A relaxation method for conservation laws via the Born-Infeld system

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1 Context and motivation

- A class of scalar conservation laws
- Need for a new relaxation scheme

2 The Jin-Xin relaxation

3 The Born-Infeld relaxation

4 Applications and extensions
A class of scalar conservation laws

- Some simplified two-phase flow models can be reduced to the equation
  \[ \partial_t u + \partial_x (u(1-u)g(u)) = 0, \quad x \in \mathbb{R}, \quad t > 0, \]  
  (1)

  where \( u(t,x) \in [0,1] \) and \( g \in C^1([0,1];\mathbb{R}) \).

- The unknown \( u \) represents a volume- or mass-fraction, while
  \[ w(u) = (1-u)g(u) \quad \text{and} \quad z(u) = -ug(u) \]  
  (2)

  play the role of convective phase velocities, since

  \[ \partial_t (u) + \partial_x (u \cdot w(u)) = 0, \]  
  (3a)

  \[ \partial_t (1-u) + \partial_x ((1-u) \cdot z(u)) = 0. \]  
  (3b)

- The slip velocity \( g(u) = w(u) - z(u) \) is assumed to keep a constant sign, i.e., \( 0 \not\in g([0,1]) \).
Need for a new relaxation scheme

- The scalar conservation law (1) is embedded in a larger system, which contains additional equations of the type

\[
\partial_t \alpha_k + \partial_x (\alpha_k \cdot w(u)) = 0, \quad (4a)
\]
\[
\partial_t \beta_\ell + \partial_x (\beta_\ell \cdot z(u)) = 0, \quad (4b)
\]

where the \( \alpha_k \)'s and \( \beta_\ell \)'s denote the species of the mixture.

- The phase velocities \( w \) and \( z \) have to be always well-defined. But standard numerical methods for (1), such as the semi-linear relaxation, do not guarantee this property.

- Design a suitable relaxation method for this problem, based (surprisingly) on a system called Born-Infeld. Can be studied per se and has interesting extensions [M3AS 19 (2009), 1–38].
Outline

1. Context and motivation

2. The Jin-Xin relaxation
   - Design principle
   - Subcharacteristic condition
   - Riemann problem

3. The Born-Infeld relaxation

4. Applications and extensions
A general-purpose relaxation strategy

Consider the scalar conservation law

\[ \partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, \ t > 0, \quad (5) \]

where \( u(t, x) \in [0, 1] \) and \( f(.) \in C^1([0, 1]; \mathbb{R}) \) is a nonlinear flux function.

To construct admissible weak solutions and to design robust numerical schemes, Jin and Xin (1995) proposed the semi-linear relaxation

\[ \partial_t U^\lambda + \partial_x F^\lambda = 0, \quad (6a) \]

\[ \partial_t F^\lambda + a^2 \partial_x U^\lambda = \lambda [f(U^\lambda) - F^\lambda], \quad (6b) \]

where \( F^\lambda \) is a full-fledged variable, maintained close to \( f(U^\lambda) \) by choosing large \( \lambda \).
The relaxation system (6) is \textbf{linear} with eigenvalues $\pm a$.

\[
\begin{align*}
\partial_t \left( \frac{U^\lambda}{2} - \frac{F^\lambda}{2a} \right) - a \partial_x \left( \frac{U^\lambda}{2} - \frac{F^\lambda}{2a} \right) &= \lambda \left[ \left( \frac{U^\lambda}{2} - \frac{f(U^\lambda)}{2a} \right) - \left( \frac{U^\lambda}{2} - \frac{F^\lambda}{2a} \right) \right] \\
\partial_t \left( \frac{U^\lambda}{2} + \frac{F^\lambda}{2a} \right) + a \partial_x \left( \frac{U^\lambda}{2} + \frac{F^\lambda}{2a} \right) &= \lambda \left[ \left( \frac{U^\lambda}{2} + \frac{f(U^\lambda)}{2a} \right) - \left( \frac{U^\lambda}{2} + \frac{F^\lambda}{2a} \right) \right]
\end{align*}
\]

It can be given a \textbf{kinetic} interpretation

\[
\partial_t K^\lambda(t, x, \xi) + \xi \partial_x K^\lambda(t, x, \xi) = \lambda \left[ k \left( \int K^\lambda(t, x, \xi) d\xi, \xi \right) - K^\lambda(t, x, \xi) \right],
\]

with $\xi \in \{-a, +a\}$ and a Maxwellian $k(., .)$ such that

\[
u = \int_{\{-a, +a\}} k(u, \xi) d\xi \quad \text{and} \quad f(u) = \int_{\{-a, +a\}} \xi k(u, \xi) d\xi. \quad (8)
\]
Approximation properties

- Inserting the Chapman-Enskog expansion

\[ F^\lambda = f(U^\lambda) + \frac{1}{\lambda} F_1^\lambda + O\left(\frac{1}{\lambda^2}\right) \]  

(9)

into the relaxation system, we obtain the equivalent equation

\[ \partial_t U^\lambda + \partial_x f(U^\lambda) = \frac{1}{\lambda} \partial_x \left\{ (a^2 - [f'(U^\lambda)]^2) \partial_x U^\lambda \right\}. \]  

(10)

- For dissipativeness, we require the subcharacteristic condition

\[ f'(u) \in [-a, +a], \quad \forall u \in [0, 1]. \]  

(11)

- Under the subcharacteristic condition and suitable assumptions on the initial data, the sequence \( U^\lambda \) can be shown to converge (in \( L^\infty \) weak*) to the entropy solution of the original scalar conservation law as \( \lambda \to +\infty \).
First-order explicit scheme

- Splitting between differential and source terms.
  - $\lambda = \infty$. Set the data at time $n$ to equilibrium, i.e.,
    \[
    F_i^n = f(u_i^n). \tag{12}
    \]
  - $\lambda = 0$. Solve the Riemann problem associated with the relaxation system at each edge $i + 1/2$. The intermediate state $(U^*, F^*)$ is subject to
    \[
    aU^* - F^* = au_L - F_L, \tag{13a}
    \]
    \[
    aU^* + F^* = au_R + F_R. \tag{13b}
    \]

- Update formulae
  \[
  u_{i+1}^n = u_i^n - \frac{\Delta t}{\Delta x} \left[ F^*(u_i^n, u_{i+1}^n) - F^*(u_{i-1}^n, u_i^n) \right], \tag{14}
  \]
  with
  \[
  F^*(u_L, u_R) = \frac{f(u_L) + f(u_R)}{2} - a \frac{u_R - u_L}{2}. \tag{15}
  \]
Maximum principles

\[ U^*(u_L, u_R) = \frac{u_L + u_R}{2} - \frac{f(u_R) - f(u_L)}{2a} \]  \hspace{1cm} (16)

satisfies the **local** maximum principle \( U^* \in [u_L, u_R] \) provided that

\[ a \geq \left| \frac{f(u_R) - f(u_L)}{u_R - u_L} \right|, \]  \hspace{1cm} (17)

which is implied by the subcharacteristic condition;

satisfies the **global** maximum principle \( U^* \in [0, 1] \) provided that

\[ a \geq \max \left\{ \frac{f(u_R) - f(u_L)}{u_R + u_L}, -\frac{f(u_R) - f(u_L)}{(1 - u_R) + (1 - u_L)} \right\}; \]  \hspace{1cm} (18)

for \( f(u) = u(1 - u)g(u) \), a simpler and sufficient condition is

\[ a \geq \max \{|g(u_L)|, |g(u_R)|\}. \]  \hspace{1cm} (19)
Troubleshoots

- Since \( F^* \neq f(U^*) = U^*(1 - U^*)g(U^*) \), the intermediate velocities

\[
W^* = \frac{F^*}{U^*} \quad Z^* = -\frac{F^*}{1 - U^*}
\]  

may become unbounded as \( u_L \) and \( u_R \) go to 0 or 1.

- However, we need \( W^* \) and \( Z^* \) in order to discretize the additional equations

\[
\partial_t \alpha_k + \partial_x (\alpha_k \cdot w(u)) = 0, \quad (21a) \\
\partial_t \beta_\ell + \partial_x (\beta_\ell \cdot z(u)) = 0. \quad (21b)
\]

- In order to achieve some maximum principle on \( w \) and \( z \), we have to take advantage of the form

\[
f(u) = u(1 - u)g(u). \quad (22)
\]
Outline

1. Context and motivation

2. The Jin-Xin relaxation

3. The Born-Infeld relaxation
   - Design principle
   - Subcharacteristic condition
   - Riemann problem
   - Numerical scheme and results

4. Applications and extensions
Change of variables

- Equilibrium variables

\[ w(u) = (1 - u)g(u) = \frac{f(u)}{u}, \quad u = \frac{z(u)}{z(u) - w(u)}, \quad (23a) \]

\[ z(u) = -ug(u) = -\frac{f(u)}{1 - u}, \quad f(u) = \frac{w(u)z(u)}{z(u) - w(u)}. \quad (23b) \]

The pair \((w, z)\) belongs to either \(\{w \geq 0, z \leq 0\}\) or \(\{w \leq 0, z \geq 0\}\).

- Relaxation variables

\[ W(U, F) = (1 - U)G = \frac{F}{U}, \quad U(W, Z) = \frac{Z}{Z - W}, \quad (24a) \]

\[ Z(U, F) = -UG = -\frac{F}{1 - U}, \quad F(W, Z) = \frac{WZ}{Z - W}. \quad (24b) \]

The pair \((W, Z)\) belongs to the same quarter-plane as \((w, z)\). The pair \((U, F)\) is not subject to the constraint \(F = f(U)\).
Definition of the relaxation system

- The Born-Infeld relaxation system for the scalar conservation law (1) is defined as

\[
\begin{align*}
\partial_t W^\lambda + Z^\lambda \partial_x W^\lambda &= \lambda [w(U(W^\lambda, Z^\lambda)) - W^\lambda], \\
\partial_t Z^\lambda + W^\lambda \partial_x Z^\lambda &= \lambda [z(U(W^\lambda, Z^\lambda)) - Z^\lambda],
\end{align*}
\]

(25a) (25b)

where \( \lambda > 0 \) is the relaxation coefficient.

- When \( \lambda = 0 \), the above system coincides with a reduced form of the Born-Infeld equations, or more accurately, with a plane-wave subset of the augmented Born-Infeld system by Brenier (2004).

- The eigenvalues \((Z^\lambda, W^\lambda)\) of (25), both linearly degenerate, are respectively associated with the strict Riemann invariants \(W^\lambda\) and \(Z^\lambda\).
From the diagonal to the conservative form

- For all $\lambda > 0$, we recover

$$\partial_t U(W^\lambda, Z^\lambda) + \partial_x F(W^\lambda, Z^\lambda) = 0 \quad (26)$$

by means of a nonlinear combination.

- For all $\lambda > 0$, the Born-Infeld relaxation system (25) is equivalent to the system

$$\partial_t U^\lambda + \partial_x (U^\lambda (1 - U^\lambda) G^\lambda) = 0, \quad (27a)$$

$$\partial_t G^\lambda + (G^\lambda)^2 \partial_x U^\lambda = \lambda [g(U^\lambda) - G^\lambda]. \quad (27b)$$

The second equation can transformed into the conservative form

$$\partial_t ((1 - 2U^\lambda)G^\lambda) - \partial_x (U^\lambda (1 - U^\lambda) (G^\lambda)^2) = \lambda (1 - 2U^\lambda) [g(U^\lambda) - G^\lambda].$$
Chapman-Enskog analysis

- Inserting the formal expansion

\[ G^\lambda = g(U^\lambda) + \lambda^{-1} g_1^\lambda + O(\lambda^{-2}) \]  \hspace{1cm} (28)

into the relaxation system yields the equivalent equation

\[ \partial_t U^\lambda + \partial_x f(U^\lambda) = \frac{1}{\lambda} \partial_x \left\{ - [f'(U^\lambda) - w(U^\lambda)][f'(U^\lambda) - z(U^\lambda)] \partial_x U^\lambda \right\}. \]

- A sufficient condition for this to be a dissipative approximation to the original equation is that the subcharacteristic condition

\[ f'(u) \in [w(u), z(u)], \]  \hspace{1cm} (29)

holds true for all \( u \) in the range of the problem at hand.

- Notation: \([a, b] = \{ra + (1 - r)b, r \in [0, 1]\} \).
Geometric interpretation

The subcharacteristic condition

\[ f'(u) \in [w(u), z(u)] = \left[ \frac{f(u) - f(0)}{u - 0}, \frac{f(1) - f(u)}{1 - u} \right] \]  (30)

can be seen as a comparison between the slopes of 3 lines.
Eligible flux functions

- Two practical criteria
  - The subcharacteristic condition (29) is satisfied at \( u \in ]0, 1[ \) if and only if the functions \( w(.) \) and \( z(.) \) are
    - decreasing at \( u \) in the case \( g(.) > 0 \),
    - increasing at \( u \) in the case \( g(.) < 0 \).
  - The subcharacteristic condition (29) is satisfied at \( u \in ]0, 1[ \) if and only if
    \[
    g'(u) \in \left[-\frac{g(u)}{u}, \frac{g(u)}{1-u}\right].
    \] (31)

- Examples
  - convex or concave functions \( f \) with \( f(0) = f(1) = 0 \) and \( 0 \not\in f([0, 1[ \) ;
  - for \( n \geq 2 \), the function
    \[
    f(u) = u(1-u) \left[1 + \frac{\sin(2\pi nu)}{2\pi n}\right]
    \] (32)
    is admissible, although \( f'' \) does not have a constant sign.
Solving Riemann problems

Starting from \( v_L = (W_L, Z_L) \) and \( v_R = (W_R, Z_R) \), we have

\[
v(t, x) = v_L 1_{\{x < s_L t\}} + \tilde{v} 1_{\{s_L t < x < s_R t\}} + v_R 1_{\{x > s_R t\}},
\]

with

\[
\begin{align*}
s_L &= \min(W_L, Z_L) \leq 0, \\
s_R &= \max(W_R, Z_R) \geq 0,
\end{align*}
\]

\[
\tilde{v} = (\tilde{W}, \tilde{Z}) = (W_L^- + W_R^+, Z_L^- + Z_R^+).
\]
Deducing maximum principles

- The intermediate velocities \( \tilde{v} = (\tilde{W}, \tilde{Z}) \) obey the local maximum principle
  \[
  \tilde{W} \in [W_L, W_R], \quad \tilde{Z} \in [Z_L, Z_R]
  \] (35)
  unconditionally.

- The intermediate fraction
  \[
  \tilde{U} = U(\tilde{W}, \tilde{Z}) = \frac{\tilde{Z}}{\tilde{Z} - \tilde{W}}
  \] (36)
  obeys the local maximum principle \( \tilde{U} \in [u_L, u_R] \) provided that
  \[
  [w(u_R) - w(u_L)][z(u_R) - z(u_L)] \geq 0,
  \] (37)
  which is implied by the subcharacteristic condition;

- The intermediate fraction \( \tilde{U} \) obeys the global maximum principle \( \tilde{U} \in [0, 1] \) unconditionally.
First-order explicit scheme

- **Update formulae**

\[
    u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} [\tilde{F}(u_{i}^{n}, u_{i+1}^{n}) - \tilde{F}(u_{i}^{n}, u_{i-1}^{n})],
\]

with \( \tilde{F}(u_{L}, u_{R}) = F(\tilde{W}, \tilde{Z}) = \)

\[
    \begin{cases}
    \frac{w(u_{L})z(u_{R})}{z(u_{R}) - w(u_{L})} & \text{if } g(.) < 0, \\
    \frac{w(u_{R})z(u_{L})}{z(u_{L}) - w(u_{R})} & \text{if } g(.) > 0.
    \end{cases}
\]

- If the subcharacteristic condition is met, then the Born-Infeld flux \( \tilde{F}(u_{L}, u_{R}) \) is monotone, i.e.,

\[
    \frac{\partial \tilde{F}}{\partial u_{L}}(u_{L}, u_{R}) \geq 0, \quad \frac{\partial \tilde{F}}{\partial u_{R}}(u_{L}, u_{R}) \leq 0.
\]
Numerical results

- The flux and data

\[ f(u) = u(1 - u)(1 + u), \quad (41a) \]

\[ (u_L, u_R) = (0.5, 1) \quad (41b) \]

give rise to a shock wave propagating at the speed

\[ \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{0 - 0.375}{1 - 0.5} = -0.75. \quad (42) \]

- We compare the Born-Infeld relaxation with two variants of the Jin-Xin relaxation, namely,

  - JX1: uniform parameter

    \[ a = \max_{i \in \mathbb{Z}} |f'(u^n_i)|; \quad (43) \]

  - JX2: local parameter

    \[ a_{i+1/2} = \max \{|f'(u^n_i)|, |f'(u^n_{i+1})|\}. \quad (44) \]
Solution snapshot

$T = 8; \text{ CFL} = 0.5$
Convergence study

\[ JX1 = 0.99992; JX2 = 1.00021; BI = 1.00573 \]
Recapitulative review

- Jin-Xin relaxation
  - designed for any $f$
  - diagonal variables have no physical meaning, except for kinetic interpretation
  - require a parameter $a$, subject to subchar. cond.
  - tunable viscosity
  - local max. princ. on $u$ thanks to subchar. cond.; global max. princ. on $u$ under cond. on $a$
  - no max. princ. on $w$ and $z$
  - monotone numerical flux
  - discrete entropy inequality

- Born-Infeld relaxation
  - designed for $f = u(1 - u)g$
  - diagonal variables have a physical meaning, but no kinetic interpretation
  - no parameter required, subchar. cond. imposed on $f$
  - “natural” viscosity
  - local max. princ. on $u$ thanks to subchar. cond. but global max. princ. on $u$ is automatic
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1. Context and motivation
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3. The Born-Infeld relaxation
4. Applications and extensions
   - Porous media and two-phase flow
   - General scalar conservation law
   - Toward larger systems
4 Applications and extensions

4.1 Porous media and two-phase flow

Porous media with discontinuous coefficients

- Simplistic model

\[ \frac{\partial_t u}{\partial x(u(1-u)k)} = 0, \quad (45a) \]
\[ \frac{\partial_t k}{\partial x} = 0, \quad (45b) \]

with \( u \in [0, 1] \) and \( k > 0 \), studied by Seguin and Vovelle (2003).

- Set

\[ w(u, k) = (1-u)k, \quad W = (1-U)K, \quad (46a) \]
\[ z(u, k) = -uk, \quad Z = -UK, \quad (46b) \]

and consider the relaxation model

\[ \frac{\partial_t W^\lambda}{\partial x} + Z^\lambda \frac{\partial_x W^\lambda}{\partial x} = \lambda [w(U(W^\lambda, Z^\lambda), k) - W^\lambda], \quad (47a) \]
\[ \frac{\partial_t Z^\lambda}{\partial x} + W^\lambda \frac{\partial_x Z^\lambda}{\partial x} = \lambda [z(U(W^\lambda, Z^\lambda), k) - Z^\lambda], \quad (47b) \]
\[ \frac{\partial_t k}{\partial x} = 0, \quad (47c) \]

where \( U(W, Z) = Z/(Z - W) \).
Porous media with discontinuous coefficients

- **Conservative form**

\[
\begin{align*}
\partial_t U^\lambda &+ \partial_x (U^\lambda (1 - U^\lambda) K^\lambda) = 0, \\
\partial_t K^\lambda &+ (K^\lambda)^2 \partial_x U^\lambda = \lambda [k - K^\lambda], \\
\partial_t k & = 0.
\end{align*}
\]

(48a)  
(48b)  
(48c)

- **Chapman-Enskog analysis**

\[
\partial_t U^\lambda + \partial_x (U^\lambda (1 - U^\lambda) k) = \lambda^{-1} \partial_x \left\{ U^\lambda (1 - U^\lambda) k^2 \partial_x U^\lambda \right\}.
\]

(49)

Dissipative approximation, no need for a subcharacteristic condition. Actually, we already have

\[
[0, (1 - 2u)k] \subset [-uk, (1 - u)k].
\]

(50)
Compressible two-phase flow

Drift-flux model in Eulerian coordinates

\begin{align}
\partial_t (\rho) + \partial_x (\rho v) &= 0, \quad (51a) \\
\partial_t (\rho v) + \partial_x (\rho v^2 + P(q)) &= 0, \quad (51b) \\
\partial_t (\rho Y) + \partial_x (\rho Y v + \rho Y (1 - Y) \phi(q)) &= 0, \quad (51c)
\end{align}

with \( q = (\rho, \rho v, \rho Y) \). Here, \( Y \in [0, 1] \) is the gas mass-fraction and \( \phi(q) \) is the slip velocity, given by a closure law.

In addition to (51), passive transport

\begin{align}
\partial_t (\rho \alpha_k) + \partial_x (\rho \alpha_k v + \rho \alpha_k (1 - Y) \phi(q)) &= 0, \quad (52a) \\
\partial_t (\rho \beta_\ell) + \partial_x (\rho \beta_\ell v - \rho \beta_\ell Y \phi(q)) &= 0, \quad (52b)
\end{align}

of various partial component-fractions.
Compressible two-phase flow

- Lagrangian velocities

\[ w(q) = \rho(1 - Y)\phi(q), \quad z(q) = -\rho Y\phi(q), \quad g(q) = \rho\phi(q). \quad (53) \]

- Switch to Lagrangian coordinates first, work out the relaxation model, then go back to Eulerian coordinates.

\[ \partial_t (\rho)^\lambda + \partial_x (\rho v)^\lambda = 0, \quad (54a) \]
\[ \partial_t (\rho v)^\lambda + \partial_x (\rho v^2 + \Pi)^\lambda = 0, \quad (54b) \]
\[ \partial_t (\rho \Pi)^\lambda + \partial_x (\rho \Pi v + a^2 v)^\lambda = \lambda \rho (P(q^\lambda) - \Pi^\lambda), \quad (54c) \]
\[ \partial_t (\rho Y)^\lambda + \partial_x (\rho Yv + Y(1 - Y)G)^\lambda = 0, \quad (54d) \]
\[ \partial_t (\rho G)^\lambda + \partial_x (\rho Gv)^\lambda + (G^\lambda)^2 \partial_x Y^\lambda = \lambda \rho (g(q^\lambda) - G^\lambda). \quad (54e) \]

- Most “real-life” hydrodynamic laws \( \phi \) satisfy the subcharacteristic condition.
General scalar conservation law

- The homogeneous Born-Infeld system

\[
\begin{align*}
\partial_t W + Z \partial_x W &= 0, \\
\partial_t Z + W \partial_x Z &= 0
\end{align*}
\] (55a) (55b)

has an entropy-entropy flux pair

\[
\partial_t U(W,Z) + \partial_x F(W,Z) = 0
\] (56)

if and only if

\[
U(W,Z) = \frac{A(W) + B(Z)}{Z - W} \quad \text{and} \quad F(W,Z) = \frac{ZA(W) + WB(Z)}{Z - W}.
\] (57)

- Goursat equation

\[
U_W - U_Z + (Z - W)U_{WZ} = 0.
\] (58)
General scalar conservation law

- The natural idea is to relax the conservation law $\partial_t u + \partial_x f(u) = 0$ by the generalized system

\[
\begin{align*}
\partial_t W^\lambda + Z^\lambda \partial_x W^\lambda &= \lambda W_F[f(U(W^\lambda, Z^\lambda)) - F(W^\lambda, Z^\lambda)], & (59a) \\
\partial_t Z^\lambda + W^\lambda \partial_x Z^\lambda &= \lambda Z_F[f(U(W^\lambda, Z^\lambda)) - F(W^\lambda, Z^\lambda)]. & (59b)
\end{align*}
\]

- The difficulty lies in obtaining close-form expressions for $W(U, F)$ and $Z(U, F)$. Moreover, the equilibrium values

\[
w(u) = W(u, f(u)), \quad z(u) = Z(u, f(u))
\]

(60)

do not always have an obvious physical meaning but have to remain bounded.

- But an abstract framework can be worked out, in which all of the results obtained for $f = u(1 - u)g$ can be extended. Most notably, the subcharacteristic condition and the monotonicity of the numerical flux.
Examples

- $A(W) = 0$ and $B(Z) = Z$ [linear BI] lead to $U = Z/(Z - W)$ and $F = WZ/(Z - W)$, the inverse of which is

$$W(U, F) = \frac{F}{U} \quad \text{and} \quad Z(U, F) = -\frac{F}{1 - U}. \quad (61)$$

If $f(u) = u(1 - u)g(u)$, where $g$ keeps a constant sign, then $w(u)$ and $z(u)$ remain well-defined.

- $A(W) = 0$ and $B(Z) = Z^2$ [quadratic BI] lead to $U = Z^2/(Z - W)$ and $F = WZ^2/(Z - W)$, the inverse of which is

$$W(U, F) = \frac{F}{U} \quad \text{and} \quad Z(U, F) = \frac{U + \sqrt{U^2 - 4F}}{2}. \quad (62)$$

If $f(u) = -uh(u)$, where $h > 0$, then $w(u)$ and $z(u)$ remain well-defined.
Examples

- **A**($W$) = 0 and **B**($Z$) = $Z^3$ [cubic BI] lead to $U = \frac{Z^3}{(Z - W)}$ and $F = \frac{WZ^3}{(Z - W)}$, the inverse of which is

  $$W(U, F) = \frac{F}{U} \quad \text{and} \quad Z(U, F) = \text{negative root of } Z^3 - UZ + F.$$  

  (63)

  If $f(u) = uh(u)$, where $h > 0$, then $w(u)$ and $z(u)$ remain well-defined.

- The function

  $$h(u) = \frac{1}{(1 + u)^\alpha}, \quad 0 \leq \alpha \leq 1,$$

  satisfies the subcharacteristic condition for the quadratic and cubic Born-Infeld relaxations.
Solution snapshot

Quadratic Born-Infeld for $f(u) = -\frac{u}{1+u}$
Solution snapshot

Cubic Born-Infeld for $f(u) = \frac{u}{1 + u}$
Rich systems

- Prototype of a $3 \times 3$ linearly degenerate rich system

\begin{align*}
\partial_t W_1 + (W_2 + W_3) \partial_x W_1 &= 0, \quad (65a) \\
\partial_t W_2 + (W_3 + W_1) \partial_x W_2 &= 0, \quad (65b) \\
\partial_t W_3 + (W_1 + W_2) \partial_x W_3 &= 0. \quad (65c)
\end{align*}

- The entropy-entropy flux pairs are of the form (Serre, 1992)

\[\partial_t (P_j(W)q(W_j)) + \partial_x (P_j(W)q(W_j)v_j(W)) = 0, \quad (66)\]

where $q(.)$ is any function,

\[P_j(W) = -\frac{1}{(W_j - W_i)(W_j - W_k)} \quad \text{and} \quad v_j(W) = W_i + W_k. \quad (67)\]

- Another entropy-entropy flux pair is

\[\partial_t (W_1 + W_2 + W_3) + \partial_x (W_2 W_3 + W_3 W_1 + W_1 W_2) = 0. \quad (68)\]
An attempt at pressureless gas

- Taking \( q(W_2) = K_0 \) and \( 2W_2K_0 \), we get

\[
\begin{align*}
&\partial_t(K_0 \mathcal{P}_2(W)) + \partial_x(K_0 \mathcal{P}_2(W) \cdot (W_3 + W_1)) = 0, \quad (69a) \\
&\partial_t(K_0 \mathcal{P}_2(W) \cdot 2W_2) + \partial_x(K_0 \mathcal{P}_2(W) \cdot 2W_2 \cdot (W_3 + W_1)) = 0. \quad (69b)
\end{align*}
\]

- If \( W_3 + W_1 = 2W_2 \), then we formally recover

\[
\begin{align*}
&\partial_t(\rho) + \partial_x(\rho u) = 0, \quad (70a) \\
&\partial_t(\rho u) + \partial_x(\rho u^2) = 0. \quad (70b)
\end{align*}
\]

Non-strictly hyperbolic, with the resonant eigenvalue \( u \).

- The idea is therefore to supplement (69) with a third equation

\[
\begin{align*}
&\partial_t(W_1 + W_2 + W_3) + \partial_x(W_2W_3 + W_3W_1 + W_1W_2) \\
&= \lambda (2W_2 - (W_3 + W_1)). \quad (71)
\end{align*}
\]
Pressureless gas... far from vacuum

- **Conservative form**

\[
\begin{align*}
\partial_t (\rho)^\lambda + \partial_x (\rho V)^\lambda &= 0, \\
\partial_t (\rho u)^\lambda + \partial_x (\rho uV)^\lambda &= 0, \\
\partial_t (V + u/2)^\lambda + \partial_x (Vu - u^2/4 - K_0/\rho)^\lambda &= \lambda (u - V)^\lambda.
\end{align*}
\]

- **Diagonal variables**

\[
\begin{align*}
W_1 &= \frac{1}{2} (V + \sqrt{(V - u)^2 + 4K_0/\rho}) \\
W_2 &= \frac{1}{2} u \\
W_3 &= \frac{1}{2} (V - \sqrt{(V - u)^2 + 4K_0/\rho})
\end{align*}
\]

- **The eigenvalues** \(\nu_j = W_i + W_k\) coincide, at equilibrium, with

\[
\begin{align*}
\nu_1 &= u - \sqrt{K_0/\rho}, \\
\nu_2 &= u, \\
\nu_3 &= u + \sqrt{K_0/\rho}.
\end{align*}
\]
The original Born-Infeld equations

- The BI system \((6 \times 6)\)

\[
\begin{align*}
\partial_t D + \nabla \times & \left( \frac{-B + D \times P}{h} \right) = 0, \quad \nabla \cdot D = 0, \\
\partial_t B + \nabla \times & \left( \frac{D + B \times P}{h} \right) = 0, \quad \nabla \cdot B = 0,
\end{align*}
\]

(75a)

(75b)

with

\[
h = \sqrt{1 + |D|^2 + |B|^2 + |D \times B|^2}, \quad P = D \times B, \quad (76)
\]

were intended (1934) to be a nonlinear correction to the Maxwell equations.

- Designed on purpose to be hyperbolic and linearly degenerate, so as to avoid shock wave. No further microscopical theory needed.

- Additional conservation laws on \(h\) (energy density) and \(P\) (Poynting vector), but \(h\) is not a uniformly convex entropy.
The augmented Born-Infeld equations

In his works on wave-particle transition, Brenier (2004) proposed to consider the conservation laws in $h$ and $P$ as part of a larger system

$$
\partial_t D + \nabla \times \frac{-B + D \times P}{h} = 0, \quad \nabla \cdot D = 0, \quad (77a)
$$

$$
\partial_t B + \nabla \times \frac{D + B \times P}{h} = 0, \quad \nabla \cdot B = 0, \quad (77b)
$$

$$
\partial_t h + \nabla \cdot P = 0, \quad (77c)
$$

$$
\partial_t P + \nabla \cdot \frac{P \otimes P - D \otimes D - B \otimes B}{h} = \nabla \frac{1}{h}. \quad (77d)
$$

The ABI system $(10 \times 10)$ is hyperbolic, linearly degenerate, and coincides with the BI system on the submanifold defined by (76).

The uniformly convex entropy $\eta = (1 + |D|^2 + |B|^2 + |P|^2)/2h$ satisfies

$$
\partial_t \eta + \nabla \cdot \frac{(\eta h - 1)P + D \times B - (D \otimes D + B \otimes B)P}{h^2} = 0. \quad (78)
$$
Plane-wave solution to the ABI system

- Choose $x = x_1$ and look for fields that depend only on $(t, x)$. After simplification, the $(h, P)$-block becomes

$$
\partial_t h + \partial_x P_1 = 0, \quad (79a)
$$

$$
\partial_t P_1 + \partial_x \frac{P_1^2 - 1}{h} = 0. \quad (79b)
$$

- The eigenvalues

$$
\nu^- = \frac{P_1 - 1}{h} \quad \text{and} \quad \nu^+ = \frac{P_1 + 1}{h} \quad (80)
$$

are linearly degenerate and are governed by $\partial_t \nu^\pm + \nu^\pm \partial_x \nu^\pm = 0$.

- Setting formally $U = (P_1 + 1)/2$ and $G = -2/h$, we recover

$$
\partial_t U + \partial_x U(1 - U)G = 0, \quad (81a)
$$

$$
\partial_t G + G^2 \partial_x U = 0. \quad (81b)
$$