The Stability of the Euler-Einstein System with a Positive Cosmological Constant

Jared Speck
jspeck@math.princeton.edu

University of Cambridge

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Indices

Latin (spatial) indices $a, b, i, j, k \in \{1, 2, 3\}$
Greek (spacetime) indices $\alpha, \beta, \kappa, \mu, \nu \in \{0, 1, 2, 3\}$
Summation convention: $u_a u^a \stackrel{\text{def}}{=} u_1 u^1 + u_2 u^2 + u_3 u^3$, etc.

$\partial_t = \partial_0$
$\partial = (\partial_t, \partial_1, \partial_2, \partial_3)$
$\bar{\partial} = (\partial_1, \partial_2, \partial_3)$

Einstein equations (for a Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$ of signature (-,+,+,+))

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

- We fix $\Lambda > 0$
- Bianchi identities imply $D_\mu T^{\mu\nu} = 0$
- Our spacetimes will have topology $[0, T] \times \mathbb{T}^3$

Why $\Lambda > 0$?

Answer # 1: Einstein was the first to envision Λ : he was looking for static solutions.

Answer # 2: In 1929, Hubble formulated an empirical “law," which was based on observations of the redshift effect, and which suggested that the universe is expanding.

Hubble’s “law:" galaxies are receding from Earth, and their velocities are proportional to their distances from it.

In the 1990’s: data from type Ia supernovae and the Cosmic Microwave Background suggested accelerated expansion.

$\Lambda > 0 \implies \exists$ solutions with accelerated expansion. e.g. $g = -dt^2 + e^{2(\sqrt{\Lambda/3})t} \sum_{a=1}^{3} (dx^a)^2$

The notion of “stability"

- Need a background solution (Ours will be FLRW type)
- Initial value problem formulation of Einstein equations; We will use wave coordinates (de Donder 1921, Choquet-Bruhat, 1952)
- Goal: show that if we slightly perturb the background data, then the resulting solution exists for all $t \geq 0$ and that the spacetime is future causally geodesically complete
- We show asymptotic stability (convergence as $t \to \infty$)
- Our proofs are based on energy estimates for quasilinear wave equations and the Euler equations

Previous stability results

For $\Lambda = 0$:

- Vacuum Einstein using maximal foliation (Christodoulou & Klainerman, 1993)
- Einstein-Maxwell (Zipser, 2000)
- Vacuum Einstein using double-null foliation (Klainerman & Nicolò, 2003)
- Einstein-scalar field using wave coordinates (Lindblad & Rodnianski, 2004-2005)
- Einstein-Maxwell-scalar field in $1 + n$ dimensions (Loizelet, 2008)
- Vacuum Einstein for more general data (Bieri, 2009)

Previous stability results cont.

For $\Lambda > 0$ using the conformal method

- Vacuum Einstein (Friedrich, 1986)
- Einstein-Maxwell & Einstein-Yang-Mills (Friedrich, 1991)
- Vacuum Einstein in $(1+n)$ dimensions, n odd (Anderson, 2005)

For $\Lambda = 0$ using wave coordinates

- Einstein-scalar field with a potential $V(\Phi)$ in $(1+n)$ dimensions (Ringström, 2008)
  $\Box\Phi = V'(\Phi)$
  $V(\Phi)$ emulates $\Lambda > 0$: $V(0) > 0$, $V'(0) = 0$, $V''(0) > 0$

Previous stability results cont.

- Cosmological-Newtonian gravity with $\Lambda > 0$ (Brauer, Rendall, & Reula, 1994)
- Irrotational Euler-Einstein with $\Lambda > 0$ (Rodnianski & Speck, 2009)

The modified Euler-Einstein system

$h_{jk} \stackrel{\text{def}}{=} e^{-2\Omega} g_{jk}, \; P \stackrel{\text{def}}{=} e^{3(1+c_s^2)\Omega} p \;\; W \stackrel{\text{def}}{=} 3c_s^2, \;\; H^2 \stackrel{\text{def}}{=} \frac{\Lambda}{3}$

$\hat{\Box}_g \stackrel{\text{def}}{=} g^{\alpha\beta}\partial_\alpha\partial_\beta;$ the "$\triangle$" terms are error terms

$$\hat{\Box}_g(g_{00}+1) = 5H\partial_t g_{00} + 6H^2(g_{00}+1) + \triangle_{00}$$

$$\hat{\Box}_g g_{0j} = 3H\partial_t g_{0j} + 2H^2 g_{0j} - Hg^{ab}\Gamma_{ajb} + \triangle_{0j}$$

$$\hat{\Box}_g h_{jk} = 3H\partial_t h_{jk} + \triangle_{jk}$$

$$u^\nu\partial_\nu(P-\bar{p}) + (1+c_s^2)\left(\frac{-1}{u_0}\right)Pu_a\partial_t u^a + (1+c_s^2)P\partial_a u^a = \triangle$$

$$u^\nu\partial_\nu u^j + \frac{c_s^2}{(1+c_s^2)P}\Pi^{j\alpha}\partial_\alpha(P-\bar{p}) = (W-2)\omega u^j + \triangle^j$$

Norms ($N \geq 3, 0 < q \ll 1$, $\Omega(t) \sim t\sqrt{\Lambda/3}$, $h_{jk} = e^{-2\Omega} g_{jk}$)

$$\underline{\mathfrak{S}}_{g_{00}+1;N} \stackrel{\text{def}}{=} e^{q\Omega}\|\partial_t g_{00}\|_{H^N} + e^{(q-1)\Omega}\|\bar{\partial} g_{00}\|_{H^N} + e^{q\Omega}\|g_{00} + 1\|_{H^N}$$

$$\underline{\mathfrak{S}}_{g_{0*};N} \stackrel{\text{def}}{=} \sum_{j=1}^{3} \left(e^{(q-1)\Omega}\|\partial_t g_{0j}\|_{H^N} + e^{(q-2)\Omega}\|\bar{\partial} g_{0j}\|_{H^N} + e^{(q-1)\Omega}\|g_{0j}\|_{H^N} \right)$$

$$\underline{\mathfrak{S}}_{h_{**};N} \stackrel{\text{def}}{=} \sum_{j,k=1}^{3} \left(e^{q\Omega}\|\partial_t h_{jk}\|_{H^N} + e^{(q-1)\Omega}\|\bar{\partial} h_{jk}\|_{H^N} + \|\bar{\partial} h_{jk}\|_{H^{N-1}} \right)$$

$$\underline{U}_N \stackrel{\text{def}}{=} \sqrt{\sum_{j=1}^{3} e^{2(1+q)\Omega}\|u^j\|_{H^N}^2}$$

$$\underline{S}_N \stackrel{\text{def}}{=} e^{\Omega}\|u^j\|_{H^N} + \|P - \bar{p}\|_{H^N}$$

$$\underline{Q}_N \stackrel{\text{def}}{=} \underline{\mathfrak{S}}_{g_{00};N} + \underline{\mathfrak{S}}_{g_{0*};N} + \underline{\mathfrak{S}}_{h_{**};N} + \underline{U}_{N-1} + \underline{S}_N$$

The global existence theorem

Theorem

(J.S. 2010) Assume that $N \geq 3$ *and* $0 < c_s^2 < 1/3$. *Then there exist* $\epsilon_0 > 0$ *and* $C > 1$ *such that for all* $\epsilon \leq \epsilon_0$, *if* $Q_N(0) \leq C^{-1}\epsilon$, *then there exists a global future causal geodesically complete solution to the modified Euler-Einstein system. Furthermore,*

$$Q_N(t) \leq \epsilon$$

holds for all $t \geq 0$.

Asymptotics of the solution

Theorem

(J.S. 2010) Under the assumptions of the global existence theorem, with the additional assumption $N \geq 5$, *there exist* $q > 0$, *a smooth Riemann metric* $g_{jk}^{(\infty)}$ *with corresponding Christoffel symbols* $\Gamma_{ijk}^{(\infty)}$ *and inverse* $g^{jk}_{(\infty)}$ *on* $\mathbb{T}^3$ *such that*

$$
\begin{aligned}
\|e^{-2\Omega}g_{jk}(t,\cdot) - g_{jk}^{(\infty)}\|_{H^N} &\leq C\epsilon e^{-qHt} \\
\|e^{2\Omega}g^{jk}(t,\cdot) - g^{jk}_{(\infty)}\|_{H^N} &\leq C\epsilon e^{-qHt} \\
\|e^{-2\Omega}\partial_t g_{jk}(t,\cdot) - 2\omega g_{jk}^{(\infty)}\|_{H^N} &\leq C\epsilon e^{-qHt} \\
\|g_{0j}(t,\cdot) - H^{-1}g^{ab}_{(\infty)}\Gamma_{ajb}^{(\infty)}\|_{H^{N-3}} &\leq C\epsilon e^{-qHt} \\
\|\partial_t g_{0j}(t,\cdot)\|_{H^{N-3}} &\leq C\epsilon e^{-qHt}
\end{aligned}
$$

Asymptotics cont.

Theorem

(Continued)

$$
\begin{aligned}
\|g_{00} + 1\|_{H^N} &\leq C_\epsilon e^{-qHt} \\
\|\partial_t g_{00}\|_{H^N} &\leq C_\epsilon e^{-qHt} \\
\|e^{-2\Omega} K_{jk}(t, \cdot) - \omega g_{jk}^{(\infty)}\|_{H^N} &\leq C_\epsilon e^{-qHt}
\end{aligned}
$$

In the above inequality, K_{jk} is the second fundamental form of the hypersurface $\{t = const\}$.
Similar results hold for the fluid variables.

The energies

Basic idea of proof: analyze energies that are $\sim$ to the norms

Lemma

If Q_N is small enough, then the energies and norms are equivalent:

$$\mathfrak{E}_{g_{00}+1;N} \sim \mathfrak{S}_{g_{00}+1;N}$$
$$\mathfrak{E}_{g_{0*};N} \sim \mathfrak{S}_{g_{0*};N}$$
$$\mathfrak{E}_{h_{**};N} \sim \mathfrak{S}_{h_{**};N}$$
$$E_N + U_{N-1} \sim S_N$$
$$\mathcal{Q}_N \sim Q_N$$

Gronwall estimates for energies

Lemma

The following system of inequalities holds for the modified equations. Furthermore, it features small data global bounds.

$$\underline{U}^2_{N-1}(t) \leq \underline{U}^2_{N-1}(t_1) + \int_{\tau=t_1}^{t} \overbrace{2(W-1+q)}^{<0} e^{(1+q)H\tau}\,\underline{U}^2_{N-1} + C\mathcal{Q}_N \underline{U}_{N-1}\,d\tau$$

$$\underline{E}^2_N(t) \leq \underline{E}^2_N(t_1) + C\int_{t_1}^{t} e^{-qH\tau}\mathcal{Q}^2_N\,d\tau$$

$$\underline{\mathfrak{E}}^2_{g_{00}+1;N}(t) \leq \underline{\mathfrak{E}}^2_{g_{00}+1;N}(t_1) + \int_{t_1}^{t} -4qH\underline{\mathfrak{E}}^2_{g_{00}+1;N} + Ce^{-qH\tau}\mathcal{Q}_N\underline{\mathfrak{E}}_{g_{00}+1;N}\,d\tau$$

$$\underline{\mathfrak{E}}^2_{g_{0*};N}(t) \leq \underline{\mathfrak{E}}^2_{g_{0*};N}(t_1) + \int_{t_1}^{t} -4qH\underline{\mathfrak{E}}^2_{g_{0*};N} + C\underline{\mathfrak{E}}_{h_{**};N}\underline{\mathfrak{E}}_{g_{0*};N} + Ce^{-qH\tau}\mathcal{Q}_N\underline{\mathfrak{E}}_{g_{0*};N}\,d\tau$$

$$\underline{\mathfrak{E}}^2_{h_{**};N}(t) \leq \underline{\mathfrak{E}}^2_{h_{**};N}(t_1) + \int_{t_1}^{t} He^{-qH\tau}\underline{\mathfrak{E}}^2_{h_{**};N} + Ce^{-qH\tau}\mathcal{Q}_N\underline{\mathfrak{E}}_{h_{**};N}\,d\tau$$

The divergence of $\dot{J}$; $\dot{W} = (\dot{P}, \dot{u}^1, \dot{u}^2, \dot{u}^3)$

$$
\begin{aligned}
\partial_\mu\left(\dot{J}^\mu(\dot{\mathbf{W}}, \dot{\mathbf{W}})\right) = {} & \left(\partial_\mu\left[\frac{u^\mu}{(1+c_s^2)P}\right]\right)\dot{P}^2 + \left(\partial_\mu\left[\frac{(1+c_s^2)Pu^\mu}{c_s^2}\right]\right) g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta \\
& + \frac{(1+c_s^2)Pu^a}{c_s^2}(\partial_a g_{\alpha\beta})\dot{u}^\alpha\dot{u}^\beta + \frac{(1+c_s^2)Pu^0}{c_s^2}(\partial_t g_{00})(\dot{u}^0)^2 \\
& + \frac{2(1+c_s^2)Pu^0}{c_s^2}(\partial_t g_{0\alpha})\dot{u}^0\dot{u}^\alpha + \frac{2(1+c_s^2)P(u^0-1)}{c_s^2}(\partial_t g_{ab})\dot{u}^a\dot{u}^b \\
& - 2\left(\partial_t\left[\frac{u_a}{u_0}\right]\right)\dot{u}^a\dot{P} + \frac{(1+c_s^2)P}{c_s^2}(\partial_t g_{ab} - 2\omega g_{ab})\dot{u}^a\dot{u}^b \\
& + \underbrace{\frac{2(1+c_s^2)P}{c_s^2}(W-1)\omega g_{ab}\dot{u}^a\dot{u}^b}_{<0} + \frac{2\mathfrak{F}}{(1+c_s^2)P}\dot{P} + \frac{2(1+c_s^2)P}{c_s^2} g_{\alpha\beta}\mathfrak{G}^\alpha\dot{u}^\beta
\end{aligned}
$$

Control of $\|\partial_\mu \dot{J}^\mu\|_{L^1}$

Using formula for $\dot{J}^\mu$, we can show:

$$\|\partial_\mu \dot{J}^\mu\|_{L^1} \leq Ce^{-q\Omega} Q_N^2$$

Comparison with flat spacetime

- Christodoulou’s monograph “The Formation of Shocks in 3-Dimensional Fluids" shows that on the Minkowski spacetime background, shock singularities can form in solutions to the Euler equations arising from data that are arbitrarily close to that of a uniform quiet state.
- Conclusion: Exponentially expanding spacetimes can stabilize fluids.

Topologies beyond that of $\mathbb{T}^3$

- The study of wave equations arising from metrics featuring accelerated expansion is a “very local" problem.
- A patching argument can “very likely" be used to allow for many of the topologies considered by Ringström: Unimodular Lie Groups different from $SU2$; $\mathbb{H}^3$; $\mathbb{H}^2 \times \mathbb{R}$; $\cdots$

Future directions

- Other equations of state
- Sub-exponential expansion rates

$g_{\mu\nu}$ building block energies

If $\alpha > 0$, $\beta \geq 0$, and ϕ is a solution to

$$\hat{\Box}_g \phi = \alpha H \partial_t \phi + \beta H^2 \phi + F,$$

then we control solutions to this equation using an energy

$$\mathcal{E}^2_{(\gamma,\delta)}[\phi, \partial\phi] \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{T}^3} \{ -g^{00}(\partial_t \phi)^2 + g^{ab}(\partial_a \phi)(\partial_b \phi) - 2\gamma H g^{00} \phi \partial_t \phi + \delta H^2 \phi^2 \} \, d^3x$$

γ, δ can be adjusted so that $\exists \eta > 0$ such that

$$\frac{d}{dt}(\mathcal{E}^2_{(\gamma,\delta)}[\phi, \partial\phi]) \leq -\eta H \mathcal{E}^2_{(\gamma,\delta)}[\phi, \partial\phi] + \int_{\mathbb{T}^3} \Big\{ -(\partial_t \phi + \gamma H \phi) F + \triangle_{\mathcal{E};(\gamma,\delta)}[\phi, \partial\phi] \Big\} \, d^3x,$$

$$\begin{aligned}
\triangle_{\mathcal{E};(\gamma,\delta)}[\phi, \partial\phi] = & -\gamma H(\partial_a g^{ab}) \phi \partial_b \phi - 2\gamma H(\partial_a g^{0a}) \phi \partial_t \phi - 2\gamma H g^{0a}(\partial_a \phi)(\partial_t \phi) \\
& - (\partial_a g^{0a})(\partial_t \phi)^2 - (\partial_a g^{ab})(\partial_b \phi)(\partial_t \phi) - \frac{1}{2}(\partial_t g^{00})(\partial_t \phi)^2 \\
& + \left(\frac{1}{2} \partial_t g^{ab} + H g^{ab} \right) (\partial_a \phi)(\partial_b \phi) - \gamma H(\partial_t g^{00}) \phi \partial_t \phi - \gamma H(g^{00} + 1)(\partial_t \phi)^2.
\end{aligned}$$

Energies for $g_{\mu\nu}$

$$\mathfrak{E}^2_{g_{00}+1;N} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N} e^{2q\Omega}\mathcal{E}^2_{(\gamma_{00},\delta_{00})}[\partial_{\vec{\alpha}}(g_{00}+1), \partial(\partial_{\vec{\alpha}}(g_{00}+1))]$$

$$\mathfrak{E}^2_{g_{0*};N} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N} \sum_{j=1}^{3} e^{2(q-1)\Omega}\mathcal{E}^2_{(\gamma_{0*},\delta_{0*})}[\partial_{\vec{\alpha}} g_{0j}, \partial(\partial_{\vec{\alpha}} g_{0j})]$$

$$\mathfrak{E}^2_{h_{**};N} \stackrel{\text{def}}{=} \sum_{|\vec{\alpha}|\leq N} \left\{ \sum_{j,k=1}^{3} e^{2q\Omega}\mathcal{E}^2_{(0,0)}[0, \partial(\partial_{\vec{\alpha}} h_{jk})] + \frac{1}{2}\int_{\mathbb{T}^3} c_{\vec{\alpha}} H^2(\partial_{\vec{\alpha}} h_{jk})\, d^3x \right\}$$