Second-order hyperbolic Fuchsian systems

Applications to Einstein vacuum spacetimes

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Fuchsian theory

Fuchsian systems:

$$Du(t,x)+N(x)u(t,x)=t\,f(t,x,u,\partial_x u,Du),$$
$$\lim_{t\searrow 0} u(t,x)=0.$$

with $D := t\partial_t$.

What does the Fuchsian theory provide?

Theory of singular PDEs which allows to construct solutions with a prescribed singular behavior.

Applications in General Relativity:

- Gowdy vacuum solutions: Kichenassamy-Rendall, Rendall.
- Polarized T^2-symmetric vacuum solutions: Isenberg-Kichenassamy.
- Cosmological models with stiff fluid/scalar field: Andersson-Rendall.
- ...

Some drawbacks of the current theory

- Often not Fuchsian equations directly. First need to identify the “leading order part” of the unknowns at the singularity.
- Often second-order equations.
- Numerical scheme for hyperbolic Fuchsian equations and proof of convergence?

Aims of the talk today:

- Tackle these drawbacks for a subset of Fuchsian equations.
- Present numerical examples.

Second-order Fuchsian equations

Class of equations

$$(D \circ D)u(t,x) + 2a(x)Du(t,x) + b(x)u(t,x) = f(t,x,u,\partial_x u, Du, \partial_x^2 u, \partial_x Du),$$

with

1. $u :]0,\delta] \times \mathbb{R} \to \mathbb{R}$, periodic in space for a $\delta > 0$,
2. a, b are smooth periodic functions on $\mathbb{R}$,
3. f is the source with certain properties, see below.

Direct generalizations:

- Systems.
- More than one spatial dimensions.

Heuristics and a singular initial value problem

Canonical two-term expansion

$$u(t,x)=\begin{cases} u_*(x)t^{-\lambda_1}+u_{**}(x)t^{-\lambda_2}+O(t^{-\lambda_2+\alpha}), & a^2(x)>b(x),\\ u_*(x)t^{-\lambda_1}\log t+u_{**}(x)t^{-\lambda_1}+O(t^{-\lambda_1+\alpha}), & a^2(x)=b(x),\end{cases}$$

at $t=0$ for $\alpha>0$, where

$$\lambda_1(x):=a(x)+\sqrt{a^2(x)-b(x)},\quad \lambda_2(x):=a(x)-\sqrt{a(x)^2-b(x)}.$$

Singular initial value problem (SIVP)

Does there exist a unique solution of the Fuchsian system in a given regularity class which obeys the canonical two-term expansion at $t=0$ for *given* **asymptotic data** $u_*(x)$, $u_{**}(x)$?

Second-order hyperbolic Fuchsian systems

Class of hyperbolic equations

Consider

$$(D \circ D)u(t,x) + 2a(x)Du(t,x) + b(x)u(t,x)$$
$$= f(t,x,u,\partial_x u,Du) + t^2 c^2(t,x)\partial_x^2 u(t,x)$$

with the speed of propagation

$$c(t,x) = t^{\beta(x)} k(t,x).$$

Here,

- $\beta(x)$ is a smooth periodic function larger than -1.
- $k(t,x)$ is a smooth positive spatially periodic function so that all derivatives have finite limits at $t = 0$.

Examples of 2nd-order hyperbolic Fuchsian equations

1. Euler–Poisson–Darboux equation:

$$(D \circ D)u - \Delta\lambda\, Du - t^2 \partial_x^2 u = 0,$$

for $\Delta\lambda \geq 0$. Equivalently $\partial_t^2 u - \partial_x^2 u = \frac{1}{t}(\Delta\lambda - 1)\partial_t u$.
SIVP: Look for solutions of the form

$$u(t,x) = \begin{cases} u_*(x) + u_{**}(x) t^{\Delta\lambda} + \dots & \Delta\lambda > 0, \\ u_*(x) \log t + u_{**}(x) + \dots & \Delta\lambda = 0, \end{cases}$$

2. (Main evolution part of the) Gowdy vacuum equations:

$$D^2 P - t^2 \partial_x^2 P = e^{2P}(DQ)^2 - t^2 e^{2P}(\partial_x Q)^2,$$

$$D^2 Q - 2k\, DQ - t^2 \partial_x^2 Q = -2(k + DP)DQ + 2t^2 \partial_x P\, \partial_x Q.$$

SIVP: Look for solutions of the form

$$P(t,x) = -k(x) \log t + P_{**}(x) + \dots,$$

$$Q(t,x) = Q_*(x) + Q_{**}(x) t^{2k(x)} + \dots.$$

Function spaces and the SIVP

The space $X_{\delta,\alpha,1}$

For a given second-order Fuchsian equation and some $\delta, \alpha > 0$, let $X_{\delta,\alpha,1}$ be the space of functions

$$w \in C^0((0,\delta], H^1(U)) \cap C^1((0,\delta], L^2(U)) \text{ and } \|w\|_{\delta,\alpha,1} < \infty,$$

with the standard norm $\|\cdot\|_{\delta,\alpha,1}$ weighted by the factor $t^{\lambda_2-\alpha}$.

SIVP

We say that u is a solution of the SIVP for given asymptotic data $u_*, u_{**} \in H^1(U)$, if u is a weak solution of the 2nd-order hyperbolic Fuchsian eq. and there exists $w \in X_{\delta,\alpha,1}$ so that

$$u(t,x) = u_{\text{canonic}}(t,x) + w(t,x).$$

The function w is called remainder.

Well-Posedness in an important case

Consider the case

$$f(t,x,u,\partial_x u,Du)=f_0(t,x)$$

for a given function f_0.

Result

For any asymptotic data $u_*,u_{**}\in H^2(U)$, there exists a unique solution of the SIVP with remainder $w\in X_{\delta,\alpha,1}$ provided:

1. we can choose $\delta,\alpha>0$ so that the matrix

$$\begin{pmatrix}\lambda_1-\lambda_2+\alpha & -\eta/2 & 0\\ -\eta/2 & \alpha & t\partial_x c-\partial_x(\lambda_1-\lambda_2)(tc\ln t)\\ 0 & t\partial_x c-\partial_x(\lambda_1-\lambda_2)(tc\ln t) & \lambda_1-\lambda_2+\alpha-1-Dc/c\end{pmatrix}$$

   is positive semidef. at each $(t,x)\in(0,\delta)\times U$ for a $\eta>0$.
2. $f_0\in X_{\delta,\alpha+\varepsilon,0}$ for some $\varepsilon>0$.
3. $\alpha<2(\beta(x)+1)-(\lambda_1(x)-\lambda_2(x))$ for all $x\in U$.

Main ingredients of the existence proof

1. Main idea: Solve a sequence of **regular initial value problems**, i.e. a sequence of approximate solutions (u_n) with initial times (τ_n) going to zero as $n \to \infty$.
2. Obtain "good" energy estimates consistent with the space $X_{\delta,\alpha,1}$ if the energy dissipation matrix is positive semidefinite.
3. Convergence:

$$\|u_n - u_m\|_{\delta,\alpha,1} \leq C|G(\tau_n) - G(\tau_m)|,$$

   where

$$G(t) := \int_0^t s^{-1}\|s^{\lambda_2-\alpha}(f_0(s) - L[u_{\text{canonic}}](s))\|_{L^2(U)} ds,$$

   and $C > 0$ independent of n. Under the hypothesis, we have $\lim_{t\to 0} G(t) = 0$. Hence (w_n) is a Cauchy sequence in $X_{\delta,\alpha,1}$.
4. Check that the limit function u satisfies the weak equation.

Main ingredients of the existence proof

Regular initial value problem (RIVP)

We say that $\tilde{u}$ is a solution of the RIVP with initial time $\tau_0 > 0$, provided (after mollification)

For $t \in (0, \tau_0)$:

$$\tilde{u}(t,x) = u_{\text{canonic}}(t,x).$$

For $t \in [\tau_0, \delta)$: $\tilde{u}$ is a smooth solution of the 2nd-order Fuchsian hyperbolic equation and initial data

$$\tilde{u}(\tau_0,x) = u_{\text{canonic}}(\tau_0,x),$$

$$\partial_t\tilde{u}(\tau_0,x) = \partial_t u_{\text{canonic}}(\tau_0,x).$$

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Comments

Comments on this result:

- Main difference to standard Fuchsian theory: approximate by smooth solutions of RIVP with an explicit convergence statement in the norm. Leads to a numerical scheme!
- Loss of regularity.

Generalizations of the rigorous results:

1. General (non-linear) sources:

   Idea: Iterate over the special case before.

   Result: If source term is locally Lipschitz in the space $X_{\delta,\alpha,1}$ and $\delta > 0$ is sufficiently small, then obtain a fixed point iteration. Strong convergence in the norm!

2. Increase regularity assumptions to control arbitrarily many derivatives in spaces $X_{\delta,\alpha,k}$ with $k \in \mathbb{N}$.

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   is positive semidef. at each $(t,x) \in (0,\delta) \times U$ for a $\eta > 0$.
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Example application: Euler-Poisson-Darboux eq.

- Consider
$$(D \circ D)u - \Delta\lambda\, Du - t^2 \partial_x^2 u = 0,$$
for a constant $\Delta\lambda \geq 0$, and find solutions of the form
$$u(t,x) = \begin{cases} u_*(x) + u_{**}(x) t^{\Delta\lambda} + \dots & \Delta\lambda > 0, \\ u_*(x) \log t + u_{**}(x) + \dots & \Delta\lambda = 0. \end{cases}$$
- It turns out: well-posedness theory applies for $0 \leq \Delta\lambda < 2$.
- Periodicity in space $\rightarrow$ explicit solutions in terms of Bessel functions. Result:

  For $0 \leq \Delta\lambda < 2$: Solutions are consistent with the canonical two-term expansion.

  For $\Delta\lambda = 2$: Solutions behave like
$$u(t,x) = u_*(x) + u_{**}(x) t^{\Delta\lambda} \log t + \dots.$$
  Second spatial derivative becomes significant at $t = 0$!

Numerics: Euler-Poisson-Darboux eq.

General numerical approach, using EPD-equation as a model:

1. Introduce time variable $\tau := \ln t$. Then equation becomes

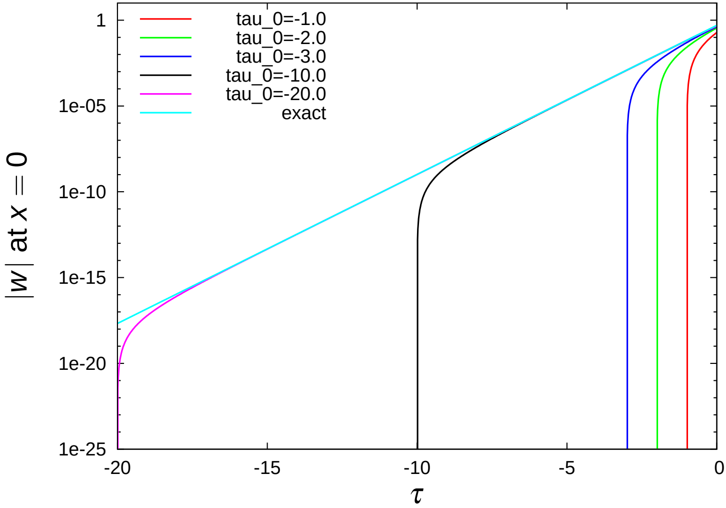
$$\partial_\tau^2 u - \Delta\lambda\, \partial_\tau u - e^{2\tau}\partial_x^2 u = 0.$$

   Singularity is "shifted to $\tau = -\infty$".
2. Write the equation for the remainder w, after having fixed asymptotic data u_*, u_{**}.
3. Solve sequence (w_n) of solutions of RIVPs with initial times $\tau_n \to -\infty$ with data

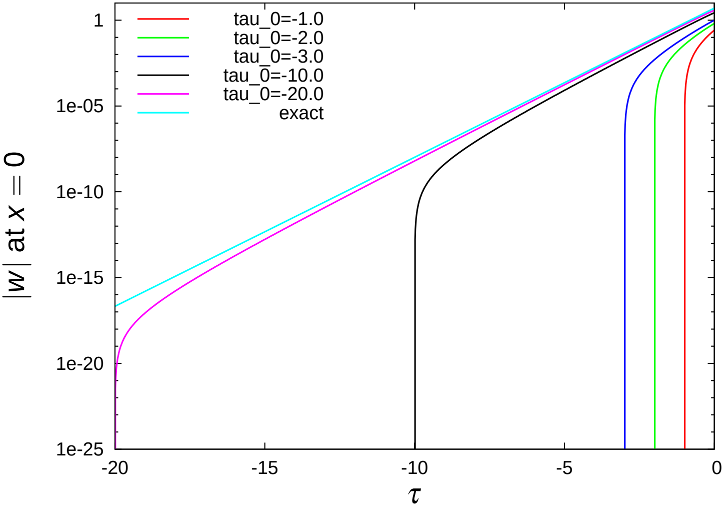
$$w_n(\tau_n) = 0, \quad \partial_\tau w_n(\tau_n) = 0,$$

   using a direct discretization of the second-order equation (as suggested by Kreiss).

Choice: $\Delta\lambda = 1.0$, $u_* = \cos x$, $u_{**} = 0$.



Choice: $\Delta\lambda = 1.9$, $u_* = \cos x$, $u_{**} = 0$.



Outlook

Desired generalizations of the current theory:

- Mixed first- and second-order hyperbolic Fuchsian systems.

Some ideas for applications:

- Gowdy coupled to a perfect fluid. Interaction of cosmological singularity and shocks?
- Numerical construction and analysis of solutions with Cauchy horizons or “pieces of Cauchy horizons” in the Gowdy case?