Second-order hyperbolic Fuchsian systems
Applications to Einstein vacuum spacetimes

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Fuchsian theory

Fuchsian systems:

\[ Du(t, x) + N(x) u(t, x) = t f(t, x, u, \partial_x u, Du), \]
\[ \lim_{t \downarrow 0} u(t, x) = 0. \]

with \( D := t \partial_t. \)

What does the Fuchsian theory provide?

Theory of singular PDEs which allows to construct solutions with a prescribed singular behavior.

Applications in General Relativity:

- Gowdy vacuum solutions: Kichenassamy-Rendall, Rendall.
- Polarized \( T^2 \)-symmetric vacuum solutions: Isenberg-Kichenassamy.
- Cosmological models with stiff fluid/scalar field: Andersson-Rendall.

...
Some drawbacks of the current theory

- Often not Fuchsian equations directly. First need to identify the “leading order part” of the unknowns at the singularity.
- Often second-order equations.
- Numerical scheme for hyperbolic Fuchsian equations and proof of convergence?

Aims of the talk today:
- Tackle these drawbacks for a subset of Fuchsian equations.
- Present numerical examples.
Second-order Fuchsian equations

Class of equations

\[(D \circ D)u(t, x) + 2a(x) Du(t, x) + b(x) u(t, x) = f(t, x, u, \partial_x u, Du, \partial_x^2 u, \partial_x Du),\]

with

1. \(u : ]0, \delta[ \times \mathbb{R} \rightarrow \mathbb{R}\), periodic in space for \(\delta > 0\),
2. \(a, b\) are smooth periodic functions on \(\mathbb{R}\),
3. \(f\) is the source with certain properties, see below.

Direct generalizations:

- Systems.
- More than one spatial dimensions.
Heuristics and a singular initial value problem

Canonical two-term expansion

\[ u(t, x) = \begin{cases} 
  u_*(x)t^{-\lambda_1} + u_{**}(x)t^{-\lambda_2} + O(t^{-\lambda_2+\alpha}), & a^2(x) > b(x), \\
  u_*(x)t^{-\lambda_1} \log t + u_{**}(x)t^{-\lambda_1} + O(t^{-\lambda_1+\alpha}), & a^2(x) = b(x), 
\end{cases} \]

at \( t = 0 \) for \( \alpha > 0 \), where

\[ \lambda_1(x) := a(x) + \sqrt{a^2(x) - b(x)}, \quad \lambda_2(x) := a(x) - \sqrt{a(x)^2 - b(x)}. \]

Singular initial value problem (SIVP)

Does there exist a unique solution of the Fuchsian system in a given regularity class which obeys the canonical two-term expansion at \( t = 0 \) for given asymptotic data \( u_*(x), u_{**}(x) \)?
Class of hyperbolic equations

Consider

\[(D \circ D)u(t, x) + 2a(x) Du(t, x) + b(x) u(t, x)\]
\[= f(t, x, u, \partial_x u, Du) + t^2 c^2(t, x) \partial^2_x u(t, x)\]

with the speed of propagation

\[c(t, x) = t^{\beta(x)} k(t, x).\]

Here,

- \(\beta(x)\) is a smooth periodic function larger than \(-1\).
- \(k(t, x)\) is a smooth positive spatially periodic function so that all derivatives have finite limits at \(t = 0\).
Examples of 2nd-order hyperbolic Fuchsian equations

1. **Euler–Poisson–Darboux equation:**

\[(D \circ D)u - \Delta \lambda Du - t^2 \partial_x^2 u = 0,\]

for \(\Delta \lambda \geq 0\). Equivalently \(\partial_t^2 u - \partial_x^2 u = \frac{1}{t}(\Delta \lambda - 1)\partial_t u\).

SIVP: Look for solutions of the form

\[u(t, x) = \begin{cases} u_*(x) + u_{**}(x)t^{\Delta \lambda} + \ldots & \Delta \lambda > 0, \\ u_*(x)\log t + u_{**}(x) + \ldots & \Delta \lambda = 0, \end{cases}\]

2. **(Main evolution part of the) Gowdy vacuum equations:**

\[D^2 P - t^2 \partial_x^2 P = e^{2P}(DQ)^2 - t^2 e^{2P}(\partial_x Q)^2,\]
\[D^2 Q - 2k DQ - t^2 \partial_x^2 Q = -2(k + DP)DQ + 2t^2 \partial_x P \partial_x Q.\]

SIVP: Look for solutions of the form

\[P(t, x) = -k(x)\log t + P_{**}(x) + \ldots,\]
\[Q(t, x) = Q_*(x) + Q_{**}(x)t^{2k(x)} + \ldots.\]
**Function spaces and the SIVP**

### The space $X_{\delta,\alpha,1}$

For a given second-order Fuchsian equation and some $\delta, \alpha > 0$, let $X_{\delta,\alpha,1}$ be the space of functions

$$w \in C^0((0, \delta], H^1(U)) \cap C^1((0, \delta], L^2(U))$$

and $\|w\|_{\delta,\alpha,1} < \infty$,

with the standard norm $\| \cdot \|_{\delta,\alpha,1}$ weighted by the factor $t^{\lambda_2 - \alpha}$.

### SIVP

We say that $u$ is a solution of the SIVP for given asymptotic data $u_*, u_{**} \in H^1(U)$, if $u$ is a weak solution of the 2nd-order hyperbolic Fuchsian eq. and there exists $w \in X_{\delta,\alpha,1}$ so that

$$u(t, x) = u_{\text{canonic}}(t, x) + w(t, x).$$

The function $w$ is called **remainder**.
Well-Posedness in an important case

Consider the case

\[ f(t, x, u, \partial_x u, Du) = f_0(t, x) \]

for a given function \( f_0 \).

**Result**

For any asymptotic data \( u_*, u_{**} \in H^2(U) \), there exists a unique solution of the SIVP with remainder \( w \in X_{\delta, \alpha, 1} \) provided:

1. we can choose \( \delta, \alpha > 0 \) so that the matrix

\[
\begin{pmatrix}
\lambda_1 - \lambda_2 + \alpha & -\eta/2 & 0 \\
-\eta/2 & \alpha & t\partial_x c - \partial_x (\lambda_1 - \lambda_2)(tc \ln t) \\
0 & t\partial_x c - \partial_x (\lambda_1 - \lambda_2)(tc \ln t) & \lambda_1 - \lambda_2 + \alpha - 1 - Dc/c \\
\end{pmatrix}
\]

is positive semidefinite at each \((t, x) \in (0, \delta) \times U\) for a \( \eta > 0 \).

2. \( f_0 \in X_{\delta, \alpha + \varepsilon, 0} \) for some \( \varepsilon > 0 \).

3. \( \alpha < 2(\beta(x) + 1) - (\lambda_1(x) - \lambda_2(x)) \) for all \( x \in U \).
Main ingredients of the existence proof

1. **Main idea:** Solve a sequence of *regular initial value problems*, i.e. a sequence of approximate solutions \((u_n)\) with initial times \((\tau_n)\) going to zero as \(n \to \infty\).

2. Obtain “good” energy estimates consistent with the space \(X_{\delta,\alpha,1}\) if the energy dissipation matrix is positive semidefinite.

3. **Convergence:**

\[
\|u_n - u_m\|_{\delta,\alpha,1} \leq C|G(\tau_n) - G(\tau_m)|,
\]

where

\[
G(t) := \int_0^t s^{-1} \left\| s^{\lambda_2 - \alpha} (f_0(s) - L[u_{\text{canonic}}](s)) \right\|_{L^2(U)} ds,
\]

and \(C > 0\) independent of \(n\). Under the hypothesis, we have \(\lim_{t \to 0} G(t) = 0\). Hence \((w_n)\) is a Cauchy sequence in \(X_{\delta,\alpha,1}\).

4. **Check that the limit function** \(u\) **satisfies the weak equation.**
Main ingredients of the existence proof

Regular initial value problem (RIVP)

We say that \( \tilde{u} \) is a solution of the RIVP with initial time \( \tau_0 > 0 \), provided (after mollification)

For \( t \in (0, \tau_0) \) :

\[
\tilde{u}(t, x) = u_{\text{canonic}}(t, x).
\]

For \( t \in [\tau_0, \delta) \) : \( \tilde{u} \) is a smooth solution of the 2nd-order Fuchsian hyperbolic equation and initial data

\[
\tilde{u}(\tau_0, x) = u_{\text{canonic}}(\tau_0, x),
\]

\[
\partial_t \tilde{u}(\tau_0, x) = \partial_t u_{\text{canonic}}(\tau_0, x).
\]
Main ingredients of the existence proof

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2. Obtain “good” energy estimates consistent with the space \(X_{\delta,\alpha,1}\) if the energy dissipation matrix is positive semidefinite.

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\|u_n - u_m\|_{\delta,\alpha,1} \leq C |G(\tau_n) - G(\tau_m)|,
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4. Check that the limit function \(u\) satisfies the weak equation.
Comments

Comments on this result:

- Main difference to standard Fuchsian theory: approximate by smooth solutions of RIVP with an explicit convergence statement in the norm. **Leads to a numerical scheme!**
- Loss of regularity.

Generalizations of the rigorous results:

1. **General (non-linear) sources:**
   - **Idea:** Iterate over the special case before.
   - **Result:** If source term is locally Lipschitz in the space $X_{\delta,\alpha,1}$ and $\delta > 0$ is sufficiently small, then obtain a fixed point iteration. Strong convergence in the norm!

2. **Increase regularity assumptions to control arbitrarily many derivatives in spaces $X_{\delta,\alpha,k}$ with $k \in \mathbb{N}$.
Consider the case
\[ f(t, x, u, \partial_x u, Du) = f_0(t, x) \]
for a given function \( f_0 \).

**Result**

For any asymptotic data \( u_*, u** \in H^2(U) \), there exists a unique solution of the SIVP with remainder \( w \in X_{\delta, \alpha, 1} \) provided:

1. we can choose \( \delta, \alpha > 0 \) so that the matrix

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\end{pmatrix}
\]

is positive semidef. at each \((t, x) \in (0, \delta) \times U\) for a \( \eta > 0 \).

2. \( f_0 \in X_{\delta, \alpha + \varepsilon, 0} \) for some \( \varepsilon > 0 \).

3. \( \alpha < 2(\beta(x) + 1) - (\lambda_1(x) - \lambda_2(x)) \) for all \( x \in U \).
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2. Increase regularity assumptions to control arbitrarily many derivatives in spaces $X_{\delta,\alpha,k}$ with $k \in \mathbb{N}$. 
Example application: Euler-Poisson-Darboux eq.

Consider

\[(D \circ D)u - \Delta \lambda \; Du - t^2 \partial_x^2 u = 0,\]

for a constant \(\Delta \lambda \geq 0\), and find solutions of the form

\[u(t, x) = \begin{cases} 
  u_{\ast}(x) + u_{\ast\ast}(x) t^{\Delta \lambda} + \ldots & \Delta \lambda > 0, \\
  u_{\ast}(x) \log t + u_{\ast\ast}(x) + \ldots & \Delta \lambda = 0. 
\end{cases}\]

It turns out: well-posedness theory applies for \(0 \leq \Delta \lambda < 2\).

Periodicity in space → explicit solutions in terms of Bessel functions. Result:

For \(0 \leq \Delta \lambda < 2\): Solutions are consistent with the canonical two-term expansion.

For \(\Delta \lambda = 2\): Solutions behave like

\[u(t, x) = u_{\ast}(x) + u_{\ast\ast}(x) t^{\Delta \lambda} \log t + \ldots.\]

Second spatial derivative becomes significant at \(t = 0\)!
General numerical approach, using EPD-equation as a model:

1. Introduce time variable $\tau := \ln t$. Then equation becomes

$$\partial_\tau^2 u - \Delta \lambda \partial_\tau u - e^{2\tau} \partial_x^2 u = 0.$$  

Singularity is “shifted to $\tau = -\infty$”.

2. Write the equation for the remainder $w$, after having fixed asymptotic data $u_*, u_{**}$.

3. Solve sequence $(w_n)$ of solutions of RIVPs with initial times $\tau_n \to -\infty$ with data

$$w_n(\tau_n) = 0, \quad \partial_\tau w_n(\tau_n) = 0,$$

using a direct discretization of the second-order equation (as suggested by Kreiss).
Choice: $\Delta \lambda = 1.0$, $u_\ast = \cos x$, $u_{**} = 0$. 

![Graph](image_url)

- $|w|$ at $x = 0$
- $\tau_0=-1.0$
- $\tau_0=-2.0$
- $\tau_0=-3.0$
- $\tau_0=-10.0$
- $\tau_0=-20.0$
- exact
Choice: $\Delta \lambda = 1.9$, $u_* = \cos x$, $u_{**} = 0$. 

![Graph showing the behavior of $|w|$ at $x = 0$ for different values of $\tau_0$. The graph includes a legend indicating the lines correspond to $\tau_0 = -1.0$, $\tau_0 = -2.0$, $\tau_0 = -3.0$, $\tau_0 = -10.0$, and $\tau_0 = -20.0$. The exact solution is also marked.]
Desired generalizations of the current theory:

- Mixed first- and second-order hyperbolic Fuchsian systems.

Some ideas for applications:

- Gowdy coupled to a perfect fluid. Interaction of cosmological singularity and shocks?
- Numerical construction and analysis of solutions with Cauchy horizons or “pieces of Cauchy horizons” in the Gowdy case?