mathbf{p}, r)$. Then, to this vector field we canonically associate a positive-definite, inner product $\mathbf{g}_{\mathbf{T}_\gamma} = \langle \cdot, \cdot \rangle_{\mathbf{T}_\gamma}$ defined in the tangent space of each point along the geodesic. We write $|A|_{\mathbf{T}_\gamma}$ or $|A|_\mathbf{T}$ for the corresponding Riemannian norm of a tensor A .

We then consider the Riemann curvature \mathbf{Rm} of the connection ∇ and, given an observer $(\mathbf{p}, \mathbf{T}_\mathbf{p})$, we compute its norm in the ball of radius r

$$\mathbf{R}_{\max}^r(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_\mathbf{p}) := \sup_\gamma |\mathbf{Rm}|_{\mathbf{T}_\gamma}, \quad (2.1)$$

where the supremum is taken over every radial geodesic from \mathbf{p} of Riemannian length r , at most. Note that $|\mathbf{Rm}|_{\mathbf{T}_\gamma}$ is evaluated with the Riemannian reference metric rather than from the Lorentzian metric. Our main assumption (1.2) is now well-defined.

Our objective in this section is constructing a foliation near \mathbf{p} , say $\bigcup_{t \leq \bar{t}} \Sigma_t$, by n -dimensional spacelike hypersurfaces $\Sigma_t \subset M$, and we require that each

slice has constant mean curvature equal to t , the range of t being specified by some functions $\underline{t} = \underline{t}(\mathbf{p})$ and $\bar{t} = \bar{t}(\mathbf{p})$. Moreover, this foliation should cover a “relatively large” part of the ball $\mathcal{B}_{\mathbf{T}}(\mathbf{p}, r)$.

Theorem 2.2 (Uniform estimates for a local CMC foliation of an observer). *There exist constants $\underline{c}, c, \bar{c}, \theta, \zeta \in (0, 1)$ with $\underline{c} < c < \bar{c}$, depending only on the dimension of the manifold such that the following property holds. Let $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_{\mathbf{p}})$ be a pointed Lorentzian manifold satisfying the curvature and injectivity radius assumptions (1.2) at some scale $r > 0$. Then, the Riemannian ball $\mathcal{B}_{\mathbf{T}}(\mathbf{p}, cr)$ can be covered by a foliation by spacelike hypersurfaces Σ_t with constant mean curvature t ,*

$$\left(\bigcup_{\underline{t} \leq t \leq \bar{t}} \Sigma_t \right) \supset \mathcal{B}_{\mathbf{T}}(\mathbf{p}, cr), \quad \underline{t} := n \frac{1 - \zeta}{sr}, \quad \bar{t} := n \frac{1 + \zeta}{sr}, \quad (2.2)$$

in which the time variable describes a range $[\underline{t}, \bar{t}]$ determined by some real $s \in [\underline{c}, \bar{c}]$ and, moreover, the unit normal vector \mathbf{N} and the second fundamental form h of the foliation satisfy

$$1 \leq -g(\mathbf{N}, \mathbf{T}) \leq 1 + \theta^{-1}, \quad \theta \leq -r^{-4} \mathbf{g}(\nabla t, \nabla t) \leq \theta^{-1}. \quad r|h| \leq \theta^{-1},$$

(Recall that the vector field \mathbf{T} is defined by parallel translating the given vector $\mathbf{T}_{\mathbf{p}}$ along radial geodesics from \mathbf{p} .)

Hence, a foliation exists in a neighborhood of the base point, in which the time variable is of order $1/r$ and describes an interval with definite size. In our construction given below, it will be important that s be chosen to be sufficiently small.

Note that the above theorem is purely geometric and does not depend explicitly on the coordinates that we are going now to introduce in order to establish the existence of the above foliation and control its geometry.

The rest of this section is devoted to giving a proof of Theorem 2.2. We first construct the CMC hypersurfaces as graphs over geodesic spheres associated with the Lorentzian metric. Geodesic spheres associated with the reference Riemannian metric will be introduced to serve as barrier functions. Indeed, each CMC hypersurface will be pinched between a Lorentzian and a Riemannian geodesic ball. The level set function describing the CMC hypersurface satisfies a nonlinear elliptic equation, whose coefficients have rather limited regularity, and this will force us to use the Nash-Moser iteration technique.

2.2 Formulation in normal coordinates

Foliation by geodesic spheres

We begin by introducing spacetime normal coordinates and by expressing the prescribed mean curvature equation in these coordinates. As we established earlier in [12], under the curvature and injectivity radius assumption (1.2) for the

observer $(\mathbf{p}, \mathbf{T}_{\mathbf{p}})$, there exist positive constants $\underline{c} < c < \bar{c} < 1$ and C depending only on the dimension of the manifold, such that the following properties hold.

First of all, the foliation by subsets \mathcal{H}_τ of geodesic spheres is defined as follows. Let γ be a future-oriented, timelike geodesic containing \mathbf{p} and let us parameterize it so that $\mathbf{p} = \gamma(cr)$. Set $\mathbf{q} := \gamma(0)$ and consider the (new) observer $(\mathbf{q}, \mathbf{T}_{\mathbf{q}})$ with $\mathbf{T}_{\mathbf{q}} := \gamma'(0)$. The constant \bar{c} is chosen sufficiently small so that the injectivity radius of the map $\mathbf{exp}_{\mathbf{q}}$ (computed for the observer $(\mathbf{q}, \mathbf{T}_{\mathbf{q}})$) is $\bar{c}r$, at least. From the point \mathbf{q} we consider normal coordinates $\mathbf{y} = (y^\alpha) = (\tau, y^j)$ determined by the family of future-oriented timelike radial geodesics from \mathbf{q} , so that the Lorentzian metric takes the form $\mathbf{g} = -d\tau^2 + \mathbf{g}_{ij} dy^i dy^j$. These coordinates cover a part of the future of the point \mathbf{q} and at least the region

$$\begin{aligned} \mathcal{C}^+(\mathbf{q}, \bar{c}r) &:= \mathbf{exp}_{\mathbf{q}}(\mathcal{C}^+(\mathbf{q}, \bar{c}r)), \\ \mathcal{C}^+(\mathbf{q}, \bar{c}r) &:= \left\{ V \in B_{\mathbf{T}_{\mathbf{q}}}(0, \bar{c}r), \quad \mathbf{g}_{\mathbf{T}_{\mathbf{q}}}(V, V) < 0, \quad \frac{\mathbf{g}_{\mathbf{T}_{\mathbf{q}}}(\mathbf{T}_{\mathbf{q}}, V)}{\mathbf{g}_{\mathbf{T}_{\mathbf{q}}}(V, V)^{1/2}} \geq 1 - \bar{c} \right\}. \end{aligned}$$

The base point \mathbf{p} is identified with $(\tau, y^1, \dots, y^n) = (cr, 0, \dots, 0)$ in these coordinates. By relying on the curvature bound, analyzing the behavior of Jacobi fields, and using standard comparison arguments from Riemannian geometry, one can establish in well-chosen coordinates [12]:

$$\begin{aligned} C^{-1} \delta_{ij} &\leq \mathbf{g}_{ij} \leq C \delta_{ij}, \\ r^{-1} \left| \frac{\partial \mathbf{g}_{ij}}{\partial \tau} \right| + r^{-2} \left| \nabla_{\frac{\partial}{\partial \tau}} \frac{\partial \mathbf{g}_{ij}}{\partial \tau} \right| &\leq C \quad \text{in } \mathcal{C}^+(\mathbf{q}, \bar{c}r) \cap \{\underline{\tau} \leq \tau \leq \bar{\tau}\}, \end{aligned} \quad (2.3)$$

where $\underline{\tau} := cr$ and $\bar{\tau} := \bar{c}r$.

The reference Riemannian metric associated with the vector field $\partial/\partial\tau$ (obtained by parallel transporting the vector $\mathbf{T}_{\mathbf{q}}$) reads $\tilde{\mathbf{g}} := d\tau^2 + \mathbf{g}_{ij} dy^i dy^j$. We use the notation $\tilde{\mathbf{d}}(\cdot, \cdot)$ and $\tilde{S}(\cdot, \cdot)$ for the distance function and the geodesic spheres associated with the metric $\tilde{\mathbf{g}}$, respectively. By construction, the function τ coincides with the distance function $\tilde{\mathbf{d}}(\mathbf{q}, \cdot)$. It will be useful also to have the following estimate of the Riemann curvature of the Lorentzian metric

$$|\mathbf{Rm}|_{\tilde{\mathbf{g}}} \leq C r^{-2} \quad \text{in } \mathcal{C}^+(\mathbf{q}, \bar{c}r) \cap \{\underline{\tau} \leq \tau \leq \bar{\tau}\}, \quad (2.4)$$

which is based on the reference metric $\tilde{\mathbf{g}}$.

In turn, the above construction provides us with a foliation (by Lorentzian geodesic spheres) of some neighborhood of \mathbf{p} (with definite size) by n -dimensional spacelike hypersurfaces \mathcal{H}_τ , hence $\mathbf{p} \in \bigcup_{\tau \in [\underline{\tau}, \bar{\tau}]} \mathcal{H}_\tau$.

Now, consider the time function τ . The standard Hessian comparison theorem for distance functions in Riemannian geometry is also useful in Lorentzian geometry and, more precisely, shows that the (restriction of the) Hessian of τ is equivalent to the induced metric:

$$\underline{k}(\tau, r) \mathbf{g}_{ij} \leq (-\nabla^2 \tau)|_{E, ij} \leq \bar{k}(\tau, r) \mathbf{g}_{ij}, \quad (2.5)$$

where $E := (\nabla\tau)^\perp$ denotes the orthogonal complement and

$$\underline{k}(\tau, r) := \frac{r^{-1}\sqrt{C}}{\tan(\tau r^{-1}\sqrt{C})}, \quad \bar{k}(\tau, r) := \frac{r^{-1}\sqrt{C}}{\tanh(\tau r^{-1}\sqrt{C})}.$$

Observe that both $\underline{k}(\tau, r)$ and $\bar{k}(\tau, r)$ behave like $1/\tau$ when $\tau \rightarrow 0$. Note also that \underline{k} will remain non-singular within the range of interest, since τ will be chosen to be a small multiple of r .

Consequently, noting that

$$-\nabla_{ij}^2\tau = \frac{1}{2} \frac{\partial \mathbf{g}_{ij}}{\partial \tau} =: A_{ij}, \quad (2.6)$$

taking the trace in the inequalities (2.5), and then using the uniform estimate (2.3), we see that the mean curvature of each slice $\tau = \text{const.}$ is close to n/τ . Our objective now is to replace these slices by constant mean curvature slices by making a small perturbation determined by solving an elliptic equation in these normal coordinates.

We will also use the *Riemannian geodesic spheres* associated with $\tilde{\mathbf{g}}$. Namely, consider the Riemannian distance function $\tilde{\mathbf{d}}(\mathbf{p}', \cdot)$ computed from some arbitrary point $\mathbf{p}' := \gamma(\tau)$ with $\tau \in [\underline{\tau}, \bar{\tau}]$. From the expression of the reference metric we find

$$\nabla^2 \tilde{\mathbf{d}}(\mathbf{p}', \cdot) = \tilde{\nabla}^2 \tilde{\mathbf{d}}(\mathbf{p}', \cdot) - 2 \frac{\partial \tilde{\mathbf{d}}}{\partial \tau} A, \quad (2.7)$$

where $\tilde{\nabla}$ is the covariant derivative associated with $\tilde{\mathbf{g}}$. Again by the Hessian comparison theorem and since $|\partial \tilde{\mathbf{d}}/\partial \tau| \leq |\tilde{\nabla} \tilde{\mathbf{d}}|_{\tilde{\mathbf{g}}} = 1$, we find after setting $\tilde{E} := (\tilde{\nabla} \tilde{\mathbf{d}})^\perp$

$$\left(\underline{k}(\tilde{\mathbf{d}}, r) - \frac{C}{r} \right) \tilde{\mathbf{g}}|_{\tilde{E}} \leq (\nabla^2 \tilde{\mathbf{d}})|_{\tilde{E}} \leq \left(\bar{k}(\tilde{\mathbf{d}}, r) + \frac{C}{r} \right) \tilde{\mathbf{g}}|_{\tilde{E}}. \quad (2.8)$$

Choosing now the time variable to be a (small) multiple of r and taking the trace of the above inequalities, we deduce that for any $a \in [\underline{c}r, \bar{c}r]$ the mean curvature $H_{\mathcal{A}(\mathbf{p}', a)}$ (computed with respect to the ambient Lorentzian metric) of the (future-oriented, spacelike, and possibly empty) intersection $\mathcal{A}(\mathbf{p}', a)$ of the $\tilde{\mathbf{g}}$ -geodesic sphere $\tilde{S}(\mathbf{p}', a)$ and the future set $\mathcal{C}^+(\mathbf{q}, \bar{c}r)$ satisfies the inequalities

$$n \underline{k}(a, r) \leq H_{\mathcal{A}(\mathbf{p}', a)} \leq n \bar{k}(a, r), \quad a \in [\underline{c}r, \bar{c}r]. \quad (2.9)$$

Hence, the mean curvature of the Riemannian slices enjoys the same inequalities as the ones of the Lorentzian slices \mathcal{H}_τ . Later in this section, we will use the graph of the Riemannian geodesic spheres as barrier functions.

This completes the discussion of a domain of coordinates \mathbf{y} covering a neighborhood of \mathbf{p} , in which we can assume that all of the above estimates are valid.

Mean curvature operator

We now search for a new foliation $\bigcup_t \Sigma_t$ in which the hypersurfaces have constant mean curvature and can be viewed as graphs, say $\Sigma_t := \{G^t(y) := (u^t(y), y)\}$, over a geodesic leaf \mathcal{H}_τ for a given value τ of time-function. Here, t is a real parameter varying in some interval of definite size and the functions $y \mapsto u^t(y)$ need to be determined. In the following, we often write $\Sigma = \Sigma_t$, $u = u^t$, and $G = G^t$. Setting $u_j := \partial u / \partial y^j$, the induced metric and its inverse are determined by projection on the slice Σ and read

$$g_{ij} = \mathbf{g}_{ij} - u_i u_j, \quad g^{ij} = \mathbf{g}^{ij} + \frac{\mathbf{g}^{ik} \mathbf{g}^{jl} u_k u_l}{1 - |\nabla u|^2},$$

and the hypersurface Σ is Riemannian if and only if

$$|\nabla u|^2 = \mathbf{g}^{ij}(u, \cdot) u_i u_j < 1.$$

We are interested here in spacelike hypersurfaces, and we denote by ∇ the covariant derivative associated with the induced Riemannian metric g_{ij} . We easily obtain

$$|\nabla u|^2 = g^{ij} u_i u_j := \frac{|\nabla u|^2}{1 - |\nabla u|^2}, \quad |\nabla u|^2 = \frac{|\nabla u|^2}{1 + |\nabla u|^2}.$$

The future-oriented unit normal to each hypersurface takes the form

$$\mathbf{N} = -\sqrt{1 + |\nabla u|^2} (1, \nabla u).$$

The second fundamental form of the slice Σ is determined by push forward (with the map G) of the coordinate vector fields $Y_j := \partial / \partial y^j$:

$$\begin{aligned} h_{ij} &:= \langle \nabla_{G_* Y_i} G_* Y_j, \mathbf{N} \rangle \\ &= \frac{1}{\sqrt{1 - |\nabla u|^2}} \left(\nabla_i \nabla_j u + \frac{1}{2} \frac{\partial \mathbf{g}_{ij}}{\partial \tau} - \frac{1}{2} \mathbf{g}^{kl} \frac{\partial \mathbf{g}_{li}}{\partial \tau} u_k u_j - \frac{1}{2} \mathbf{g}^{kl} \frac{\partial \mathbf{g}_{lj}}{\partial \tau} u_k u_i \right) \\ &= \frac{1}{\sqrt{1 + |\nabla u|^2}} \left(\nabla_i \nabla_j u + A_{ij} \right), \end{aligned} \tag{2.10}$$

where

$$\nabla_i \nabla_j u = \frac{\partial^2 u}{\partial y^i \partial y^j} - \Gamma_{ij}^k(u, y) \frac{\partial u}{\partial y^k}$$

and Γ_{ij}^k are the Christoffel symbols of \mathbf{g} . The tensor field $\nabla_i \nabla_j u$ is the space-time Hessian of the function u (restricted to the hypersurface $\tau = u$), while $\nabla_i \nabla_j u$ is the fully spatial Hessian defined from the intrinsic metric g_{ij} .

The mean curvature of a slice is the trace of h_{ij} , that is, in intrinsic form

$$\mathcal{M}u := h_{ij} g^{ij} = \frac{1}{\sqrt{1 + |\nabla u|^2}} (\Delta u + A_j^j),$$

where Δ is the Laplace operator in the hypersurface, or equivalently in local coordinates

$$\begin{aligned} \mathcal{M}u &= \frac{1}{\sqrt{\mathbf{g}(u, \cdot)}} \frac{\partial}{\partial y^i} \left(\sqrt{\mathbf{g}(u, \cdot)} \nu(\nabla u) \mathbf{g}^{ij}(u, \cdot) \frac{\partial u}{\partial y^j} \right) \\ &\quad + \left(\nu(\nabla u)^{-1} \mathbf{g}^{ij}(u, \cdot) + \nu(\nabla u) \mathbf{g}^{ik}(u, \cdot) \mathbf{g}^{jl}(u, \cdot) u_k u_l \right) \frac{1}{2} \frac{\partial \mathbf{g}_{ij}}{\partial \tau}(u, \cdot), \end{aligned}$$

where we have introduced the nonlinear function

$$\nu(\nabla u) := \frac{1}{\sqrt{1 - |\nabla u|^2}} = \sqrt{1 + |\nabla u|^2} = \nu(\nabla u).$$

Note that, in fact, ν depends also on u .

Local formulation of the prescribed mean curvature problem

We are now ready to introduce a formulation of the problem of interest, in terms of the reference Riemannian metric $\tilde{\mathbf{g}}$. Recall that γ is a fixed, future-oriented, timelike curve passing through \mathbf{p} . Assuming for definiteness that $2c + 4c^2 < \bar{c}$, from now on we fix some $s \in [c, 2c]$ and we introduce the point $\mathbf{p}_s := \gamma((s + s^2)r)$ which lies in the future of the base point \mathbf{p} since γ is a timelike future-oriented curve passing through \mathbf{p} for the parameter value cr . We then introduce the subset $\Omega_s \subset \{\tau = sr\}$ whose *boundary is defined* by the condition

$$\partial\Omega_s := \mathcal{A}(\mathbf{p}_s, (s^2 + s^3)r) \cap \{\tau = sr\}$$

and which fills up its interior. This choice is essential for the mean curvature equation (discussed below) to admit the Riemannian slices as barrier functions. Observe that

$$B_{sr}(\gamma(sr), s^{5/2}r/2) \subset \Omega_s \subset B_{sr}(\gamma(sr), 2s^{5/2}r). \quad (2.11)$$

Here, $B_{sr}(\gamma(sr), a)$ is the geodesic ball of radius a which lies *in the slice* $\tau = sr$ and is determined by the metric \mathbf{g}_{ij} induced on the geodesic leaf.

Finally, given $\alpha \in (0, 1)$ and a bounded function H of class C^α defined on Ω_s and satisfying the restriction

$$n\bar{k}(sr, r) \leq H \leq n\underline{k}(2s^2r, r),$$

we seek for a spacelike hypersurface with mean curvature H and boundary $\partial\Omega_s$. Analytically, this is equivalent to solving the Dirichlet problem

$$\begin{aligned} \mathcal{M}u &= H && \text{in } \Omega_s, \\ u &= sr && \text{in } \partial\Omega_s, \end{aligned} \quad (2.12)$$

in which, therefore, we have prescribed both the boundary of the unknown hypersurface and its mean curvature. In the present paper, we are mainly interested in the case that H is a constant function. We also assume that s is sufficiently small.

2.3 Statements of the uniform estimates

To establish the existence of CMC hypersurfaces as graphs over a given geodesic leaf \mathcal{H}_τ , the main difficulty is to bound $|\nabla u|$ away from 1, for all functions u satisfying $u = \tau$ on the boundary and having their mean curvature pinched in some interval. Precisely, the rest of this section is devoted to the proof of the following result.

Recall that $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_\mathbf{p})$ denotes a pointed Lorentzian manifold satisfying the curvature and injectivity radius assumptions (1.2) at some scale $r > 0$ and that γ is a future-oriented timelike geodesic satisfying $\gamma(cr) = \mathbf{p}$.

Proposition 2.3 (Uniform estimates for CMC hypersurfaces). *There exist constants $c, \theta > 0$ depending on the dimension n only such that the following property holds with the notation introduced in this section. For any $s \in [c, 2c]$ and $t \in [n\bar{k}(sr, r), n\underline{k}(2s^2r, r)]$ there exists a solution u to the Dirichlet problem (2.12) associated with the (constant) mean curvature function $H \equiv t$ and such that*

$$\sup_{\Omega_s} |\nabla u| \leq 1 - \theta, \quad \sup_{\Omega'_s} r|h| \leq \theta^{-1},$$

where $\Omega'_s = B_{sr}(\gamma(sr), s^{5/2}r/4) \subset \{\tau = sr\}$.

Observe that the bound on the second fundamental form holds only in a subset of Ω_s , whose diameter, however, is also of the order r . Theorem 2.2 is immediate once we establish Proposition 2.3. The proof of Proposition 2.3 will follow from several preliminary results. The first lemma below is a direct consequence of the maximum principle for elliptic operators. The other lemmas will be established in Subsection 2.4.

Lemma 2.4 (Comparison principle). *Given two functions u, w satisfying $\mathcal{M}u \geq \mathcal{M}w$ in their domain of definition and $u \leq w$ along the boundary, one has either $u < w$ in the interior of their domain of definition or else $u \equiv w$. In particular, if $\mathcal{M}u \geq n\bar{k}(\bar{\tau}, r)$ everywhere and $u \leq \bar{\tau}$ along the boundary, then $u \leq \bar{\tau}$. Similarly, if $\mathcal{M}u \leq n\underline{k}(\underline{\tau}, r)$ everywhere and $u \geq \underline{\tau}$ along the boundary, then $u \geq \underline{\tau}$.*

Lemma 2.5 (Boundary gradient estimate). *For any solution u of (2.12) with mean curvature function satisfying $n\bar{k}(sr, r) \leq \mathcal{M}u \leq n\underline{k}(2s^2r, r)$ one has*

$$|\nabla u| < \frac{1}{2} \quad \text{on the boundary } \partial\Omega_s. \quad (2.13)$$

Lemma 2.6 (Global gradient estimate). *Under the assumptions of Lemma 2.5 one has*

$$\sup_{\Omega_s} |\nu(\nabla u)| \leq C_1(n),$$

where the constant $C_1(n)$ depends on the dimension, only.

Now, in view of Lemmas 2.5 and 2.6 and by standard arguments [16], one can check that for each $t \in [\bar{k}(sr, r), \underline{k}(2s^2r, r)]$ the Dirichlet problem (2.12) admits a smooth solution u determining a slice with constant mean curvature t .

Note that, by Lemma 2.6, the induced metric on Σ_t is equivalent to the metric \mathbf{g}_{ij} on the domain Ω so that we can use, for instance, Sobolev inequalities on Σ_t .

Lemma 2.7 (Interior estimates for the second fundamental form). *Under the assumptions of Lemma 2.5, for all $q \in [1, \infty)$ there exist positive constants $C_2(n)$ and $C_3(n, q)$ such that for every $p' \in \Sigma \setminus \partial\Sigma$*

$$|h(p')| \leq \frac{C_2(n)}{d(p', \partial\Sigma)},$$

$$\left(\frac{1}{d(p', \partial\Sigma)^n} \int_{B(p', d(p', \partial\Sigma)/4)} |\nabla h|^q dv_\Sigma \right)^{1/q} \leq \frac{C_3(n, q)}{d(p', \partial\Sigma)^2},$$

where dv_Σ is the induced volume form on Σ and $d(p', \partial\Sigma)$ is the distance to the boundary $\partial\Sigma$ associated with the induced metric g_{ij} on Σ .

Observe that the upper bound in the above lemma blows-up if the point p' approaches the boundary of the CMC slice, and that for $p' \in \Omega'_s$ the factor $d(p', \partial\Sigma)$ is of order r , as required for Proposition 2.3.

Lemma 2.8 (Time-derivative of the level function). *Under the assumptions of Lemma 2.5 there exist constants $C_4(n), C_5(n) > 0$ such that*

$$C_4(n)r^2 \leq -\frac{\partial u}{\partial t} \leq C_5(n)r^2 \quad \text{on } \Omega'_s.$$

2.4 Derivation of the uniform estimates

Proof of Lemma 2.5. We use here the maximum principle stated in Lemma 2.4. The part of the Riemannian geodesic sphere $\tilde{S}(\mathbf{p}_s, (s^2 + s^3)r)$ (defined by $\tilde{\mathbf{g}}$) “below” Ω_s , that is the part corresponding to $\tau \leq sr$, is the graph $y \mapsto (\underline{u}(y), y)$ of a function \underline{u} over Ω_s whose boundary values are sr on $\partial\Omega_s$. Since s is sufficiently small, one easily checks that, for instance,

$$|\nabla \underline{u}| < s^{1/2} \quad \text{on } \Omega_s \tag{2.14}$$

and, in particular, \underline{u} satisfies (2.13) along the boundary $\partial\Omega_s$.

Suppose now that there exists a C^2 spacelike hypersurface $(u(y), y)$ defined over Ω_s having the same boundary values as the function \underline{u} and such that its mean curvature H remains bounded in the interval $[n\bar{k}(sr, r), n\underline{k}((s^2 + s^{5/2})r, r)]$. Let us set

$$\bar{m} = \sup_{y \in \Omega_s} u(y), \quad \underline{m} = \sup_{y \in \Omega_s} \tilde{\mathbf{d}}((u(y), y), \mathbf{p}_s)$$

and use the following comparison technique.

Note that the range of the function u lies between $(s - 2s^3)r$ and $(s + 2s^3)r$, since u is spacelike (so the norm of its gradient can not exceed 1), its boundary value is sr , and the diameter of the set Ω_s is $4s^3r$ at most. We are going to show the pinching property

$$\underline{u} - sr \leq u - sr \leq 0,$$

which immediately implies the desired boundary gradient estimate (2.13).

First of all, we claim that $\bar{m} = sr$. If this were not true, then the maximum of u would be achieved at some point y_0 in the interior of Ω_s . Since the graph of u is below the graph of $\tau \equiv \bar{m}$ and both graphs are tangent at the point (y_0, \bar{m}) , we conclude that at the point $(y_0, u(y_0))$ the mean curvature of u is less or equal to that of $\tau \equiv \bar{m}$. However, in view of the Hessian estimate (2.5) this is a contradiction if $\bar{m} > sr$.

Considering next the lower bound for u , we claim that $u \geq \underline{u}$ on Ω_s . Otherwise, by contradiction there would exist a point $y_1 \in \Omega_s$ such that (at least)

$$\tilde{\mathbf{d}}((u(y_1), y_1), \mathbf{p}_s) = \underline{m} < (s^2 + 2s^3)r.$$

By comparing, at the base point $(u(y_1), y_1)$, the mean curvature of the graph u and the one of the sphere $S(\mathbf{q}', \tilde{\mathbf{d}}((u(y_1), y_1), \mathbf{p}_s))$, we find that

$$\mathcal{M}u(y_1) > nk((s^2 + 2s^3)r, r),$$

which contradicts our assumption $\mathcal{M}u \leq nk(2s^2r, r)$. \square

Proof of Lemma 2.6. Step 1. We will first show that, for some sufficiently large p , the sup norm of $\nu(\nabla u)$ is bounded by its L^p norm. By scaling, we may assume $r = 1$ from now on. Here, as in the rest of this paper, the main difficulty is making sure that all constants arising in the following arguments depend on the injectivity radius and curvature bounds, only. It will be convenient to work with the intrinsic form of the mean-curvature operator \mathcal{M} , but the expression in coordinates will be also used in the end of the argument in order to control certain Sobolev constants.

Recall that, on the hypersurface Σ ,

$$\Delta u + A_j^j = \nu(\nabla u) H, \tag{2.15}$$

where Δ denotes the Laplace operator on the slice and H is the prescribed mean curvature function. Observe that since the second fundamental form of the geodesic sphere is bounded, we have

$$|A_j^j| = \left| g^{ij} \frac{\partial \mathbf{g}_{ij}}{\partial \tau} \right| \leq C g^{ij} \mathbf{g}_{ij} \leq \frac{C}{1 - |\nabla u|^2},$$

hence

$$|\Delta u| \leq C |\nu(\nabla u)|^2. \tag{2.16}$$

Observe that the coefficients of the Laplace operator on Σ are nothing but metric coefficients on which, at this stage of the analysis, we have an L^∞ control, only.

We are going to use the classical Weitzenböck identity applied to the level function u

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2\langle\nabla u, \nabla\Delta u\rangle + 2\text{Ric}(\nabla u, \nabla u). \quad (2.17)$$

First, recalling that $G = (u, y)$ we estimate the Ricci curvature term by relying on the Gauss formula

$$\begin{aligned} R_{ijkl} &= \mathbf{R}_{\alpha\beta\gamma\delta} G_i^\alpha G_j^\beta G_k^\gamma G_l^\delta - (h_{ik}h_{jl} - h_{il}h_{jk}) \\ &= \mathbf{R}_{ijkl} + \mathbf{R}_{0jkl}u_i + \mathbf{R}_{i0kl}u_j + \mathbf{R}_{ij0l}u_k + \mathbf{R}_{ijk0}u_l + u_i u_k \mathbf{R}_{0j0l} \\ &\quad + u_i u_l \mathbf{R}_{0jk0} + u_j u_k \mathbf{R}_{i00l} + u_j u_l \mathbf{R}_{i0k0} - (h_{ik}h_{jl} - h_{il}h_{jk}), \end{aligned}$$

where $G_i = G_*(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial y^i} + \frac{\partial u}{\partial y^i} \frac{\partial}{\partial \tau}$. The spacetime curvature being uniformly bounded, we find

$$\begin{aligned} \mathbf{R}_{\alpha\beta\gamma\delta} G_i^\alpha G_j^\beta G_k^\gamma G_l^\delta g^{jl} &\geq -C(1 + |\nabla u|^2) \mathbf{g}_{ik} \\ &= -C(1 + |\nabla u|^2) (g_{ik} + u_i u_k). \end{aligned}$$

Taking the trace of the Gauss formula, we obtain a lower bound for the Ricci curvature of the hypersurface:

$$R_{ik} \geq h_{il}h_{kj}g^{lj} - Hh_{ik} - C(1 + |\nabla u|^2)(g_{ik} + u_i u_k),$$

and therefore

$$\text{Ric}(\nabla u, \nabla u) \geq -C(1 + |\nabla u|^2)^3.$$

In turn, from (2.17) we deduce the key inequality

$$\Delta|\nabla u|^2 - 2|\nabla^2 u|^2 \geq 2\langle\nabla u, \nabla(\Delta u)\rangle - C(1 + |\nabla u|^2)^3, \quad (2.18)$$

which is an *intrinsic* statement written on the hypersurface Σ and the constant C depends on the spacetime curvature bound, only.

Next, to estimate the gradient of u we consider the function

$$v = v(\nabla u) := (1 + |\nabla u|^2 - k)_+,$$

where k is chosen suitably large so that, thanks to the boundary gradient estimate in Lemma 2.5, the function $v(|\nabla u|)$ vanishes on the boundary of the hypersurface Σ . Multiplying (2.18) by v^q for $q \geq 1$ integrating over the hypersurface, and using Green's formula we obtain

$$\begin{aligned} &\int_{\Sigma} \left(q v^{q-1} |\nabla v|^2 + 2v^q |\nabla^2 u|^2 \right) dv_{\Sigma} \\ &\leq \int_{\Sigma} \left(2q v^{q-1} \langle\nabla v, \nabla u\rangle \Delta u + 2v^q |\Delta u|^2 + C(v^{q+3} + v^q) \right) dv_{\Sigma}. \end{aligned}$$

At this juncture, we observe that the higher-order term Δu is controlled by the prescribed mean curvature equation (2.16). We obtain

$$\int_{\Sigma} \left(q v^{q-1} |\nabla v|^2 + 2v^q |\nabla^2 u|^2 \right) dv_{\Sigma} \leq (q+1) C \int_{\Sigma} (v^{q+3} + v^{q-1}) dv_{\Sigma}$$

and thus

$$\int_{\Sigma} \left| \nabla(v^{(q+1)/2}) \right|^2 dv_{\Sigma} \leq (q+1)^2 C \int_{\Sigma} (v^{q+3} + v^{q-1}) dv_{\Sigma}. \quad (2.19)$$

To make use of (2.19) it is convenient to return to our notation in coordinates, by observing that

$$\begin{aligned} \sqrt{\det(g)} &= \sqrt{1 - |\nabla u|^2} \sqrt{\det(\mathbf{g})}, \\ |\nabla(v^{(q+1)/2})|^2 &\geq \left(\frac{q+1}{2}\right)^2 v^{q-1} |\nabla v|^2, \end{aligned}$$

so that

$$C |\nabla(v^{(q+1)/2})|^2 \sqrt{\det(g)} \geq |\nabla v^{(q+\frac{1}{2})/2}|^2 \sqrt{\det(\mathbf{g})},$$

where we used (if $v > 0$)

$$C v^{-1/2} \geq \sqrt{1 - |\nabla u|^2} \geq C' v^{-1/2}.$$

Therefore, provided we now assume that $q \geq 3/2$, (2.19) takes the following coordinate-dependent form:

$$\int_{\Omega_s} |\nabla(v^{(q+\frac{1}{2})/2})|^2 dy \leq (q+1)^2 C \int_{\Omega_s} (v^{q+3-1/2} + v^{q-1-1/2}) dy. \quad (2.20)$$

By Sobolev's inequality in the local coordinates under consideration, we have

$$\left(\int_{\Omega_s} w^{2n/(n-1)} dy \right)^{(n-1)/n} \leq C \int_{\Omega_s} (|\nabla w|^2 + w^2) dy,$$

which we apply to the function $w := v^{(q+1/2)/2}$. Recalling that $r = 1$ (after normalization) and observing that the domain of integration in y is bounded, we deduce from (2.20) that

$$\left(\int_{\Omega_s} v^{(q+\frac{1}{2})n/(n-1)} dy \right)^{(n-1)/n} \leq C (q+1)^2 \int_{\Omega_s} (v^{q+3-1/2} + v^{q-1-1/2}) dy$$

or, equivalently, for all $p > 2$

$$\left(\int_{\Omega_s} v^{pn/(n-1)} dy \right)^{(n-1)/(pn)} \leq C^{1/p} p^{2/p} \left(\int_{\Omega_s} (v^{p+2} + v^{p-2}) dy \right)^{1/p}. \quad (2.21)$$

Without loss of generality, we may assume that $\|v\|_{L^\infty(\Omega_s)} \geq 1$, for otherwise the result is immediate. Then, (2.21) leads to the main estimate

$$\begin{aligned} & \max\left(1, \left(\int_{\Omega_s} v^{pn/(n-1)} dy\right)^{(n-1)/(pn)}\right) \\ & \leq C^{1/p} p^{2/p} \|v\|_{L^\infty(\Omega_s)}^{2/p} \max\left(1, \left(\int_{\Omega_s} v^p dy\right)^{1/p}\right). \end{aligned}$$

It remains to iterate the above estimate, which yields

$$\begin{aligned} \|v\|_{L^\infty(\Omega_s)} & \leq C' \|v\|_{L^\infty(\Omega_s)}^\alpha \left(\int_{\Omega_s} v^{p_0} dy\right)^{1/p_0}, \\ \alpha & := \frac{2}{p_0} \sum_{k=0}^{\infty} (1 - 1/n)^k = \frac{2n}{p_0}. \end{aligned}$$

In conclusion, provided that $p_0 > 2n$ the sup norm of v is uniformly bounded by its L^{p_0} norm.

Step 2. It remains to derive an estimate for some L^{p_0} norm. Following [14], we return to the inequality (2.16) satisfied by the function u and, for every λ , we write

$$\begin{aligned} \Delta(e^{\lambda u}) & = \lambda e^{\lambda u} \Delta u + \lambda^2 e^{\lambda u} |\nabla u|^2 \\ & \geq -C \lambda e^{\lambda u} (\nu(\nabla u))^2 + \lambda^2 e^{\lambda u} |\nabla u|^2. \end{aligned}$$

Combining this estimate with a direct calculation from (2.18) (similar to the one in Step 1 above), we obtain

$$\begin{aligned} \Delta(v^q e^{\lambda u}) & \geq \lambda^2 v^{q+1} e^{\lambda u} + \lambda v^{q-1} e^{\lambda u} (\langle \nabla u, \nabla v \rangle - C v (v+1)) \\ & \quad + q v^{q-1} e^{\lambda u} (2|\nabla^2 u|^2 - C v^3 + 2\langle \nabla u, \nabla(\Delta u) \rangle) \\ & \quad + q(q-1) v^{q-2} e^{\lambda u} |\nabla v|^2. \end{aligned}$$

Then, by integrating over the hypersurface Σ , integrating by parts, using (2.16) to control the term Δu , and finally choosing λ sufficiently large, we arrive at

$$\int_{\Sigma} |\nabla u|^q dv_{\Sigma} \leq C'_q.$$

This completes the proof of Lemma 2.6. \square

Proof of Lemma 2.7. Step 1. We are going to control the sup norm of h , and to this end we will use Nash-Moser's iteration technique. Note that the elliptic equation satisfied by the second fundamental form a priori has solely L^∞ coefficients. By scaling we can assume $r = 1$. We consider an arbitrary point $p' \in \Sigma$

and we set $\delta := d(p', \partial\Sigma)$. Simons' identity [28] for the hypersurface reads

$$\begin{aligned}\Delta h_{ij} &= \Delta h_{ij} - (trh)_{ij} \\ &= |h|^2 h_{ij} - (trh)h_{ik}h_{lj}g^{kl} - \mathbf{R}_{ipjq}h_{kl}g^{pk}g^{ql} + \mathbf{R}_{jplq}h_{ik}g^{pq}g^{kl} \\ &\quad + \nabla_p(\mathbf{R}_{qj\mathbf{N}i})g^{pq} - \nabla_j(\mathbf{R}_{i\mathbf{N}}),\end{aligned}\tag{2.22}$$

in which the Hessian $(trh)_{ij}$ vanishes since Σ has constant mean curvature. Recall here that \mathbf{N} is the future-oriented normal to Σ . Thanks to (2.22) we obtain

$$\begin{aligned}\Delta|h|^2 &\geq 2|\nabla h|^2 + 2|h|^4 - C(|\mathbf{Rm}|_{\mathbf{N}} + 1)|h|^3 \\ &\quad + 2\langle \nabla_p(\mathbf{R}_{qj\mathbf{N}i})g^{pq} - \nabla_j(\mathbf{R}_{i\mathbf{N}}), h_{ij} \rangle.\end{aligned}\tag{2.23}$$

Let φ be a smooth, non-negative, non-increasing cut-off function which equals 1 in the interval $[0, 1/2]$ and 0 in $[1, \infty)$. Then, the function $\psi := \varphi \circ \kappa$ with $\kappa := (d(p', \cdot)/\delta)$ is a cut-off function on the CMC hypersurface Σ which vanishes near the boundary $\partial\Sigma$.

Fix some $q \in [1, \infty)$. Multiplying (2.23) by $\psi|h|^q$, integrating over Σ , and then integrating by parts, we arrive at

$$\begin{aligned}&\int_{\Sigma} \psi|h|^q \Delta|h|^2 dv_{\Sigma} \\ &\geq \int_{\Sigma} \left(\psi|h|^q (2|\nabla h|^2 + 2|h|^4 - C|h|^3 - C|\mathbf{Rm}|_{\mathbf{N}}(q+1)|\nabla h|) \right. \\ &\quad \left. - C|\mathbf{Rm}|_{\mathbf{N}}|\nabla\psi||h|^{q+1} \right) dv_{\Sigma}.\end{aligned}$$

Using

$$\int_{\Sigma} \psi|h|^q \Delta|h|^2 dv_{\Sigma} \leq \int_{\Sigma} 2|\nabla\psi||h|^{q+1}|\nabla h| dv_{\Sigma} - \int_{\Sigma} 2q\psi|h|^q|\nabla|h||^2 dv_{\Sigma}$$

and Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}&\int_{\Sigma} \psi|\nabla h|^2|h|^q + \int_{\Sigma} \psi|h|^{q+4} dv_{\Sigma} \\ &\leq C \int_{\Sigma} \left(\left(\frac{|\varphi'|^2}{\delta^2\varphi} + \varphi \right) \circ \kappa |h|^{q+2} + (q+1)^2 \psi|h|^q + \frac{1}{\delta} |\varphi' \circ \kappa| |h|^{q+1} \right) dv_{\Sigma},\end{aligned}\tag{2.24}$$

in which we can always choose φ so that $|\varphi'|^2 \leq C|\varphi|$.

Then, by Hölder's inequality we have

$$\left(\int_{\Sigma} \psi|h|^{q+4} dv_{\Sigma} \right)^{1/(q+4)} \leq C_q \delta^{\frac{n}{q+4}-1}.\tag{2.25}$$

In view of Lemma 2.6, the hypersurface is uniformly spacelike and so we have the Sobolev inequality

$$\left(\int_{\Sigma} |\psi|h|^{q+2}|^{\frac{n-1}{n-1}} dv_{\Sigma} \right)^{\frac{n-1}{n}} \leq C' \int_{\Sigma} |\nabla(\psi|h|^{q+2})| dv_{\Sigma}.$$

Combining this with (2.24) and suitably choosing the function φ , we find that for all $i = 1, 2, \dots$

$$\begin{aligned} & \left(\int_{B(p', \frac{\delta}{2} + \frac{\delta}{2^{i+1}})} |h|^{\frac{n(q+2)}{n-1}} dv_\Sigma \right)^{\frac{n-1}{n(q+2)}} \\ & \leq \left(2^i \frac{C'(2+q)^2}{\delta} \right)^{\frac{1}{q+2}} \left(\int_{B(p', \frac{\delta}{2} + \frac{\delta}{2^i})} |h|^{q+2} dv_\Sigma \right)^{\frac{1}{q+2}}. \end{aligned} \quad (2.26)$$

Using the Nash-Moser's iteration technique we deduce that

$$\sup_{B(p', \delta/2)} |h| \leq C_q \delta^{-\frac{n}{q+2}} \left(\int_{B(p', 3\delta/4)} |h|^{q+2} dv_\Sigma \right)^{\frac{1}{q+2}}. \quad (2.27)$$

Finally, choosing $q = 0$ in (2.25) and $q = 2$ in (2.27), we find

$$\sup_{B(p', \delta/2)} |h| \leq \frac{C''}{\delta}.$$

Step 2. Next, by relying on the sup norm estimate that we just established, we can estimate the covariant derivative of h . We need an L^p estimate for the equation (2.22). From Gauss equation we see that the curvature of the hypersurface is bounded by $C'\delta^{-2}$ on the ball $B(p', \delta/2)$. By introducing harmonic coordinates on the (Riemannian) slice Σ , as in [18], we see that the metric coefficients belong to the Hölder space $C^{1,\alpha}$. Since the right-hand side of (2.22) belongs to the Sobolev space $W^{-1,q}$ for any $q \in (1, \infty)$, thanks to the Sobolev regularity property for elliptic operators (in fixed coordinates) we find

$$\left(\frac{1}{\delta^n} \int_{B(p', \delta/4)} |\nabla h|^q dv_\Sigma \right)^{1/q} \leq \frac{C_q}{\delta^2}$$

for some constant $C_q > 0$, which completes the proof of Lemma 2.7. \square

Proof of Lemma 2.8. We need now to estimate the time-derivative of the level set function u . Set $\mathbf{p}'' = (u(\gamma(sr)), \gamma(sr)) \in \Sigma$ and consider the geodesic distance function $\rho = \rho(\mathbf{p}'', \cdot)$ associated with the *induced metric* on the CMC hypersurface Σ . By the Gauss equation, the Ricci curvature is bounded (especially from below) by

$$R_{ij} \geq -\frac{C'}{r^2} g_{ij}.$$

Hence, thanks to the Laplacian comparison theorem, the distance function ρ is a supersolution for the operator $-\Delta + C'/\rho$, that is, in the weak sense

$$\Delta \rho \leq \frac{C'}{\rho}.$$

Let φ be the (non-increasing) cut-off function introduced in the proof of Lemma 2.7. Then, by differentiating with respect to t the equation (2.12) satisfied by the solution u and in view of the bounds on h in Lemma 2.7, we obtain

$$\left(\Delta - |h|^2 - \mathbf{Ric}(\mathbf{N}, \mathbf{N})\right) \left(\nu(\nabla u) \frac{\partial u}{\partial t} + \epsilon \varphi \left(\frac{4\rho}{s^{5/2}r}\right)\right) \geq 1 - \epsilon \frac{C'}{r} \geq 0, \quad (2.28)$$

where we have set $\epsilon := r^2/C'$. Finally, applying the maximum principle (see (2.30) below for the ellipticity property) we conclude that

$$-C' r^2 \leq \frac{\partial u}{\partial t} \leq -\frac{1}{C'} r^2 \quad \text{on } \Omega'_s = B_{sr}(\gamma(sr), s^{5/2}r/4) \subset \{\tau = sr\}.$$

□

Proof of Proposition 2.3 and Theorem 2.2. Step 1. The first variation $\mathcal{LM}(X)$ of the mean curvature along an arbitrary vector field X reads

$$\mathcal{LM}(X) = \Delta \langle X, \mathbf{N} \rangle - (|h|^2 + \mathbf{Ric}(\mathbf{N}, \mathbf{N})) \langle X, \mathbf{N} \rangle - \langle X, \nabla H \rangle, \quad (2.29)$$

where we recall that \mathbf{N} is the unit normal vector field to the hypersurface and h is its second fundamental form. In the case of graphs, the linearization of the mean curvature equation around a *constant* mean curvature hypersurface reads

$$\mathcal{LM}(\varphi) = \Delta(\nu(\nabla u) \varphi) - (|h|^2 + \mathbf{Ric}(\mathbf{N}, \mathbf{N})) \nu(\nabla u) \varphi.$$

Under our assumptions, this operator is uniformly invertible, since

$$|\mathbf{Ric}(\mathbf{N}, \mathbf{N})| \leq C(n)r^{-2}, \quad |h|^2 \geq \frac{H^2}{n} \geq s^{-1/2}r^{-2} \quad (2.30)$$

and we choose s sufficiently small so that the term $|h|^2$ dominates $\mathbf{Ric}(\mathbf{N}, \mathbf{N})$.

More precisely, we have the following important conclusion.

By the implicit function theorem, the linearized mean curvature operator \mathcal{LM} on the space of all spacelike $C^{2,\alpha}$ functions u , with fixed boundary value cr on $\partial\Omega_s$ and with $\alpha \in (0, 1)$, is locally invertible around any smooth hypersurface of constant mean curvature. In consequence, starting from any fixed spacelike hypersurface u with constant mean curvature

$$2\bar{k}(sr, r) \in [\bar{k}(sr, r), \underline{k}(2s^2r, r)]$$

and using the implicit function theorem, we can find a smooth family of spacelike hypersurfaces u^t with constant mean curvature t varying in an ϵ -neighborhood of $2\bar{k}(sr, r)$. From Lemma 2.6 and Schauder's estimate, we then deduce higher-order uniform estimates for u^t . So, the function u^t is smooth and we can take a convergent subsequence of values t converging to the end-points of the interval.

Next, by continuation we may still use the implicit function theorem and extend the smooth family under consideration for all mean curvature parameter values in the interval $[\bar{k}(sr, r), \underline{k}(2s^2r, r)]$. In other words, we conclude that

for any $t \in [\bar{k}(sr, r), \underline{k}(2s^2r, r)]$ we can solve the Dirichlet problem (2.12) of prescribed mean curvature equal to t , and, moreover, the solution u depends smoothly upon the mean curvature t .

Next, differentiating with respect to t the conditions satisfied by the solution u we obtain

$$\begin{aligned} \left(\Delta - (|h|^2 + \mathbf{Ric}(\mathbf{N}, \mathbf{N})) \right) \left(\nu(\nabla u) \frac{\partial u}{\partial t} \right) &= 1 && \text{in } \Omega_s, \\ \frac{\partial u}{\partial t} &= 0 && \text{along } \partial\Omega_s. \end{aligned} \quad (2.31)$$

As already observed, by the maximum principle we have

$$-Cr^2 < \frac{\partial u}{\partial t} < 0.$$

This property shows that the family of CMC hypersurfaces forms a foliation of the region under consideration.

Step 2. In this last part of the construction we choose the geodesic slice over which the CMC foliation should be based. We observe that the foliation constructed in Step 1 need not pass through the given observer $\mathbf{p} = \gamma(cr)$. To cope with this difficulty, we now vary the parameter s in order to ensure that the foliation contains a neighborhood of \mathbf{p} . We proceed as follows.

For any $s \in [c, 2c]$ we use the notation $u^{(s)}$ for the function describing the CMC hypersurface constructed over the reference domain $\Omega_s \subset \{\tau = sr\}$ for the chosen value of the mean curvature

$$2\bar{k}(sr, r) \in [\bar{k}(sr, r), \underline{k}(2s^2r, r)].$$

Now, we emphasize that $u^{(s)}$ depends continuously upon the parameter s . Indeed, in view of (2.29) we can apply the implicit function theorem and we see that the solution depends smoothly upon the parameters arising in the domain of definition and upon the boundary values. Therefore, recalling the result in Lemma 2.6, given any function $u^{(c)}$ we may extend it to a whole family $u^{(s)}$ smoothly for all $s \in [c, 2c]$.

Then, by Lemma 2.4 we have $u^{(s)} \geq \tau(s)r$ for some $\tau(s)$ satisfying $2\bar{k}(sr, r) = \underline{k}(\tau(s)r, r)$, which implies that $\tau(2c) > c$, at least when c is suitably small. Since we have $u^{(c)}(\gamma(cr)) < cr$, by continuity there is some $s_0 \in [c, 2c]$ such that $u^{(s_0)}(\gamma(cr)) = cr$. Therefore, we have constructed a family of CMC hypersurfaces u^t with constant mean curvature

$$t \in [n\bar{k}(s_0r, r), n\underline{k}(2s_0^2r, r)] \quad \text{for some } s_0 \in [c, 2c]$$

over some geodesic slice $\Omega_{s_0} \subset \{\tau = s_0r\}$. Most importantly, the point $\gamma(cr)$ lies in the CMC hypersurface with mean curvature $2\bar{k}(s_0r, r)$.

In addition, by a direct computation we obtain

$$\nabla t = \sqrt{1 - |\nabla u|^2} \left(\frac{\partial u}{\partial t} \right)^{-1} \frac{(-1, \nabla u)}{\sqrt{1 - |\nabla u|^2}}$$

and, in view of Lemma 2.8, the proof is now completed. \square

2.5 Further geometric estimates

For any $p' \in \Sigma_t$ with $\delta r = d(p', \partial\Sigma_t)$ and thanks to our estimate of the second fundamental form in $B(p', \delta/2)$ and Gauss equation, we see that the curvature of the hypersurface is bounded by $C\delta^{-2}r^{-2}$. By choosing harmonic coordinates as in [18] and using the L^p estimates in (2.31), the function $\lambda = -\nu(\nabla u) \frac{\partial u}{\partial t}$ satisfies (for any $q \in [1, \infty)$)

$$\left(\frac{1}{\delta^{n_r n}} \int_{B(p', \delta/4)} |\nabla^2 \lambda|^q dv_\Sigma \right)^{1/q} \leq \frac{C_q}{\delta^2 r^2}. \quad (2.32)$$

In addition, let us investigate the geometry of the boundary $\partial\Sigma$ of the foliation leaves. More precisely, we can estimate its second fundamental form $\Pi_{\partial\Sigma}$, as follows.

Proposition 2.9 (Boundary of the CMC foliation). *The CMC hypersurfaces constructed in the proof of Theorem 2.2 also satisfy the uniform estimate*

$$|\Pi_{\partial\Sigma}| \leq \frac{C(n)}{r}.$$

Proof. Recall that, for any tangent vector fields X, Y along $\partial\Sigma$, the scalar $\Pi_{\partial\Sigma}(X, Y)$ is defined as $\mathbf{g}(\nabla_X Y, N_{\partial\Sigma})$, where $N_{\partial\Sigma}$ is the normal vector field of $\partial\Sigma$ in the hypersurface Σ .

On the other hand, since $\partial\Sigma = \partial\Omega$ is obtained by the intersection of two level surfaces $\mathcal{H} := \{\tau = \text{const.}\}$ and $\tilde{S} := \{\tilde{\mathbf{d}} = \text{const.}\}$, $\partial\Sigma$ may be regarded as a hypersurface of codimension 1 in either \mathcal{H} or \tilde{S} . The second fundamental form $\Pi_{\partial\Sigma}^{\mathcal{H}}$ of $\partial\Sigma$ in \mathcal{H} reads

$$\Pi_{\partial\Sigma}^{\mathcal{H}}(X, Y) = \mathbf{g}(\nabla_X Y, N_{\mathcal{H}}) = \frac{1}{|\nabla \tilde{\mathbf{d}}|} \nabla_{\mathcal{H}}^2 \tilde{\mathbf{d}}(X, Y), \quad (2.33)$$

where $\tilde{\mathbf{d}}$ is regarded as a function on \mathcal{H} and $\nabla_{\mathcal{H}}$ denotes the covariant derivative associated with the induced metric on \mathcal{H} , while $N_{\mathcal{H}}$ is the normal vector of $\partial\Sigma$ in \mathcal{H} . Similarly, we have

$$\Pi_{\partial\Sigma}^{\tilde{S}}(X, Y) = \mathbf{g}(\nabla_X Y, N_{\tilde{S}}) = \frac{1}{|\nabla \tau|} \nabla_{\tilde{S}}^2 \tau(X, Y). \quad (2.34)$$

By a direct computation we find that

$$\nabla_{\mathcal{H}}^2 \tilde{\mathbf{d}}(X, Y) = \nabla^2 \tilde{\mathbf{d}}(X, Y) - \mathbf{g}\left(\nabla \tilde{\mathbf{d}}, \frac{\partial}{\partial \tau}\right) A(X, Y).$$

So, we have

$$\frac{1}{c^2 C' r} \mathbf{g}(X, Y) \leq \nabla_{\mathcal{H}}^2 \tilde{\mathbf{d}}(X, Y) \leq \frac{C'}{c^2 r} \mathbf{g}(X, Y) \quad (2.35)$$

and similar inequalities for $\nabla_{\tilde{\mathbf{g}}}^2 \tau(X, Y)$. On the other hand, by the triangle comparison theorem for the Riemannian metric $\tilde{\mathbf{g}}$, there exists a constant C'' (depending on c) such that

$$\langle N_{\mathcal{H}}, N_{\tilde{\mathcal{S}}} \rangle_{\mathbf{g}} \geq 1 + \frac{1}{C''}.$$

This implies the existence of two functions a, b that are bounded by some uniform constant C and satisfy $N_{\partial\Sigma} = a N_{\mathcal{H}} + b N_{\tilde{\mathcal{S}}}$. This completes the proof of Proposition 2.9. \square

3 Local coordinates ensuring the optimal regularity

3.1 Main statements for this section

From now on we assume that the manifold M satisfies the Einstein vacuum equations. We will now prove:

Theorem 3.1 (Local coordinates ensuring the optimal regularity). *Given $\epsilon > 0$ and $q \in [1, \infty)$ there exists a constant $c(n, \epsilon)$ satisfying $\lim_{\epsilon \rightarrow 0} c(n, \epsilon, q) = 0$ such that the following property holds. Let $(\mathbf{M}, \mathbf{g}, \mathbf{p}, \mathbf{T}_{\mathbf{p}})$ be a pointed Lorentzian manifold satisfying the curvature and injectivity radius bounds (1.2) at some scale $r > 0$, together with Einstein field equation $\mathbf{Ric} = 0$. Then, there exists a local coordinate chart $\mathbf{x} = (x^\alpha)$ satisfying $x^\alpha(p) = 0$, defined for all $|\mathbf{x}|^2 := (x^0)^2 + (x^1)^2 + \dots + (x^n)^2 < r_1^2$ with $r_1 := c_1(n, \epsilon)r$, and such that*

$$\begin{aligned} \sup_{|\mathbf{x}| \leq r_1} \left(|\mathbf{g}_{\alpha\beta} - \boldsymbol{\eta}_{\alpha\beta}| + r |\partial \mathbf{g}_{\alpha\beta}| \right) &\leq \epsilon, \\ \frac{1}{r^{n+1-2q}} \int_{|\mathbf{x}| \leq r_1} |\partial^2 \mathbf{g}_{\alpha\beta}|^q dx &\leq C(\epsilon, q), \end{aligned}$$

where $\boldsymbol{\eta}_{\alpha\beta}$ is the Minkowski metric in these local coordinates.

The proof of this theorem will be given at the end of this section, after establishing several preliminary results of independent interest.

The main observation made in the present section is that the time function t associated with the CMC foliation constructed in the previous section admits well-controlled covariant derivatives up to third-order; cf. Proposition 3.4 below. Consequently, by following our arguments given earlier in [12, Proposition 9.1] we are led to the desired optimal regularity result in Theorem 3.1. Recall that in the earlier work [12] we relied on a coordinate system in which the metric coefficients \mathbf{g} had well-controlled first-order derivatives only; indeed, the time function in [12] was simply taken to be the geodesic distance function, which is only twice differentiable under the curvature and injectivity radius bounds. In contrast, in the present paper we have constructed a more regular time function

t (the mean curvature of the spacelike slices) which turns out to have third-order regularity.

Consider the constant mean curvature foliation Σ_t given by Theorem 2.2, and observe that the time function t (together with the Lorentzian metric \mathbf{g}) provides us with a natural flow Φ^m associated with the vector field

$$\frac{\nabla t}{-\mathbf{g}(\nabla t, \nabla t)},$$

such that the parameter m may differ from t by a constant. Denote by τ the normal time function introduced in Section 2 and recall that, in the interior of the slice,

$$\frac{\partial u}{\partial \tau} < 0, \quad \left\langle \frac{\nabla t}{-\mathbf{g}(\nabla t, \nabla t)}, \nabla \tau \right\rangle < 0,$$

while the vector field ∇t vanishes identically on the boundary. By starting from any arbitrary point $\mathbf{p}_t \in \Sigma_t$, the integral curve $\Phi^m(\mathbf{p}_t)$ intersects each CMC slice exactly once and the flow Φ^m preserves the CMC foliation.

Let $y = (y^i)$ be spatial coordinate chosen arbitrary on a given slice Σ_{t_0} , and let us use the flow Φ^m , in order to transport these coordinates to any other slice Σ_t . Together with the mean curvature function t , these spatial coordinates provide us with spacetime coordinates $\mathbf{y} = (t, y^i)$. The metric \mathbf{g} then takes the form

$$\mathbf{g} = -\lambda(t, y)^2 dt^2 + g_{ij}(t, y) dy^i dy^j \quad (3.1)$$

and satisfies the ADM equations

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2\lambda k_{ij}, \\ \frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \lambda - \lambda g^{pq} k_{ip} k_{qj} + \lambda \mathbf{R}_{iNjN}, \end{aligned} \quad (3.2)$$

where $\lambda > 0$ is the lapse function and k_{ij} is the second fundamental form of Σ_t expressed in the coordinates under consideration in this section

From now on, without loss of generality we set $r = 1$. The central technical estimate of the present section concerns the lapse function and is stated in the following lemma.

Lemma 3.2 (Second-order estimates for the lapse function). *Under the assumption of Theorem 3.1 and with the above notation, the function λ satisfies*

$$\int_{\Sigma} \left(|\nabla^2 \lambda|^2 + \left| \frac{\partial \lambda}{\partial t} \right|^2 + \left| \frac{\partial^2 \lambda}{\partial t^2} \right|^2 \right) dv_{\Sigma} \leq C(n).$$

For any $\delta > 0$ we set $\Sigma^{\delta} := \{x \in \Sigma / d(x, \partial \Sigma) \geq \delta\}$ and, away from the boundary of the slices, we can improve Lemma 3.2, as follows.

Lemma 3.3 (Higher-order interior estimates for the lapse function). *Under the assumption of Theorem 3.1 and with the above notation, for any $\delta > 0$ and $q \in [1, \infty)$ one has*

$$\begin{aligned} \sup_{\Sigma^\delta} \left(|k| + |\nabla\lambda| + |\nabla^2\lambda| + \left| \frac{\partial\lambda}{\partial t} \right| + \left| \nabla \frac{\partial\lambda}{\partial t} \right| + \left| \frac{\partial^2\lambda}{\partial t^2} \right| \right) &\leq C(n, \delta), \\ \int_{\Sigma^\delta} \left(|\nabla k|^q + |\nabla^3\lambda|^q + \left| \nabla^2 \frac{\partial\lambda}{\partial t} \right|^q + \left| \nabla \frac{\partial^2\lambda}{\partial t^2} \right|^q \right) dv_\Sigma &\leq C(n, q, \delta). \end{aligned}$$

Finally, based on Lemma 3.3 we prove that the time function t admits well-controlled third-order derivatives. Here, we use the covariant derivative $\widehat{\nabla}$ associated with the reference Riemannian metric in the coordinates under consideration, that is, $\widehat{\mathbf{g}} := \lambda(t, y)^2 dt^2 + g_{ij}(t, y) dy^i dy^j$.

Proposition 3.4 (Third-order estimates for the time-function). *Under the assumption of Theorem 3.1 and with the above notation, for all $q \in [1, \infty)$ one has*

$$\begin{aligned} \sup_{\Sigma^\delta} \left(|\nabla^2 t| + |\widehat{\nabla}^2 t| + |\widehat{\nabla}^2 \lambda| \right) &\leq C(n, \delta), \\ \int_{\Sigma^\delta} \left(|\nabla^3 t|^q + |\widehat{\nabla}^3 t|^q \right) dv_\Sigma &\leq C(n, q, \delta), \end{aligned}$$

where all the norms are computed with the reference metric $\widehat{\mathbf{g}}$.

3.2 Derivation of the key estimates on the lapse function

This section is devoted to the proof of Lemma 3.2.

Step 1. Zero-order estimates in time. By integrating Weitzenböck identity (2.17) and observing that $|\Delta u| + |\nabla u| \leq C$ on a CMC slice Σ , we obtain

$$\int_{\Sigma} |\nabla^2 u|^2 dv_\Sigma \leq C + \int_{\partial\Sigma} (\nabla^2 u)(n_{\partial\Sigma}, n_{\partial\Sigma}), \quad (3.3)$$

where $n_{\partial\Sigma} := \frac{\nabla u}{|\nabla u|}$ is the unit normal vector field of the boundary $\partial\Sigma$ on Σ . Since

$$\begin{aligned} (\nabla^2 u)(n_{\partial\Sigma}, n_{\partial\Sigma}) &= \Delta u - \text{tr}_g(\nabla^2 u) \\ &= \Delta u - |\nabla u| \text{tr}_g(\mathbb{H}_{\partial\Sigma}) \quad \text{along } \Sigma, \end{aligned} \quad (3.4)$$

we conclude with the boundary estimate in Proposition 2.9 that the second fundamental form k is uniformly bounded in the L^2 norm

$$\int_{\Sigma} |k|^2 dv_\Sigma \leq C. \quad (3.5)$$

Observe that this estimate covers the whole slice up to its boundary (in contrast with the interior sup-norm estimate given by Lemma 2.7).

We use the notation introduced in Section 2. Recall that the Riemannian distance function $\tilde{\mathbf{d}} = \tilde{\mathbf{d}}(\gamma(s+s^2), \cdot)$ takes the constant value $c_0 = s^2 + s^3$ on the

boundary $\partial\Sigma$, and that $c_0 - \tilde{\mathbf{d}}$ is proportional to the intrinsic distance function to the boundary of the slice, i.e.

$$\frac{1}{C'} d(\cdot, \partial\Sigma) \leq c_0 - \tilde{\mathbf{d}} \leq C' d(\cdot, \partial\Sigma). \quad (3.6)$$

Moreover, by the Laplacian comparison lemma for distance functions and relying on our curvature assumption we have also

$$\frac{C'}{c^2} \geq \Delta \tilde{\mathbf{d}} \geq \frac{1}{c^2 C'} \quad \text{on the hypersurface } \Sigma. \quad (3.7)$$

Now, taking the trace of the second identity in (3.2) and recalling that $\Sigma = \Sigma_t$ has constant mean curvature we obtain the *elliptic equation satisfied by the lapse function*

$$\Delta \lambda = -1 + (|k|^2 + \mathbf{Ric}(N, N)) \lambda. \quad (3.8)$$

In view of (3.7) and recalling that $\lambda > 0$ we deduce

$$\Delta(\lambda + c^2 C' \tilde{\mathbf{d}}) \geq 0$$

so that, thanks to the maximum principle,

$$0 \leq \lambda \leq c^2 C' (c_0 - \tilde{\mathbf{d}}). \quad (3.9)$$

In particular, in view of (3.6) this implies the desired gradient estimate *along the boundary* at least

$$\sup_{\partial\Sigma} |\nabla \lambda| \leq C'. \quad (3.10)$$

Next, by (3.5) and (3.9), the right-hand side of (3.8) belongs to L^2 , which yields us a bound for the Laplacian of the lapse function

$$\int_{\Sigma} |\Delta \lambda|^2 dv_{\Sigma} \leq C'. \quad (3.11)$$

On the other hand, by multiplying (3.8) by the function λ and integrating by parts, we find the L^2 gradient estimate

$$\int_{\Sigma} |\nabla \lambda|^2 dv_{\Sigma} \leq C'. \quad (3.12)$$

Finally, by observing that

$$\int_{\Sigma} |\Delta \lambda|^2 \leq C', \quad \Delta \lambda|_{\partial\Sigma} = -1, \quad \mathbf{Ric}(\nabla \lambda, \nabla \lambda) \geq -C' |\nabla \lambda|^2,$$

integrating Bochner formula

$$\Delta |\nabla \lambda|^2 = 2|\nabla^2 \lambda|^2 + 2\langle \nabla \lambda, \nabla \Delta \lambda \rangle + 2\mathbf{Ric}(\nabla \lambda, \nabla \lambda), \quad (3.13)$$

and then using (3.10) together with a similar calculation as in (3.4), we arrive at an estimate of *all second-order spatial derivatives* of λ :

$$\int_{\Sigma} |\nabla^2 \lambda|^2 dv_{\Sigma} \leq C', \quad (3.14)$$

which is one of the estimates stated in Lemma 3.2.

Furthermore, we can also control certain nonlinear functions. Multiplying the identity

$$\Delta \lambda^2 = 2(|k|^2 + \mathbf{Ric}(N, N)) \lambda^2 - 2\lambda + 2|\nabla \lambda|^2, \quad (3.15)$$

by $|\nabla \lambda|^2 \lambda^{-(1-\epsilon)}$ on one hand and by $|k|^2$ on the other hand, for all $\epsilon \in (0, 1)$ we find

$$\int_{\Sigma} \lambda^{-1+\epsilon} |\nabla \lambda|^4 dv_{\Sigma} \leq C_{\epsilon}, \quad \int_{\Sigma} |k|^2 |\nabla \lambda|^2 dv_{\Sigma} \leq C. \quad (3.16)$$

Thus, by multiplying (3.13) by $|\nabla \lambda|^2$ and then using (3.16), we obtain

$$\int_{\Sigma} \left(|\nabla^2 \lambda|^2 |\nabla \lambda|^2 + |k_{ij} \nabla_i \lambda|^2 |\nabla \lambda|^2 \right) dv_{\Sigma} \leq C, \quad (3.17)$$

where we used the Gauss equation for the expression of Ric. In particular, this provides us with a control of

$$\int_{\Sigma} |\nabla |\nabla \lambda|^4| dv_{\Sigma} \leq C.$$

Moreover, since the boundary values of $|\nabla \lambda|$ are uniformly bounded, by Sobolev inequality we also have

$$\int_{\Sigma} |\nabla \lambda|^{\frac{4n}{n-1}} dv_{\Sigma} \leq C.$$

Step 2. First-order estimates in time. This is the first instance where we use our assumption that the manifold is Ricci-flat. By differentiating (3.8) in time, we obtain that

$$\begin{aligned} \Delta \left(\frac{\partial \lambda}{\partial t} \right) &= \left\langle \frac{\partial g_{ij}}{\partial t}, \nabla_i \nabla_j \lambda \right\rangle + (|k|^2 + \mathbf{Ric}(N, N)) \frac{\partial \lambda}{\partial t} + 2\lambda \left\langle \frac{\partial k_{ij}}{\partial t}, k_{ij} \right\rangle \\ &\quad - 2 \frac{\partial g_{ij}}{\partial t} k_{kl} k_{rs} g^{sl} g^{ik} g^{jr} \lambda + \mathbf{Ric}(\nabla_{\frac{\partial}{\partial t}} N, N) \lambda + (\nabla_{\frac{\partial}{\partial t}} \mathbf{Ric})(N, N) \lambda \\ &\quad + t |\nabla \lambda|^2 - 2g^{il} g^{kj} k_{ij} \nabla_l \lambda \nabla_k \lambda + 2\lambda g^{kl} \nabla_l \lambda \mathbf{R}_{k0}, \end{aligned} \quad (3.18)$$

where we used

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma_{ij}^k &= -g^{kl} \left(\nabla_i \lambda k_{lj} + \nabla_j \lambda k_{li} - \nabla_l \lambda k_{ij} + \lambda (\nabla_i k_{lj} + \nabla_j k_{li} - \nabla_l k_{ij}) \right), \\ -g^{ij} \frac{\partial}{\partial t} \Gamma_{ij}^k &= 2\lambda g^{kl} g^{ij} \nabla_i k_{lj} + 2g^{ij} g^{kl} k_{li} \nabla_j \lambda - t g^{kl} \nabla_l \lambda \\ &= -2\lambda g^{kl} \mathbf{R}_{l0} + 2g^{ij} g^{kl} k_{li} \nabla_j \lambda - t g^{kl} \nabla_l \lambda, \end{aligned} \quad (3.19)$$

as well as Codazzi equation $\nabla_i k_{lj} - \nabla_l k_{ij} = \mathbf{R}_{il0j}$. Plugging in the vacuum Einstein equation $\mathbf{Ric} = 0$, we arrive at the equation satisfied by the derivative of the lapse function

$$\begin{aligned} \Delta \left(\frac{\partial \lambda}{\partial t} \right) - |k|^2 \frac{\partial \lambda}{\partial t} &= 2 \langle \mathbf{R}_{iNjN}, k_{ij} \rangle \lambda^2 - 4 \lambda \langle k_{ij}, \nabla_i \nabla_j \lambda \rangle \\ &\quad + 2 k_{ij} k_{kl} k_{rs} g^{sl} g^{ik} g^{jr} \lambda^2 + t |\nabla \lambda|^2 - 2 g^{il} g^{kj} k_{ij} \nabla_l \lambda \nabla_k \lambda \\ &= -Q + \nabla_l V^l, \end{aligned} \tag{3.20}$$

in which, thanks to our estimates in Step 1,

$$\int_{\Sigma} \left(|Q|^2 + |V|^4 \right) dv_{\Sigma} \leq \bar{C}, \quad V^l = -2g^{il} g^{kj} k_{ij} \lambda \nabla_k \lambda.$$

By multiplying (3.20) by $\left(\frac{\partial \lambda}{\partial t} \right)^{1+\epsilon}$ on both sides, using Sobolev inequality for the function $\left(\frac{\partial \lambda}{\partial t} \right)^{1+\epsilon/2}$, recalling $\frac{\partial \lambda}{\partial t} |_{\partial \Sigma} = 0$, and finally integrating by parts, we conclude that

$$\begin{aligned} C'' \left(\int_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right|^{\frac{n}{n-2}(2+\epsilon)} dv_{\Sigma} \right)^{(n-2)/n} &\leq \int_{\Sigma} \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 \left| \frac{\partial \lambda}{\partial t} \right|^{\epsilon} dv_{\Sigma} + \int_{\Sigma} |k|^2 \left| \frac{\partial \lambda}{\partial t} \right|^{2+\epsilon} dv_{\Sigma} \\ &\leq \left(\int_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right|^{2(1+\epsilon)} \right)^{1/2} \left(\int_{\Sigma} |Q|^2 dv_{\Sigma} \right)^{1/2} \\ &\quad + \left(\int_{\Sigma} \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 \left| \frac{\partial \lambda}{\partial t} \right|^{\epsilon} dv_{\Sigma} \right)^{1/2} \left(\int_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right|^{2\epsilon} dv_{\Sigma} \right)^{1/4} \left(\int_{\Sigma} |V|^4 dv_{\Sigma} \right)^{1/4}. \end{aligned} \tag{3.21}$$

So, we should take $\epsilon \leq \frac{4}{n-4}$ if $n \geq 5$, but can take arbitrary $\epsilon > 0$ if $n \leq 4$. We conclude that

$$\begin{aligned} \int_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right|^{\frac{2n}{n-4}} dv_{\Sigma} &\leq C \quad \text{if } n \geq 5, \\ \int_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right|^q dv_{\Sigma} &\leq C_q \quad \text{if } n = 4, \\ \sup_{\Sigma} \left| \frac{\partial \lambda}{\partial t} \right| &\leq C \quad \text{if } n \leq 3. \end{aligned} \tag{3.22}$$

In the case $n \leq 3$ above, we used once more Nash-Moser's iteration technique.

Next, multiplying Bochner formula

$$\Delta \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 = \left| \nabla^2 \frac{\partial \lambda}{\partial t} \right|^2 + 2 \langle \nabla \frac{\partial \lambda}{\partial t}, \nabla \Delta \frac{\partial \lambda}{\partial t} \rangle + 2 Ric \left(\nabla \frac{\partial \lambda}{\partial t}, \nabla \frac{\partial \lambda}{\partial t} \right)$$

by $(c_0 - \tilde{\mathbf{d}})^2$, using

$$\int_{\Sigma} \left(\left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 + (c_0 - \tilde{\mathbf{d}})^2 \left| \Delta \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} \leq C,$$

and then integrating by parts, we find

$$\int_{\Sigma} (c_0 - \tilde{\mathbf{d}})^2 \left| \nabla^2 \left(\frac{\partial \lambda}{\partial t} \right) \right|^2 dv_{\Sigma} \leq C. \quad (3.23)$$

Multiplying (3.15) by $|\nabla \frac{\partial \lambda}{\partial t}|^2$ and integrating by parts, we have

$$\begin{aligned} \int_{\Sigma} 2|\nabla \lambda|^2 \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 dv_{\Sigma} &\leq C + \int_{\Sigma} 4\lambda \left| \nabla^2 \frac{\partial \lambda}{\partial t} \right| \left| \nabla \frac{\partial \lambda}{\partial t} \right| |\nabla \lambda| dv_{\Sigma} \\ &\leq C + \int_{\Sigma} |\nabla \lambda|^2 \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 dv_{\Sigma} + 4 \int_{\Sigma} \lambda^2 \left| \nabla^2 \frac{\partial \lambda}{\partial t} \right|^2 dv_{\Sigma} \end{aligned} \quad (3.24)$$

and, after using (3.9) and (3.23),

$$\int_{\Sigma} |\nabla \lambda|^2 \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 dv_{\Sigma} \leq C. \quad (3.25)$$

Now, multiplying Bochner formula (3.13) by $|\frac{\partial \lambda}{\partial t}|^2$ and then integrating by parts, we obtain

$$\begin{aligned} &\int_{\Sigma} \left(2|\nabla^2 \lambda|^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 + 2 Ric(\nabla \lambda, \nabla \lambda) \left| \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} \\ &\leq C' \int_{\Sigma} \left(|\nabla \lambda|^2 \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 + |\nabla^2 \lambda|^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} + C' \int_{\Sigma} (\Delta \lambda)^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 dv_{\Sigma} \end{aligned} \quad (3.26)$$

and, thanks to (3.25) and (3.21),

$$\int_{\Sigma} \left(|\nabla^2 \lambda|^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 + |k_{ij} \nabla_j \lambda|^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} \leq C.$$

Step 3. Second-order estimates in time. We now turn to the most involved estimate concerning the function $\frac{\partial^2 \lambda}{\partial t^2}$ which, we claim, satisfies an equation of the form

$$\Delta \left(\frac{\partial^2 \lambda}{\partial t^2} \right) - |k|^2 \frac{\partial^2 \lambda}{\partial t^2} = \nabla_i V^i + f_1 + f_2 + f_3, \quad (3.27)$$

where

$$\int_{\Sigma} |V^i|^2 dv \leq C$$

and $f_1 \in L^1$ has the form $f_1 = k * f'_1$ (that a linear combination of such products) with

$$\int_{\Sigma} |f'_1|^2 \leq C, \quad \int_{\Sigma} |f_2|^{\frac{2n}{n+2}} \leq C,$$

and f_3 is bounded pointwise by $C' |\nabla^2 \lambda|^2$. This is one of the key observations in the present paper.

To establish (3.27) we differentiate (3.20) in time. It is not hard to show that all terms arising in the right-hand side of the equation, except those of form $\nabla_i V^i$ with $\int_\Sigma |V|^2 \leq \bar{C}$, belongs to L^1 uniformly. We emphasize that we may arrange the other terms by introducing new terms V^i 's so that they all have the desired form in (3.27).

Let us now deduce from (3.27) that

$$\int_\Sigma \left(\left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right|^2 + \left| \frac{\partial^2 \lambda}{\partial t^2} \right|^2 \right) dv_\Sigma \leq C''. \quad (3.28)$$

Namely, by multiplying both sides of (3.27) by $\frac{\partial^2 \lambda}{\partial t^2}$, then integrating by parts, and using $\frac{\partial^2 \lambda}{\partial t^2} |_{\partial \Sigma} = 0$, we obtain

$$\begin{aligned} & \int_\Sigma \left(\left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right|^2 + |k|^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right|^2 \right) dv_\Sigma \\ & \leq \int_\Sigma \left(|V| \left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right| + \left| \frac{\partial^2 \lambda}{\partial t^2} \right| |k| |f_1| \right) dv_\Sigma \\ & \quad + \left(\int_\Sigma \left| \frac{\partial^2 \lambda}{\partial t^2} \right|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \left(\int_\Sigma |f_2|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} + C \int_\Sigma \left| \frac{\partial^2 \lambda}{\partial t^2} \right| |\nabla^2 \lambda|^2 dv_\Sigma. \end{aligned} \quad (3.29)$$

Since $Ric \geq -Cg$, by multiplying Bochner formula (3.13) by $\left| \frac{\partial^2 \lambda}{\partial t^2} \right|$ we obtain

$$\begin{aligned} 2 \int_\Sigma |\nabla^2 \lambda|^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| dv_\Sigma & \leq \int_\Sigma \left(|\nabla |\nabla \lambda|^2| \left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right| + 2(\Delta \lambda)^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| \right) dv_\Sigma \\ & \quad + \int_\Sigma \left(|\nabla \lambda \Delta \lambda| \left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right| + C |\nabla \lambda|^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| \right) dv_\Sigma. \end{aligned} \quad (3.30)$$

By Cauchy-Schwartz inequality and Sobolev inequality, we then have

$$\begin{aligned} & \int_\Sigma \left(\left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right|^2 + \left| \frac{\partial^2 \lambda}{\partial t^2} \right|^2 \right) dv_\Sigma \\ & \leq C \int_\Sigma |f_1|^2 dv_\Sigma + C \left(\int_\Sigma |f_2|^{\frac{2n}{n+2}} dv_\Sigma \right)^{\frac{n+2}{n}} + C \int_\Sigma |\nabla^2 \lambda|^2 |\nabla \lambda|^2 dv_\Sigma \\ & \quad + \int_\Sigma \left(\frac{(\Delta \lambda)^4}{|k|^2} + |\nabla \lambda \Delta \lambda|^2 + |\nabla \lambda|^4 \right) dv_\Sigma. \end{aligned} \quad (3.31)$$

Hence, by combining with (3.5), (3.17), (3.8), (3.9), and (3.16) we obtain

$$\int_\Sigma \left(|\nabla^2 \lambda|^2 |\nabla \lambda|^2 + \frac{(\Delta \lambda)^4}{|k|^2} + |\nabla \lambda \Delta \lambda|^2 + |\nabla \lambda|^4 \right) dv_\Sigma \leq C,$$

which gives the desired estimate (3.28).

In summary, by combining the estimates already established in Lemma 2.7 and in this proof, we thus have

$$\begin{aligned}
& \sup_{\Sigma} \lambda |k| + \int_{\Sigma} |k|^2 \left(\left| \frac{\partial \lambda}{\partial t} \right|^2 + |\nabla \lambda|^2 + 1 \right) dv_{\Sigma} \\
& + \int_{\Sigma} \left(|\nabla^2 \lambda|^2 (|\nabla \lambda|^2 + \left| \frac{\partial \lambda}{\partial t} \right|^2 + 1) + \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} \\
& + \int_{\Sigma} \left(\lambda^2 \left| \nabla^2 \frac{\partial \lambda}{\partial t} \right|^2 + |\nabla \lambda|^4 + |\nabla \lambda|^2 \left| \nabla \frac{\partial \lambda}{\partial t} \right|^2 \right) dv_{\Sigma} \\
& + \int_{\Sigma} \left(|k_{ij} \nabla_j \lambda|^2 \left| \frac{\partial \lambda}{\partial t} \right|^2 + \left| \frac{\partial k_{ij}}{\partial t} \right|^2 \right) dv_{\Sigma} \leq C.
\end{aligned} \tag{3.32}$$

In the rest of this proof, we use the notation $A \approx B$ when $A - B$ is controled by the left-hand side of (3.32).

We can compute

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\Delta \frac{\partial \lambda}{\partial t} - |k|^2 \frac{\partial \lambda}{\partial t} \right) - \left(\Delta \frac{\partial^2 \lambda}{\partial t^2} - |k|^2 \frac{\partial^2 \lambda}{\partial t^2} \right) \\
& = \nabla_i \left(2 \lambda k_{ij} \nabla_j \frac{\partial \lambda}{\partial t} \right) - t \langle \nabla \lambda, \nabla \frac{\partial \lambda}{\partial t} \rangle - 4 \lambda k_{ij} k_{jk} k_{ki} \frac{\partial \lambda}{\partial t} - 2 \frac{\partial k_{ij}}{\partial t} k_{ij} \frac{\partial \lambda}{\partial t} \\
& \approx 2 \langle \nabla_i \nabla_j \lambda, k_{ij} \rangle \frac{\partial \lambda}{\partial t} \approx 0
\end{aligned} \tag{3.33}$$

thanks to (3.32), and

$$\begin{aligned}
\frac{\partial}{\partial t} \left(2 \langle \mathbf{R}_{i\mathbf{N}j\mathbf{N}}, k_{ij} \rangle \lambda^2 \right) & = 4 \langle \mathbf{R}_{i\mathbf{N}j\mathbf{N}}, k_{ij} \rangle \lambda \frac{\partial \lambda}{\partial t} + 8 \mathbf{R}_{i\mathbf{N}q\mathbf{N}} k_{pq} k_{ip} \lambda^3 \\
& + 2 \langle \mathbf{R}_{i\mathbf{N}j\mathbf{N}}, \frac{\partial k_{ij}}{\partial t} \rangle \lambda^2 + 2 \langle \frac{\partial \mathbf{R}_{i\mathbf{N}j\mathbf{N}}}{\partial t}, k_{ij} \rangle \lambda^2 \\
& \approx 2 \langle \frac{\partial \mathbf{R}_{i\mathbf{N}j\mathbf{N}}}{\partial t}, k_{ij} \rangle \lambda^2.
\end{aligned} \tag{3.34}$$

We can also compute the following:

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(-4 \lambda \langle k_{ij}, \nabla_i \nabla_j \lambda \rangle \right) \\
& = -4 \langle k_{ij}, \nabla_i \nabla_j \lambda \rangle \frac{\partial \lambda}{\partial t} - 4 \lambda \left\langle \frac{\partial k_{ij}}{\partial t}, \nabla_i \nabla_j \lambda \right\rangle - 4 \lambda \langle k_{ij}, \nabla_i \nabla_j \frac{\partial \lambda}{\partial t} \rangle \\
& \quad - 8 \lambda k_{ij} k_{jk} \nabla_i \lambda \nabla_k \lambda + 4 \lambda |k_{ij}|^2 |\nabla \lambda|^2 + 4 \lambda^2 k_{ij} \nabla_i k_{kj} \nabla_k \lambda \\
& \approx \nabla_i (4 \lambda^2 k_{ij} k_{kj} \nabla_k \lambda) - 8 \lambda k_{ij} k_{kj} \nabla_i \lambda \nabla_k \lambda - 4 \lambda^2 k_{ij} k_{kj} \nabla_i \nabla_k \lambda,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(2 k_{ij} k_{kl} k_{rs} g^{sl} g^{ik} g^{jr} \lambda^2 \right) \\
& = 4 k_{ij} k_{jk} k_{ki} \lambda \frac{\partial \lambda}{\partial t} + 12 \lambda^3 k_{ij} k_{jk} k_{ks} k_{si} + 6 \lambda^2 \frac{\partial k_{ij}}{\partial t} k_{js} k_{si} \approx 0,
\end{aligned} \tag{3.36}$$

$$\frac{\partial}{\partial t} \left(t |\nabla \lambda|^2 \right) = |\nabla \lambda|^2 + t 2\lambda k_{ij} \nabla_i \lambda \nabla_j \lambda + 2 \nabla \left(t \lambda \nabla \frac{\partial \lambda}{\partial t} \right) - 2t \lambda \Delta \frac{\partial \lambda}{\partial t} \approx 0, \quad (3.37)$$

and finally

$$\begin{aligned} & \frac{\partial}{\partial t} \left(-2 g^{il} g^{kj} k_{ij} \nabla_l \lambda \nabla_k \lambda \right) \\ &= -8 \lambda k_{ij} k_{jk} \nabla_i \lambda \nabla_k \lambda - 2 \frac{\partial k_{ij}}{\partial t} \nabla_i \lambda \nabla_j \lambda - 4 k_{ij} \nabla_i \frac{\partial \lambda}{\partial t} \nabla_j \lambda \\ &\approx \nabla_i (\lambda \nabla_i |\nabla \lambda|^2) - \lambda \Delta |\nabla \lambda|^2 \approx 0. \end{aligned} \quad (3.38)$$

To deal with the curvature term in (3.33) we need some recall property of the curvature. We have

$$\frac{\partial}{\partial t} \mathbf{R}_{ikjl} = \nabla_0 \mathbf{R}_{ikjl} + \Gamma_{0,i}^\alpha(\mathbf{g}) \mathbf{R}_{\alpha kjl} + \Gamma_{0,k}^\alpha(\mathbf{g}) \mathbf{R}_{i\alpha jl} + \Gamma_{0,j}^\alpha(\mathbf{g}) \mathbf{R}_{ik\alpha l} + \Gamma_{0,l}^\alpha(\mathbf{g}) \mathbf{R}_{ikj\alpha}$$

with

$$\begin{aligned} \Gamma_{00}^0(\mathbf{g}) &= \frac{1}{\lambda} \frac{\partial \lambda}{\partial t}, & \Gamma_{0i}^0(\mathbf{g}) &= \frac{1}{\lambda} \frac{\partial \lambda}{\partial y^i}, & \Gamma_{ij}^0(\mathbf{g}) &= \frac{1}{2\lambda^2} \frac{\partial g_{ij}}{\partial t}, \\ \Gamma_{00}^k(\mathbf{g}) &= -\lambda g^{kl} \frac{\partial \lambda}{\partial y^l}, & \Gamma_{i0}^k(\mathbf{g}) &= \frac{1}{2} g^{kl} \frac{\partial g_{li}}{\partial t}, & \Gamma_{ij}^k(\mathbf{g}) &= \Gamma_{ij}^k, \end{aligned}$$

$$\nabla_j \mathbf{R}_{ik0l} = \nabla_j \mathbf{R}_{ik0l} - \Gamma_{ji}^0(\mathbf{g}) \mathbf{R}_{0k0l} - \Gamma_{jk}^0(\mathbf{g}) \mathbf{R}_{i00l} - \Gamma_{j0}^\alpha(\mathbf{g}) \mathbf{R}_{ik\alpha l}.$$

Recall also the second Bianchi identity,

$$\nabla_0 \mathbf{R}_{ikjl} = -\nabla_j \mathbf{R}_{ik0l} - \nabla_l \mathbf{R}_{ikj0}.$$

We obtain

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{R}_{iNjN} &= -\left(\frac{\partial}{\partial t} \mathbf{R}_{ikjl} \right) g^{kl} + \lambda g^{-2} * \mathbf{Rm} * k \\ &= \lambda g^{-1} \nabla(\mathbf{R}_{N***}) + \lambda g^{-2} * k * \mathbf{Rm} + g^{-2} * \nabla \lambda * k * \mathbf{R}_{N***} \\ &\quad + \lambda \mathbf{R}_{N*N*} * k * g^{-2}. \end{aligned}$$

So, we have

$$\frac{\partial}{\partial t} \left(\langle \mathbf{R}_{iNjN}, k_{ij} \rangle \lambda^2 \right) \approx 0,$$

which completes the proof of Lemma 3.2.

3.3 Proofs of the main statements

Proof of Lemma 3.3. Since the second fundamental form is bounded in each slice Σ^δ , then according to Gauss equation the intrinsic curvature of Σ^δ is also uniformly bounded by $C\delta^{-2}$. Hence, from the injectivity radius theorem of Cheeger, Gromov, and Taylor for Riemannian manifolds [11], it follows that the injectivity radius of Σ^δ is uniformly bounded from below by $C\delta$. Next,

using the theorem of Jost and Karcher [18], we can find a fixed number of harmonic coordinate charts covering Σ^δ and in which the metric is equivalent to the Euclidean metric and has $W^{2,q}$ regularity for each $q \in [1, \infty)$. In addition, by Sobolev's embedding theorem, the metric coefficients also belong to the Hölder space $C^{1,\alpha}$ for all $\alpha \in (0, 1)$.

Next, using an L^q estimate from the equation (2.22) satisfied by the second fundamental form in these harmonic coordinates, we deduce that $k_{ij} \in W^{1,q}$ for all q . We also observe that the Christoffel symbols are of class C^α , so that this also provides us that $\nabla k \in L^q$. All implied constants are uniform and only depend on the dimension n and the distance δ to the boundary of the slice.

Then, using a standard $W^{2,q}$ regularity estimate for equation (3.8) satisfied by the lapse function (see, for instance, [16]) and noting that $g \in W^{2,p}$, we deduce that $\partial^3 \lambda \in L^q$. Here, $\partial^3 \lambda$ stands for any natural derivative in the constructed harmonic coordinates. Since $\partial \Gamma \in L^q$, we obtain the third-order covariant derivatives $|\nabla^3 \lambda| \in L^q$. Finally, we emphasize that for the spatial regularity of $\frac{\partial \lambda}{\partial t}$ and $\frac{\partial^2 \lambda}{\partial t^2}$, we need the crucial estimates established earlier in Lemma 3.2. We use an $W^{2,q}$ regularity estimate to equations (3.20) and use Lemma 3.2; this leads to the desired estimate for $\frac{\partial \lambda}{\partial t}$. Finally, the above estimates imply $\Delta \frac{\partial^2 \lambda}{\partial t^2} - |k|^2 \frac{\partial^2 \lambda}{\partial t^2} \in W^{-1,q}$, and we use again Lemma 3.2 and an L^p regularity estimate in order to control $\frac{\partial^2 \lambda}{\partial t^2}$. \square

Proof of Proposition 3.4. We now want to control the covariant derivatives of the function t . Since this question is independent of the choice of coordinates, then on the fixed slice Σ^δ we choose finitely many spatially harmonic coordinates patches as in the previous proof, and we use them our new coordinates y^i . Then, on this fixed time slice, the spatial metric belongs to $W^{2,q}$ for all $q \in [1, \infty)$. In particular, the Christoffel symbols $\Gamma(\mathbf{g}) = (\Gamma_{ij}^k)$ are uniformly bounded and $\partial_y \Gamma(g) \in L^q$. Combining (3.1) and Lemma 3.3 together, we see that $\Gamma(\mathbf{g})$ is bounded and $|\partial_{t,y} \Gamma(\mathbf{g})| \in L^q$ at slice Σ^δ in these particular harmonic coordinates and at this fixed time.

Now, we calculate the covariant derivatives of t in the coordinates chosen above. Since $\nabla_{\alpha\beta}^2 t = -\Gamma(\mathbf{g})_{\alpha\beta}^0$, we have $\sup_{\Sigma^\delta} |\nabla^2 t| \leq C$. For the third-order derivative, we write $\nabla^3 t = \partial_{t,y} \Gamma(\mathbf{g}) + \Gamma(\mathbf{g}) * \Gamma(\mathbf{g}) \in L^q$. A direct computation yields us

$$\begin{aligned} \Gamma_{00}^0(\mathbf{g}) - \Gamma_{00}^0(\widehat{\mathbf{g}}) &= 0, & \Gamma_{0i}^0(\mathbf{g}) - \Gamma_{0i}^0(\widehat{\mathbf{g}}) &= 0, & \Gamma_{ij}^0(\mathbf{g}) - \Gamma_{ij}^0(\widehat{\mathbf{g}}) &= -\frac{2}{\lambda} k_{ij}, \\ \widetilde{\Gamma}_{00}^k &= 2\lambda g^{kl} \frac{\partial \lambda}{\partial x^l}, & \Gamma_{i0}^k(\mathbf{g}) - \Gamma_{i0}^k(\widehat{\mathbf{g}}) &= 0, & \Gamma_{ij}^k(\mathbf{g}) - \Gamma_{ij}^k(\widehat{\mathbf{g}}) &= 0. \end{aligned}$$

Since $\widehat{\nabla}^2 t - \nabla^2 t = -\frac{2}{\lambda}k$, we have $\sup_{\Sigma_\delta} |\widehat{\nabla}^2 t| \leq C_\delta$, and

$$\begin{aligned} \nabla_l k_{ij} &= \nabla_l k_{ij}, & \nabla_0 k_{ij} &= -\nabla_i \nabla_j \lambda + \lambda g^{lp} k_{pj} k_{li} + \lambda \mathbf{R}_{iNjN}, \\ \nabla_{ij}^2 \lambda &= \nabla_i \nabla_j \lambda - \frac{1}{\lambda} k, & \nabla_{0j}^2 \lambda &= \nabla_i \frac{\partial \lambda}{\partial t} + \lambda k * \nabla \lambda + \lambda^{-1} \frac{\partial \lambda}{\partial t} \nabla \lambda, \\ \nabla_{00}^2 \lambda &= \frac{\partial^2 \lambda}{\partial t^2} - \frac{1}{\lambda} \left(\frac{\partial \lambda}{\partial t} \right)^2 - \lambda |\nabla \lambda|^2. \end{aligned} \quad (3.39)$$

Therefore, we have

$$\begin{aligned} \widehat{\nabla}^3 t &= (\widehat{\nabla} - \nabla) \widehat{\nabla}^2 t + \nabla(\widehat{\nabla}^2 t - \nabla^2 t) + \nabla^3 t \\ &= \left(\frac{1}{\lambda} k + \frac{1}{\lambda} \nabla \lambda \right) * \left(\frac{1}{\lambda} k + \frac{1}{\lambda} \nabla \lambda \right) + \nabla \left(\frac{k}{\lambda} \right) + \nabla^3 t, \end{aligned}$$

and the result follows from (3.39) and Lemma 3.3. \square

Proof of Theorem 3.1. By a direct computation (see for instance [12]) one can check that the Riemannian curvature of the metric $\widehat{\mathbf{g}}$ on $\bigcup_t \Sigma_t$ is uniformly bounded and, actually,

$$\sup_{\Sigma_\delta} |\widehat{R}_{\alpha\beta\gamma\delta} - \mathbf{R}_{\alpha\beta\gamma\delta}| \leq C_\delta (|\nabla^2 \lambda| + |k|^2 + |\nabla \lambda|^2),$$

hence $\sup_{\Sigma_\delta} |\widehat{R}_{\alpha\beta\gamma\delta}| \leq C'_\delta$. According to the injectivity estimate for Riemannian manifolds established in [11], the injectivity radius of the metric $\widehat{\mathbf{g}}$ at the point \mathbf{p} is uniformly bounded from below, i.e. $\text{inj}(\mathbf{M}, \widehat{\mathbf{g}}, \mathbf{p}) \geq c'_\delta$. Therefore, according to Jost and Karcher [18], we may choose harmonic coordinates \mathbf{x}^α of $\widehat{\mathbf{g}}$ around \mathbf{p} .

Noting that $\widehat{\mathbf{g}} = \mathbf{g} + 2\lambda^2 dt \otimes dt$, we obtain

$$\begin{aligned} \widehat{\nabla}^2 \mathbf{g} &= 2 \widehat{\nabla}^2 \lambda^2 \otimes \widehat{\nabla} t \otimes \widehat{\nabla} t + 2 \lambda^2 \widehat{\nabla}^3 t \otimes \widehat{\nabla} t \\ &\quad + 2 \lambda^2 \widehat{\nabla} t \otimes \widehat{\nabla}^3 t + 4 \lambda^2 \widehat{\nabla}^2 t \otimes \widehat{\nabla}^2 t. \end{aligned}$$

By combining with the result (3.4), we see that the coefficients of \mathbf{g} in harmonic coordinates \mathbf{x}^α for the Riemannian metric $\widehat{\mathbf{g}}$ belongs to $W^{2,q}$. \square

4 CMC–harmonic coordinates of an observer

4.1 Construction of local coordinates

Preliminaries

In Theorem 3.1, we constructed coordinates in which the Lorentzian metric coefficients have optimal regularity. However, one inconvenient of these coordinates is that they are not consistent with the CMC foliation constructed in Section 2. In the present section, we show that both strategies can be combined and we construct a new coordinate system which is based on the CMC foliation and has

optimal regularity as stated in Theorem 3.1. The basic idea is to now choose spatial harmonic coordinates on *each* CMC hypersurface. This strategy goes back to Anderson [4] who, however, used the time function given by a distance function. In contrast, in our present construction, the time function coincides with the mean curvature function of CMC slices and has much better regularity.

In view of Theorem 2.2, since the second fundamental form is bounded in each slice Σ_t and the spacetime curvature is bounded, Gauss equation implies that the intrinsic curvature of the slice is also bounded. So, according to [11] there exists a constant $\eta = \eta(n) > 0$ so that the injectivity radius of the slice is bounded below, that is, $\mathbf{Inj}(\Sigma_t, p_t) \geq 2\eta r$, where p_t is the orbit of the base point \mathbf{p} along the above flow. By a theorem established by Jost and Karcher for Riemannian manifolds [18], there exists a constant $\eta' = \eta'(n) > 0$ such that a harmonic coordinate system $|y| \leq \eta' r$ exists around \mathbf{p} on the slice $\Sigma_{t(p)}$, with $\mathbf{p} = (0, \dots, 0)$ and, on that slice,

$$\frac{1}{2} \delta_{ij} \leq g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \leq 2\delta_{ij}. \quad (4.1)$$

By using the above mentioned flow, the coordinate functions y^i can be extended to other slices Σ_t and, together with the time function t , yield a space-time coordinate system. Then, the Lorentzian metric \mathbf{g} takes the form $\mathbf{g} = -\lambda(t, y)^2 dt^2 + g_{ij}(t, y) dy^i dy^j$. From the estimate of $|\nabla t|^2$ given by Theorem 2.2 and in view of the expression $\nu(\nabla u) \frac{\partial u}{\partial t} = -\lambda$, we deduce that

$$\frac{\sqrt{\theta}}{r^2} \leq \lambda \leq \frac{1}{r^2 \sqrt{\theta}}. \quad (4.2)$$

Moreover, in view of the results in [18] and thanks to (2.31) and (2.22), we have the uniform control

$$\nabla K, \nabla^2 \lambda \in L^q(\Sigma_t), \quad q \in [1, \infty).$$

Almost linear coordinates

We now construct the coordinates of interest in this section. We can assume $r = 1$. For each $i = 1, \dots, n$ and for *each slice* Σ_t let x^i be the solution of the Dirichlet problem

$$\begin{aligned} \Delta_t x^i &= 0 & \text{in } \Sigma_t \cap \{y : |y| < \eta'\}, \\ x^i &= y^i & \text{on } |y| = \eta'. \end{aligned} \quad (4.3)$$

Let \bar{t} be such that the slice $\mathbf{p} \in \Sigma_{\bar{t}}$ has mean curvature \bar{t} .

By applying the maximum principle for the operator Δ we can derive some basic properties of the above functions. First of all, at the time \bar{t} one has

$$n \leq \Delta_{\bar{t}} |y|^2 = 2 \sum_{i=1}^n g^{ii} \leq 4n \quad \text{on the slice } \Sigma_{\bar{t}}, \quad (4.4)$$

where we have solely used that the metric coefficients are uniformly bounded. Since

$$\left| \frac{\partial}{\partial t} (\Delta_t |y|^2) \right| = |\nabla \lambda * k * \nabla |y|^2| \leq C_\delta \quad \text{on the slice } \Sigma_{\bar{t}},$$

and $\nabla \lambda$ and k are uniformly bounded, we deduce from (4.4) that

$$\frac{n}{2} \leq \Delta_t |y|^2 \leq 8n \quad \text{on any slice } \Sigma_t \cap \{|y| \leq \eta'\}$$

for all $|t - \bar{t}| \leq \frac{1}{C_\delta C''n}$. From now on we drop the subscript t in the notation.

Then, by the same arguments as the ones above we find

$$|\Delta y^i| \leq \epsilon \quad \text{on } \Sigma_t \cap \{|y| \leq \eta'\} \quad (4.5)$$

for all $|t - \bar{t}| \leq \frac{\epsilon}{C_\delta}$. Now, since $\Delta(x^i - y^i) = -\Delta y^i$, by the maximum principle we obtain

$$C''\epsilon\eta'(\eta' - |y|) \geq x^i - y^i \geq -C''\epsilon\eta'(\eta' - |y|)$$

on the slice $\Sigma_t \cap \{|y| \leq \eta'\}$ for all $|t - \bar{t}| \leq \frac{\epsilon}{C_\delta C''n}$. In particular, along the boundary $\{|y| = \eta'\}$ the above property implies

$$\sup_{|y|=\eta'} |\nabla(x^i - y^i)| \leq C(n)\epsilon\eta' \quad \text{for } |t - \bar{t}| \leq \frac{\epsilon}{C_\delta C''n}.$$

Next, we can also estimate $\sup_{|y| \leq \eta'} |\nabla(x^i - y^i)|$ from the equation satisfied by the coordinates, as follows. By integration by parts we obtain

$$\int_{|y| \leq \eta'} |\nabla(x^i - y^i)|^2 \leq C(n) \epsilon (\eta')^{n+1}. \quad (4.6)$$

Second, let $w = \max(0, |\nabla(x^i - y^i)|^2 - C(n)\epsilon\eta')$ and consider Bochner formula

$$\begin{aligned} & \Delta |\nabla(x^i - y^i)|^2 \\ &= 2|\nabla^2(x^i - y^i)|^2 + 2\langle \nabla(x^i - y^i), -\nabla \Delta y^i \rangle + 2\text{Ric}(\nabla(x^i - y^i), \nabla(x^i - y^i)), \end{aligned}$$

multiply it by w^a for $a > 0$, and integrate by parts. Then, using (4.5) together with Sobolev inequality and Nash-Moser technique, we arrive at the sup-norm gradient estimate

$$\sup_{|y| \leq \eta'} w \leq \frac{C}{\eta'^n} \int_{|y| \leq \eta'} w \, dy \leq C(n)\eta'\epsilon \quad (4.7)$$

for all $|t - \bar{t}| \leq \frac{\epsilon}{C_\delta C''n}$. The latter inequality follows from (4.6).

By choosing ϵ suitably small (depending upon the dimension only), (4.7) implies that the harmonic map $\Psi = (x^1, \dots, x^n)$ is a local diffeomorphism from $\{|y| \leq \eta'\} \cap \Sigma_t$ onto its image. By the maximum principle, Ψ is a map from $\{|y| \leq \eta'\} \cap \Sigma_t$ to $\{|x| \leq \eta'\}$, which leaves invariant the boundary. Hence, Ψ is a diffeomorphism from $\{|y| \leq \eta'\} \cap \Sigma_t$ to the Euclidean ball $\{|x| \leq \eta'\}$, and

$x = (x^1, \dots, x^n)$ with $\{|x| \leq \eta'\}$ is a harmonic coordinate system. Moreover, by choosing ϵ sufficiently small in (4.7) we find

$$\frac{1}{4} \delta_{ij} \leq \frac{1}{2} g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \leq g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \leq 2g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \leq 4 \delta_{ij},$$

as inequalities between symmetric tensors.

The ADM formulation

Including also $x^0 = t$ in the coordinates, we have therefore constructed local spacetime coordinates (x^0, x^1, \dots, x^n) covering a neighborhood of the point \mathbf{p} . Recall that \mathbf{N} denotes the unit normal vector along slices Σ_t .

Note that the function t appears as the time coordinate in two different coordinate systems, that is, $\Psi' = (y^0, y^1, \dots, y^n)$ and $\Psi = (x^0, x^1, \dots, x^n)$ with $y^0 = x^0 = t$. It is easy to see that

$$\lambda N = \Psi'^{-1} \left(\frac{\partial}{\partial y^0} \right) = \frac{\partial}{\partial x^0} + \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^0} - \xi,$$

and so $\frac{\partial}{\partial x^0} = \lambda N + \xi$, where we refer to $\xi = \sum_{i=1}^n \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$ as the shift vector. Define $A_{ij} = \left(\frac{\partial x^i}{\partial y^j} \right)$, and $\tilde{g}_{kl} = g_{ij}(y, t) A^{-1}_{ik} A^{-1}_{jl}$. It is not hard to see that $dy^0 = dx^0$ and $dy^i = A^{-1}_{ik} \left(dx^k - \frac{\partial x^k}{\partial y^0} dx^0 \right)$, hence in the coordinates (x^0, x^1, \dots, x^n) the metric \mathbf{g} has the form

$$\begin{aligned} \mathbf{g} &= -\lambda(t, y)^2 dt^2 + g_{ij}(t, y) dy^i dy^j \\ &= -\lambda^2 (dx^0)^2 + \tilde{g}_{ij}(x^0, x) (dx^i + \xi^i dx^0) (dx^j + \xi^j dx^0). \end{aligned} \quad (4.8)$$

For simplicity in the notation, we drop the tilde from \tilde{g}_{ij} and simply write the *metric decomposition* as

$$\mathbf{g} = -\lambda^2 (dx^0)^2 + g_{ij}(x^0, x) (dx^i + \xi^i dx^0) (dx^j + \xi^j dx^0). \quad (4.9)$$

Recall that the second fundamental form is defined by $k_{ij} = \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, N \rangle$, where ∇ is the covariant derivative associated with the metric \mathbf{g} , and recall Gauss-Codazzi equations

$$\begin{aligned} \mathbf{R}_{ijkl} &= R_{ijkl}^{\Sigma} + k_{ik} k_{jl} - k_{il} k_{kj}, \\ \nabla_l k_{ij} - \nabla_i k_{lj} &= \mathbf{R}_{liNj}. \end{aligned} \quad (4.10)$$

The geometry of the slice is determined by the induced metric g_{ij} and the second fundamental form k_{ij} , both, satisfying the following evolution equations:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^0} &= -2\lambda k_{ij} + \mathcal{L}_{\xi} g_{ij}, \\ \frac{\partial k_{ij}}{\partial x^0} &= -\nabla_i \nabla_j \lambda + \mathcal{L}_{\xi} k_{ij} - \lambda g^{pq} k_{ip} k_{qj} + \lambda \mathbf{R}_{iNjN}. \end{aligned} \quad (4.11)$$

Note also that since x^1, \dots, x^n are harmonic coordinates on Σ_t , we have

$$g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} g_{ij} + Q(\partial g, \partial g) = -2R_{ij}, \quad (4.12)$$

where $Q_{ij}(\partial g, \partial g)$ is some quadratic expression in ∂g with coefficients depending on the inverse metric g^{-1} .

Estimating the shift vector

Next, we derive the equation for the shift vector ξ . By differentiating the harmonic equation $\Delta x^k = 0$ with respect to x^0 , and using (4.11), we get

$$\begin{aligned} 0 &= g^{kl} g^{ij} \left(\nabla_i (-2\lambda k_{lj} + \nabla_j \xi_l + \nabla_l \xi_j) + \nabla_j (-2\lambda k_{li} + \nabla_i \xi_l + \nabla_l \xi_i) \right. \\ &\quad \left. - \nabla_l (-2\lambda k_{ij} + \nabla_j \xi_i + \nabla_i \xi_j) \right) \\ &= 2 \left(\Delta \xi^k + g^{ki} R_{ij} \xi^j + g^{kl} \nabla_l (\lambda \operatorname{tr} K) - 2g^{kl} g^{ij} k_{li} \nabla_j \lambda - 2\lambda (\operatorname{tr} k)_l + 2\lambda g^{kl} \mathbf{R}_{lN} \right), \end{aligned}$$

where $\Delta \xi^k$ is the k -th component of $\Delta \xi$. By combining this result with the constant mean curvature equation, this gives us the *elliptic equation satisfied by the shift vector*

$$\Delta \xi^k = -g^{ki} R_{ij} \xi^j - (\operatorname{tr} k) g^{kl} \nabla_l \lambda + 2g^{kl} g^{ij} k_{li} \nabla_j \lambda - 2\lambda g^{kl} \mathbf{R}_{lN}. \quad (4.13)$$

It is easy to see $\Delta |\xi| \geq -C$ for some constant C depending only on the dimension. By choosing a sufficiently large constant we obtain $\Delta (|\xi| + C'|x|^2) \geq 0$, hence by the maximum principle we arrive at the following sup norm estimate for the shift vector

$$|\xi| \leq C(n)(\eta' - |x|). \quad (4.14)$$

4.2 Proof of the main theorem

We are now in a position to give the proof of Theorem 1.1. By scaling, we may assume $r = 1$.

Step 1. Spatial derivative estimate. We are going to use (2.22) (4.12), (3.8), (4.13), together with elliptic regularity estimates, and establish a bound for the spatial derivatives of the metric \mathbf{g} .

By choosing other harmonic coordinates on each slice, letting η' suitably small, and recalling the L^p regularity estimates for uniformly elliptic operators, we find for all $q \in [1, \infty)$

$$\int_{|x| \leq \eta'} |\nabla^2(x^i - y^i)|^q \leq C_q. \quad (4.15)$$

This implies that

$$\int_{|x| \leq \eta'} \left| \frac{\partial g_{ij}}{\partial x^k} \right|^q \leq C_q \quad (4.16)$$

and, therefore, for all $\alpha \in (0, 1)$, $\|g\|_{C^\alpha\{|x| \leq \eta'\}} \leq C_\alpha$. In view of (4.12), we obtain

$$\int_{|x| \leq \eta'} \left| g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} \left((\eta'^2 - |x|^2) g_{ij} \right) \right|^q \leq C_q \quad (4.17)$$

and using L^p estimate, since the coefficient of the Laplacian operator are Hölder continuous and the function under consideration vanishes on the boundary,

$$\begin{aligned} \left| \frac{\partial}{\partial x^k} g_{ij} \right| &\leq \frac{C(n)}{\eta' - |x|}, \\ \int_{|x| \leq \eta'} \left| \frac{\partial^2}{\partial x^k \partial x^l} \left((\eta'^2 - |x|^2) g_{ij} \right) \right|^q &\leq C_q. \end{aligned} \quad (4.18)$$

Note that in the expression $\Delta \xi = g^{kl} \frac{\partial^2 \xi}{\partial x^k \partial x^l} + \Gamma * \nabla \xi + \partial \Gamma * \xi$, we have $\Gamma \in L^q$ (thanks to (4.16)) and

$$|\partial \Gamma * \xi| \leq C (|\partial^2 g| + |\partial g|^2) (\eta'^2 - |x|^2)$$

thanks to (4.14). The latter term belongs to L^q in view of (4.18) and therefore L^p regularity estimates applied to (4.13) yield

$$\sup |\partial_x \xi| + \int_{|x| \leq \eta'} |\partial_x^2 \xi^k|^q \leq C_q$$

or, in covariant form, we have estimated the first- and second-order derivatives of the shift vector

$$\sup |\nabla \xi| + \int_{|x| \leq \eta'} |\nabla^2 \xi|^q \leq C_q. \quad (4.19)$$

In addition, since $\partial_x k = \nabla k + \Gamma * k$ and $|\nabla k| \in L^q$ by Lemma 3.3, we also find

$$\int_{|x| \leq \eta'} |\partial_x k|^q \leq C_q.$$

Similarly, since $\partial_x^2 \lambda = \nabla^2 \lambda + \Gamma * \nabla \lambda$ and in view of Lemma 3.3, we also obtain

$$\int_{|x| \leq \eta'} |\partial_x^2 \lambda|^q \leq C_q.$$

In summary, we have now control the spatial derivatives (up to second order) of the metric, the lapse function, and the shift vector:

$$(\eta'^2 - |x|^2) g_{ij}, \lambda, \xi^i \in W^{2,q}(\{|x| \leq \eta'\}). \quad (4.20)$$

Step 2. Estimates of first-order time derivatives. The strategy now is to differentiate the equations (2.22), (4.12), (3.8), and (4.13) with respect to t and then use the elliptic regularity property.

First of all, thanks to Step 1 we have $\mathcal{L}_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i \in W_x^{1,q}$, $\lambda k_{ij} \in W_x^{1,q}$ for all $q \in [1, \infty)$. By (4.11) we have $\frac{\partial g_{ij}}{\partial x^0} \in W_x^{1,q}$ and, in particular, $\frac{\partial^2 g_{ij}}{\partial x \partial x^0} \in L_x^q$, i.e. in other words for all $q \in [1, \infty)$

$$\sup_{|x| \leq \eta'} \left| \frac{\partial g_{ij}}{\partial x^0} \right| + \int_{|x| \leq \eta'} \left| \frac{\partial^2 g_{ij}}{\partial x \partial x^0} \right|^q \leq C_q. \quad (4.21)$$

In view of Step 1 and (4.11) again, we have $\frac{\partial k_{ij}}{\partial x^0} \in L_x^q$ for all $q \in [1, \infty)$. Then, from Lemma 3.3 we deduce

$$\begin{aligned} & \sup_{|x| \leq \eta'} \left(\left| \frac{\partial \lambda}{\partial t} \right| + \left| \nabla \frac{\partial \lambda}{\partial t} \right| + \left| \frac{\partial^2 \lambda}{\partial t^2} \right| + \left| \nabla^2 \lambda \right| \right) \\ & + \int_{|x| \leq \eta'} \left(|\nabla k|^q + |\nabla^3 \lambda|^q + \left| \nabla^2 \frac{\partial \lambda}{\partial t} \right|^q + \left| \nabla \frac{\partial^2 \lambda}{\partial t^2} \right|^q \right) \leq C_q. \end{aligned} \quad (4.22)$$

Since $\frac{\partial \lambda}{\partial x^0} = \frac{\partial \lambda}{\partial t} + \langle \xi, \nabla \lambda \rangle$, we have

$$\nabla \frac{\partial \lambda}{\partial x^0} = \nabla \frac{\partial \lambda}{\partial t} + \nabla \xi * \nabla \lambda + \xi * \nabla^2 \lambda$$

and

$$\nabla^2 \frac{\partial \lambda}{\partial x^0} = \nabla^2 \frac{\partial \lambda}{\partial t} + \nabla^2 \xi * \nabla \lambda + \xi * \nabla^3 \lambda + \nabla \xi * \nabla^2 \lambda.$$

Then by (4.22), (4.19), and Step 1, we find

$$\sup_{|x| \leq \eta'} \left| \frac{\partial \lambda}{\partial x^0} \right| + \left| \nabla \frac{\partial \lambda}{\partial x^0} \right| + \int_{|x| \leq \eta'} \left| \nabla^2 \frac{\partial \lambda}{\partial x^0} \right|^q \leq C_q. \quad (4.23)$$

Next, note that

$$\frac{\partial}{\partial x^0} \nabla_i \nabla_j \xi^k = \nabla_i \nabla_j \frac{\partial \xi^k}{\partial x^0} + \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-1} * \nabla \xi + g^{-1} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \xi.$$

By differentiating (4.13) with respect to the time variable x^0 , we obtain

$$\Delta \frac{\partial \xi^k}{\partial x^0} = A_1 + A_2 + A_3, \quad (4.24)$$

with

$$\begin{aligned} A_1 & := -g^{ki} R_{ij} \frac{\partial \xi^j}{\partial x^0} - \text{trk} g^{kl} \nabla_l \frac{\partial \lambda}{\partial x^0} + 2g^{kl} g^{ij} k_{li} \nabla_j \frac{\partial \lambda}{\partial x^0} + 2g^{kl} g^{ij} \frac{\partial k_{li}}{\partial x^0} \nabla_j \lambda, \\ A_2 & := -\frac{\partial R_{ij}}{\partial x^0} g^{ki} \xi^j - 2\lambda g^{kl} \left(\nabla_{\frac{\partial}{\partial x^0}} \mathbf{Ric} \right) \left(\frac{\partial}{\partial x^l}, N \right) - 2\lambda g^{kl} \mathbf{Ric} \left(\nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^l}, N \right) \\ & \quad - 2\lambda g^{kl} \mathbf{Ric} \left(\frac{\partial}{\partial x^l}, \nabla_{\frac{\partial}{\partial x^0}} N \right) + \frac{\partial g_{rs}}{\partial x^0} \left(g^{kr} g^{si} R_{ij} \xi^j + \text{trk} g^{kr} g^{ls} \nabla_l \lambda \right. \\ & \quad \left. - 2g^{kr} g^{sl} g^{ij} k_{li} \nabla_j \lambda - 2g^{kl} g^{ir} g^{js} k_{li} \nabla_j \lambda + 2\lambda g^{kr} g^{ls} \mathbf{R}_{lN} + g^{ir} g^{js} \nabla_i \nabla_j \xi^k \right), \end{aligned}$$

and

$$A_3 := \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-2} * \nabla \xi + g^{-2} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \xi.$$

Since the spacetime under consideration satisfies the vacuum Einstein equation, we obtain

$$\begin{aligned} \Delta \frac{\partial \xi^k}{\partial x^0} &= -g^{ki} R_{ij} \frac{\partial \xi^j}{\partial x^0} + g^{-2} * \left(k * \nabla \frac{\partial \lambda}{\partial x^0} + \frac{\partial k}{\partial x^0} * \nabla \lambda \right) + g^{-2} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \xi \\ &\quad + \frac{\partial Ric}{\partial x^0} * g^{-1} * \xi + \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-2} * \nabla \xi \\ &\quad + \frac{\partial g}{\partial x^0} * \left(g^{-2} * Ric * \xi + k * g^{-3} * \nabla \lambda + g^{-2} * \nabla^2 \xi \right). \end{aligned} \quad (4.25)$$

It is a classical observation that

$$-2 \frac{\partial R_{ij}}{\partial x^0} = \Delta_L \left(\frac{\partial g_{ij}}{\partial x^0} \right) + \nabla_i V_j + \nabla_j V_i, \quad (4.26)$$

where

$$\Delta_L \left(\frac{\partial g_{ij}}{\partial x^0} \right) = \Delta \left(\frac{\partial g_{ij}}{\partial x^0} \right) + 2R_{ikjl} \frac{\partial g_{kl}}{\partial x^0} - R_{ik} \frac{\partial g_{kj}}{\partial x^0} - R_{jk} \frac{\partial g_{ki}}{\partial x^0}$$

is the Lichnerowicz Laplacian, and

$$V_i := \frac{1}{2} \nabla_i (g^{kl} \frac{\partial g_{kl}}{\partial x^0}) - g^{kl} \nabla_k \frac{\partial g_{il}}{\partial x^0}.$$

Since $\nabla \frac{\partial g}{\partial x^0} \in L_x^q$, $\nabla^2 \frac{\partial g}{\partial x^0} = \partial \nabla \frac{\partial g}{\partial x^0} + \Gamma * \nabla \frac{\partial g}{\partial x^0}$, we see that $\frac{\partial R_{ij}}{\partial x^0} \in W_x^{-1,q}$ for all $q \in [1, \infty)$. Note that $\frac{\partial \lambda}{\partial x^0} \in W_x^{2,q}$, $\frac{\partial k_{ij}}{\partial x^0} \in L_x^q$, $\nabla \lambda \in W_x^{1,q}$, $\nabla_i \nabla_j \xi^k \in L_x^q$, $\nabla \xi \in L_x^q$, $\nabla \left(\frac{\partial g}{\partial x^0} \right) \in L_x^q$, $\nabla^2 \left(\frac{\partial g}{\partial x^0} \right) = \partial \nabla \left(\frac{\partial g}{\partial x^0} \right) + \Gamma * \nabla \left(\frac{\partial g}{\partial x^0} \right) \in W_x^{-1,q}$.

Now at the boundary $|x| = \eta'$, we have

$$\frac{\partial \xi^k}{\partial x^0} = \frac{\partial^2 x^k}{\partial t^2} + \xi(\xi^k) = 0$$

where we used $\xi|_{|x|=\eta'} = 0$. By applying the L^p estimate, we conclude that $\frac{\partial \xi^k}{\partial x^0} \in W_x^{1,q}$ for all $q \in [1, \infty)$. In particular,

$$\frac{\partial}{\partial x^0} \nabla^2 \xi^k = \nabla^2 \frac{\partial \xi^k}{\partial x^0} + \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-1} * \nabla \xi + g^{-1} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \xi \in W_x^{-1,q}. \quad (4.27)$$

In summary, we have proved that the first order (in time) derivatives of the metric, lapse function, and shift vector have well-controlled spatial derivatives up to first (or even second) order:

$$\frac{\partial g_{ij}}{\partial x^0} \in W_x^{1,q}, \quad \frac{\partial \lambda}{\partial x^0} \in W_x^{2,q}, \quad \frac{\partial \xi^k}{\partial x^0} \in W_x^{1,q} \quad (4.28)$$

for all $q \in [1, \infty)$.

Step 3. Second-order time derivative of the metric and lapse function.

First of all, by differentiating (4.11) we find

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial x^{02}} &= \lambda \frac{\partial k}{\partial x^0} + \frac{\partial \lambda}{\partial x^0} k + \frac{\partial g}{\partial x^0} * \nabla \xi + \nabla \left(\frac{\partial \xi}{\partial x^0} \right) * g + g^{-1} * \nabla \left(\frac{\partial g}{\partial x^0} \right) * \xi * g, \\
\frac{\partial^2 k_{ij}}{\partial x^{02}} &= \nabla^2 \frac{\partial \lambda}{\partial x^0} + \mathcal{L}_\xi \frac{\partial k}{\partial x^0} + \nabla k * \frac{\partial \xi}{\partial x^0} + k * \nabla \left(\frac{\partial \xi}{\partial x^0} \right) \\
&\quad + \lambda \left(k * \frac{\partial k}{\partial x^0} * g^{-1} + g^{-2} * \frac{\partial g}{\partial x^0} * k^2 + \frac{\partial Ric}{\partial x^0} + k \right) \\
&\quad + g^{-1} * \nabla \left(\frac{\partial g}{\partial x^0} \right) * (\nabla \lambda + k * \xi) + (k^2 * g^{-1} + Ric) \frac{\partial \lambda}{\partial x^0}.
\end{aligned} \tag{4.29}$$

Recalling that $g_{ij}, \xi^i, \lambda \in W_x^{2,q}$, $\frac{\partial g_{ij}}{\partial x^0} \in W_x^{1,q}$, $\frac{\partial \lambda}{\partial x^0} \in W_x^{2,q}$, $\frac{\partial \xi^k}{\partial x^0} \in W_x^{1,q}$, $k \in W_x^{1,q}$, $\frac{\partial k}{\partial x^0} \in L_x^q$, $\frac{\partial Ric}{\partial x^0} \in W_x^{-1,q}$, and combining together with (4.12), (4.28), and (4.20), we get the following bounds for the metric and the second fundamental form

$$\frac{\partial^2 g_{ij}}{\partial x^{02}} \in L_x^q, \quad \frac{\partial^2 k_{ij}}{\partial x^{02}} \in W_x^{-1,q} \tag{4.30}$$

for all $q \in [1, \infty)$.

To handle the lapse function we note that $\frac{\partial \lambda}{\partial x^0} = \frac{\partial \lambda}{\partial t} + \langle \xi, \nabla \lambda \rangle$ and $\frac{\partial}{\partial x^0} = \lambda N + \xi$, so that

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial x^{02}} &= \frac{\partial}{\partial x^0} (\lambda N \lambda) + \frac{\partial}{\partial x^0} \langle \xi, \nabla \lambda \rangle \\
\frac{\partial}{\partial x^0} \langle \xi, \nabla \lambda \rangle &= g^{-2} * \frac{\partial g}{\partial x^0} * \xi * \nabla \lambda + g^{-1} * \frac{\partial \xi}{\partial x^0} * \nabla \lambda + g^{-1} * \xi * \nabla \frac{\partial \lambda}{\partial x^0} \\
\frac{\partial}{\partial x^0} (\lambda N \lambda) &= \frac{\partial^2 \lambda}{\partial t^2} + \xi \left(\frac{\partial \lambda}{\partial x^0} \right) - \xi \langle \xi, \nabla \lambda \rangle \\
&= \frac{\partial^2 \lambda}{\partial t^2} + \langle \xi, \nabla \frac{\partial \lambda}{\partial x^0} \rangle - \langle \nabla_\xi \xi, \nabla \lambda \rangle - \nabla^2 \lambda (\xi, \xi),
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
\nabla \frac{\partial^2 \lambda}{\partial x^{02}} &= \nabla \frac{\partial^2 \lambda}{\partial t^2} \\
&\quad + g^{-2} * \nabla \frac{\partial g}{\partial x^0} * \xi * \nabla \lambda + g^{-2} * \frac{\partial g}{\partial x^0} * \nabla \xi * \nabla \lambda + g^{-2} * \frac{\partial g}{\partial x^0} * \xi * \nabla^2 \lambda \\
&\quad + g^{-1} * \frac{\partial \xi}{\partial x^0} * \nabla^2 \lambda + g^{-1} * \nabla \frac{\partial \xi}{\partial x^0} * \nabla \lambda + g^{-2} * \nabla^3 \lambda * \xi^2 \\
&\quad + g^{-2} * \nabla^2 \xi * \xi * \nabla \lambda + g^{-2} * \nabla \xi * \nabla \xi * \nabla \lambda + g^{-2} * \nabla \xi * \xi * \nabla^2 \lambda \\
&\quad + g^{-1} * \nabla \xi * \nabla \frac{\partial \lambda}{\partial x^0} + g^{-1} * \xi * \nabla^2 \frac{\partial \lambda}{\partial x^0}.
\end{aligned} \tag{4.32}$$

Hence, combining together (4.19), (4.22), and (4.28), we arrive at the following control of the lapse function

$$\int_{|x| \leq \eta'} \left| \nabla \frac{\partial^2 \lambda}{\partial x^{02}} \right|^q + \left| \frac{\partial^2 \lambda}{\partial x^{02}} \right|^q \leq C_q$$

for all $q \in [1, \infty)$.

Step 4. Second-order time derivative of the lapse function.

It remains to derive the second-order time estimate for the shift function. By differentiating (4.24) in time, we have

$$\Delta \frac{\partial^2 \xi^k}{\partial x^{02}} = B_1 + B_2 + B_3, \quad (4.33)$$

with

$$\begin{aligned} B_1 &:= -g^{ki} R_{ij} \frac{\partial^2 \xi^j}{\partial x^{02}} + Ric * \left(\frac{\partial \xi}{\partial x^0} * \frac{\partial g}{\partial x^0} * g^{-2} + g^{-3} * \left(\frac{\partial g}{\partial x^0} \right)^2 * \xi \right) \\ &\quad + g^{-2} * k * \nabla \frac{\partial^2 \lambda}{\partial x^{02}} + \left(g^{-2} * \frac{\partial k}{\partial x^0} + g^{-3} * \frac{\partial g}{\partial x^0} * k \right) * \nabla \frac{\partial \lambda}{\partial x^0} \\ &\quad + \left(g^{-2} * \frac{\partial^2 k}{\partial x^{02}} + g^{-3} * \frac{\partial g}{\partial x^0} * \frac{\partial k}{\partial x^0} + g^{-4} * \left(\frac{\partial g}{\partial x^0} \right)^2 * k \right) * \nabla \lambda, \\ B_2 &:= + \frac{\partial^2 Ric}{\partial x^{02}} * g^{-1} * \xi + \frac{\partial Ric}{\partial x^0} * \left(\frac{\partial \xi}{\partial x^0} * g^{-1} + g^{-2} * \frac{\partial g}{\partial x^0} * \xi \right) \\ &\quad + \frac{\partial^2 g}{\partial x^{02}} * \left(g^{-2} * Ric * \xi + k * g^{-3} * \nabla \lambda + g^{-2} * \nabla^2 \xi \right) \\ &\quad + \left(\frac{\partial}{\partial x^0} \nabla^2 \xi \right) * g^{-2} * \frac{\partial g}{\partial x^0} + \nabla^2 \xi * g^{-3} * \left(\frac{\partial g}{\partial x^0} \right)^2, \\ B_3 &:= \left(\left(\nabla \frac{\partial g}{\partial x^0} \right)^2 + \frac{\partial g}{\partial x^0} * \nabla^2 \frac{\partial g}{\partial x^0} + \nabla^2 \frac{\partial^2 g}{\partial x^{02}} * g \right) * \xi * g^{-3} \\ &\quad + g^{-2} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \frac{\partial \xi}{\partial x^0} + \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-2} * \nabla \frac{\partial \xi}{\partial x^0} \\ &\quad + \nabla \left(\frac{\partial^2 g}{\partial x^{02}} \right) * g^{-2} * \nabla \xi + \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-3} * \frac{\partial g}{\partial x^0} * \nabla \xi. \end{aligned}$$

Note that we already have $\partial_{x,t} \mathbf{g}, \partial_{x,t}^2 \mathbf{g} \in L_x^q$ for all $q \in [1, \infty)$, except that we

do not control $\frac{\partial^2 \xi}{\partial x^{02}}$ yet. Therefore, we can write

$$\begin{aligned}
\Delta \frac{\partial^2 \xi^k}{\partial x^{02}} &= -g^{ki} R_{ij} \frac{\partial^2 \xi^j}{\partial x^{02}} + g^{-2} * \frac{\partial^2 k}{\partial x^{02}} * \nabla \lambda \\
&+ \frac{\partial^2 Ric}{\partial x^{02}} * g^{-1} * \xi + \frac{\partial Ric}{\partial x^0} * \left(\frac{\partial \xi}{\partial x^0} * g^{-1} + g^{-2} * \frac{\partial g}{\partial x^0} * \xi \right) \\
&+ \left(\frac{\partial}{\partial x^0} \nabla^2 \xi \right) * g^{-2} * \frac{\partial g}{\partial x^0} + g^{-2} * \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \frac{\partial \xi}{\partial x^0} \\
&+ g^{-3} * g * \nabla^2 \left(\frac{\partial^2 g}{\partial x^{02}} \right) * \xi + \nabla \left(\frac{\partial^2 g}{\partial x^{02}} \right) * g^{-2} * \nabla \xi \quad \text{mod. } L_x^q.
\end{aligned} \tag{4.34}$$

In view of (4.20), (4.28), (4.27), and (4.30) we have

$$\begin{aligned}
\frac{\partial Ric}{\partial x^0}, \frac{\partial^2 k}{\partial x^{02}}, \nabla \frac{\partial^2 \lambda}{\partial x^{02}}, \frac{\partial}{\partial x^0} \nabla^2 \xi, \nabla \left(\frac{\partial^2 g}{\partial x^{02}} \right), \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) &\in W_x^{-1,q}, \\
\nabla^2 \frac{\partial^2 g}{\partial x^{02}} &\in W_x^{-2,q},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 Ric}{\partial x^{02}} &= g^{-1} * \left(\nabla^2 \frac{\partial^2 g}{\partial x^{02}} + Rm * \frac{\partial^2 g}{\partial x^{02}} \right. \\
&+ \nabla^2 \left(\frac{\partial g}{\partial x^0} \right) * \left(\frac{\partial g}{\partial x^0} \right) * g^{-1} + \nabla \left(\frac{\partial g}{\partial x^0} \right) * \nabla \left(\frac{\partial g}{\partial x^0} \right) * g^{-1} \left. \right) \\
&= \nabla^2 \left(g^{-1} * \frac{\partial^2 g}{\partial x^{02}} \right) + \nabla \left(\nabla \frac{\partial g}{\partial x^0} * \frac{\partial g}{\partial x^0} * g^{-2} \right) \quad \text{mod. } L_x^q
\end{aligned}$$

for all $q \in [1, \infty)$. Consequently, we have

$$\begin{aligned}
\Delta \frac{\partial^2 \xi^k}{\partial x^{02}} + g^{ki} R_{ij} \frac{\partial^2 \xi^j}{\partial x^{02}} &= \nabla_i \nabla_j (f_m^{kij} \xi^m) + \nabla_i f^{ki} + f^k \\
&= \partial_i \partial_j (F_m^{kij}) \xi^m + \partial_i F^{ki} + F^k,
\end{aligned} \tag{4.35}$$

where for fixed k , f_m^{kij} etc. are tensors, and ∇ and ∂ are covariant derivatives and partial derivatives in the coordinates x^i , respectively, with moreover

$$\int_{|x| \leq \eta'} \left(|f^{kij}|^q + |f^{ki}|^q + |f^k|^q + |F_m^{kij}|^q + |F^{ki}|^q + |F^k|^q \right) \leq C_q.$$

For the second equality, we used $\xi \partial^2 g \in L^q$. Now, we will use the L^p regularity estimates in the following manner.

Since the coefficients of the elliptic operator $g^{ab} \partial_a \partial_b$ belong to C^γ on the closed ball $\{|x| \leq \eta'\}$, we can solve the equation $g^{ab} \partial_a \partial_b u^{kij} = F^{kij}$ on $|x| < \eta'$ with the trivial boundary condition $u^{kij}|_{|x|=\eta'} = 0$. We then apply the L^p regularity estimate and obtain

$$\int_{|x| \leq \eta'} \left(|u^{kij}|^q + |\partial u^{kij}|^q + |\partial^2 u^{kij}|^q \right) \leq C_q.$$

Next, we observe that

$$\begin{aligned} \partial_i \partial_j (F_m^{kij}) \xi^m &= g^{ab} \partial_a \partial_b (\partial_i \partial_j u_m^{kij} \xi^m) + \xi \partial^2 g * \partial^2 u_m^{kij} \\ &\quad + \partial(\partial g * \partial^2 u_m^{kij} * \xi + \partial^2 u_m^{kij} * \partial \xi) + \partial^2 u_m^{kij} * \partial^2 \xi * g^{-1} \end{aligned}$$

and so, for some new terms F^{ki} , we obtain

$$\Delta \frac{\partial^2 \xi^k}{\partial x^{02}} + g^{ki} R_{ij} \frac{\partial^2 \xi^j}{\partial x^{02}} = g^{ab} \partial_a \partial_b (u_m^k \xi^m) + \partial_i F^{ki} + F^k \quad (4.36)$$

for some

$$\int_{|x| \leq \eta'} |u_m^k|^q \leq C_q. \quad (4.37)$$

Since

$$\begin{aligned} \Delta (u_m^k \xi^m) &= g^{ab} \partial_a \partial_b (u_m^k \xi^m) + \Gamma * \partial(u * \xi) + \partial^2 g * u * \xi \\ &= g^{ab} \partial_a \partial_b (u_m^k \xi^m) + \partial(\Gamma * u * \xi) + \partial^2 g * u * \xi, \end{aligned}$$

and $\xi \partial^2 g \in L^q$ for all q , by modifying F^{ki} and F^k , we can show

$$\Delta \left(\frac{\partial^2 \xi^k}{\partial x^{02}} - u_m^k \xi^m \right) + g^{ki} R_{ij} \left(\frac{\partial^2 \xi^j}{\partial x^{02}} - u_m^j \xi^m \right) = \partial_i F^{ki} + F^k, \quad (4.38)$$

where the notation Δ stands here for the covariant Laplacian of a vector field.

It is not hard to see $\frac{\partial^2 \xi^k}{\partial x^{02}} \Big|_{|x|=\eta'} = 0$. Let $v^k = \frac{\partial^2 \xi^k}{\partial x^{02}} - u_m^k \xi^m$. Integrating $\Delta |v^k|^{2q+2}$ with the induced (intrinsic) volume form, using $v^k \Big|_{|x|=\eta'} = 0$ and (4.38), and finally applying Hölder inequality, we find

$$\int_{|x| \leq \eta'} |v^k|^{2q} |\nabla v^k|^2 \leq C_q \int_{|x| \leq \eta'} |v^k|^{2q+2} + C_q \quad (4.39)$$

for all $q \in [1, \infty)$. This implies, in particular, $\int_{|x| \leq \eta'} |v^k|^2 q \leq C$ by Sobolev inequalities. Combining this result with (4.37), we arrive at the estimate for the shift vector

$$\int_{|x| \leq \eta'} \left| \frac{\partial^2 \xi^k}{\partial x^{02}} \right|^q \leq C_q$$

In summary, we have obtain the following uniform control of the second-order time derivatives of the metric, lapse function, and shift vector:

$$\frac{\partial^2 g_{ij}}{\partial x^{02}} \in L_x^q, \quad \frac{\partial^2 \lambda}{\partial x^{02}} \in L_x^q, \quad \frac{\partial^2 \xi^k}{\partial x^{02}} \in L_x^q \quad (4.40)$$

for all $q \in [1, \infty)$. By combining (4.20) with (4.28) and (4.40), the proof of Theorem 1.1 is now completed.

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References

- [1] M.T. ANDERSON, On stationary vacuum solutions to the Einstein equations, *Ann. Henri Poincaré* 1 (2000), 977–994.
- [2] M.T. ANDERSON, On the structure of solutions to the static vacuum Einstein equations, *Ann. Henri Poincaré* 1 (2000), 995–1042.
- [3] M.T. ANDERSON, On long-time evolution in general relativity and geometrization of 3-manifolds, *Commun. Math. Phys.* 222 (2001), 533–567.
- [4] M.T. ANDERSON, Regularity for Lorentz metrics under curvature bounds, *Jour. Math. Phys.* 44 (2003), 2994–3012.
- [5] L. ANDERSSON, Constant mean curvature foliations of flat space-times, *Comm. Anal. Geom.* 10 (2002), 1125–1150.
- [6] L. ANDERSSON, Bel-Robinson energy and constant mean-curvature foliations, *Ann. H. Poincaré* 5 (2004), 235–244.
- [7] L. ANDERSSON AND V. MONCRIEF, Elliptic-hyperbolic systems and the Einstein equations, *Ann. Inst. Henri Poincaré* 4 (2003), 1–34.
- [8] L. ANDERSSON AND V. MONCRIEF, Future complete vacuum spacetimes, in “The Einstein equations and the large scale behavior of gravitational fields”, Birkhäuser, Basel, 2004, pp. 299–330.
- [9] R. BARTNIK AND L. SIMON, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Commun. Math. Phys.* 87 (1982), 131–152.
- [10] A. BESSE, *Einstein manifolds*, *Ergebnisse Math. Series 3*, Springer Verlag, 1987.
- [11] J. CHEEGER, M. GROMOV, AND M. TAYLOR, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Diff. Geom.* 17 (1982) 15–53.
- [12] B.-L. CHEN AND P.G. LEFLOCH, Injectivity radius estimates for Lorentzian manifolds, *Commun. Math. Phys.* 278 (2008), 679–713.
- [13] D.M. DETURCK AND J.L. KAZDAN, Some regularity theorems in Riemannian geometry. *Ann. Sci. École Norm. Sup.* 14 (1981), 249–260.

- [14] C. GERHARDT, H-surfaces in Lorentzian manifolds, *Commun. Math. Phys.* 89 (1983), 523–533.
- [15] C. GERHARDT, *Curvature problems*, in “Series in Geometry and Topology”, vol. 39, International Press, Somerville, MA 2006.
- [16] D. GILBARG AND N.S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.
- [17] S. HAWKING AND G.F. ELLIS, *The large scale structure of spacetime*, Cambridge Univ. Press, 1973.
- [18] J. JOST AND H. KARCHER, Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen, *Manuscripta Math.* 40 (1982), 27–77.
- [19] S. KLAINERMAN AND I. RODNIANSKI, Ricci defects of microlocalized Einstein metrics, *J. Hyperbolic Differ. Equa.* 1 (2004), 85–113.
- [20] S. KLAINERMAN AND I. RODNIANSKI, Rough solutions of the Einstein-vacuum equations, *Ann. of Math.* 161 (2005), 1143–1193.
- [21] S. KLAINERMAN AND I. RODNIANSKI, On the radius of injectivity of null hypersurfaces, *J. Amer. Math. Soc.* 21 (2008), 775–795.
- [22] S. KLAINERMAN AND I. RODNIANSKI, On the breakdown criterion in general relativity, preprint, 2008.
- [23] B. O’NEILL, *Semi-Riemannian geometry with applications to relativity*, Acad. Press, New York, 1983.
- [24] R. PENROSE, *Techniques of differential topology in relativity*, CBMS-NSF Region. Conf. Series Appl. Math., Vol. 7, 1972.
- [25] P. PETERSEN, Convergence theorems in Riemannian geometry, in “Comparison Geometry” (Berkeley, CA, 1992–93), MSRI Publ. 30, Cambridge Univ. Press, 1997, pp. 167–202.
- [26] M. REIRIS, The constant mean curvature Einstein flow and the Bel-Robinson energy, Preprint ArXiv:0705.3070.
- [27] M. REIRIS, The ground state and the long-time evolution in the CMC Einstein flow, Preprint ArXiv:0809.3444.
- [28] J. SIMONS, Minimal varieties in Riemannian manifolds, *Ann. of Math.* 88 (1968), 62–105.