A constrained scheme for Einstein equations in numerical relativity

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based on collaboration with
S. Bonazzola, I. Cordero-Carrión, P. Cerdá-Durán, H. Dimmelmeier,
É. Gourgoulhon & J.L. Jaramillo.

1 Introduction
- Constraints issues in 3+1 formalism
- Motivation for a fully-constrained scheme

2 Description of the formulation and strategy
- Covariant 3+1 conformal decomposition
- Einstein equations in Dirac gauge and maximal slicing
- Integration strategy

3 Non-uniqueness problem
- CFC and FCF
- A cure in CFC
- New constrained formulation
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3+1 FORMALISM

Decomposition of spacetime and of Einstein equations

**Evolution Equations:**

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = -D_i D_j N + N R_{ij} - 2N K_{ik} K^k_j + N [K K_{ij} + 4\pi((S - E)\gamma_{ij} - 2S_{ij})]\]

\[
K_{ij} = \frac{1}{2N} \left( \frac{\partial \gamma_{ij}}{\partial t} + D_i \beta^i + D^i \beta_i \right).
\]

**Constraint Equations:**

\[
R + K^2 - K_{ij} K^{ij} = 16\pi E,
\]

\[
D_j K^{ij} - D^i K = 8\pi J^i.
\]

\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 \, dt^2 + \gamma_{ij} \left( dx^i + \beta^i \, dt \right) \left( dx^j + \beta^j \, dt \right)
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**Evolution equations:**

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**Constraint violation**

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

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\[\downarrow\]

Appearance of constraint violating modes

Some cures have been investigated (and work):
- constraint-preserving boundary conditions (Lindblom *et al.* 2004)
- constraint projection (Holst *et al.* 2004)
- Using of constraint damping terms and adapted gauges
  $\Rightarrow$ BSSN or Generalized Harmonic approaches.
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Some reasons not to solve constraints

- Computational cost of usual elliptic solvers...
- Few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems
- Definition of boundary conditions at finite distance and at black hole excision boundary
Motivations for a Fully-Constrained Scheme

“Alternate” approach (although most straightforward)


⇒ Rather popular for 2D applications, but disregarded in 3D
Still, many advantages:
  - constraints are verified!
  - elliptic systems have good stability properties
  - easy to make link with initial data
  - evolution of only two scalar-like fields ...
Description of the formulation and strategy

Bonazzola et al. (2004)
Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura formalism)

Dynamical degrees of freedom of the gravitational field:
York (1972) : they are carried by the conformal “metric”

\[ \hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \quad \text{with} \quad \gamma := \det \gamma_{ij} \]

Problem
\[ \hat{\gamma}_{ij} = \text{tensor density of weight } -2/3 \]
not always easy to deal with tensor densities... not really covariant!
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INTRODUCTION OF A FLAT METRIC

We introduce $f_{ij}$ (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of $\gamma_{ij}$, and $\mathcal{D}_i$ the associated covariant derivative.

**Define:**

$$\tilde{\gamma}_{ij} := \Psi^{-\frac{1}{4}} \gamma_{ij} \text{ or } \gamma_{ij} := \Psi^{\frac{1}{4}} \tilde{\gamma}_{ij}$$

with

$$\Psi := \left( \frac{\gamma}{f} \right)^{1/12}$$

$$f := \det f_{ij}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of $\gamma_{ij}$ and verifies $\det \tilde{\gamma}_{ij} = f$

$\Rightarrow$ no more tensor densities: only tensors.

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.
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**Generalized Dirac gauge**

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

\[
D_j \tilde{\gamma}^{ij} = D_j h^{ij} = 0
\]

where \( D_j \) denotes the covariant derivative with respect to the flat metric \( f_{ij} \).

Compare

- minimal distortion (Smarr & York 1978): \( D_j (\partial \tilde{\gamma}^{ij} / \partial t) = 0 \)
- pseudo-minimal distortion (Nakamura 1994): \( D^j (\partial \tilde{\gamma}_{ij} / \partial t) = 0 \)

*Notice:* Dirac gauge \( \iff \) BSSN connection functions vanish: \( \tilde{\Gamma}^i = 0 \)
**Generalized Dirac gauge properties**

- $h^{ij}$ is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, $h^{ij}$ is asymptotically traceless
- $3R_{ij}$ is a simple Laplacian in terms of $h^{ij}$
- $3R$ does not contain any second-order derivative of $h^{ij}$
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($D_j \left[ (\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is locally strongly well posed
- the Conformal Flatness Condition (CFC) verifies the Dirac gauge $\Rightarrow$ possibility to easily use initial data for binaries now available
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EINSTEIN EQUATIONS

Dirac gauge and maximal slicing ($k = 0$)

**Hamiltonian constraint**

\[
\Delta (\Psi^2 N) = \Psi^6 N \left(4\pi S + \frac{3}{4} \tilde{A}_{kl} A^{kl}\right) - h^{kl} D_k D_l (\Psi^2 N) + \Psi^2 \left[N \left(\frac{1}{16} \tilde{\gamma}^{kl} D_k h^{ij} D_l \tilde{\gamma}_{ij} - \frac{1}{8} \tilde{\gamma}^{kl} D_k h^{il} D_j \tilde{\gamma}_{kl} + 2 \tilde{D}_k \ln \Psi \tilde{D}^k \ln \Psi\right) + 2 \tilde{D}_k \ln \Psi \tilde{D}^k N\right]
\]

**Momentum constraint**

\[
\Delta \beta^i + \frac{1}{3} D^i \left(D_j \beta^j\right) = 2 A^{ij} D_j N + 16 \pi N \Psi^4 j^i - 12 N A^{ij} D_j \ln \Psi - 2 \Delta^i_{kl} N A^{kl}
\]

\[
- h^{kl} D_k D_l \beta^i - \frac{1}{3} h^{ik} D_k D_l \beta^l
\]

**Trace of dynamical equations**

\[
\Delta N = \Psi^4 N \left[4\pi (E + S) + \tilde{A}_{kl} A^{kl}\right] - h^{kl} D_k D_l N - 2 \tilde{D}_k \ln \Psi \tilde{D}^k N
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**Dirac Gauge and Maximal Slicing ($K = 0$)**

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Dirac gauge and maximal slicing \((K = 0)\)

Evolution equations

\[
\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4} \Delta h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta \mathcal{L}_\beta h^{ij} = S^{ij}
\]

6 components - 3 Dirac gauge conditions - \((\det \tilde{\gamma}^{ij} = 1)\)

2 degrees of freedom

\[
- \frac{\partial^2 A}{\partial t^2} + \Delta A = S_A
\]

\[
- \frac{\partial^2 \tilde{B}}{\partial t^2} + \Delta \tilde{B} = S_{\tilde{B}}
\]

with \(A\) and \(\tilde{B}\) two scalar potentials representing the degrees of freedom.
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Dirac gauge and maximal slicing ($K = 0$)

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**Dirac gauge and maximal slicing** \((K = 0)\)

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with \(A\) and \(\tilde{B}\) two scalar potentials representing the degrees of freedom.
**Integration procedure**

Everything is known on slice $\Sigma_t$

\[ \Downarrow \]

Evolution of $A$ and $\tilde{B}$ to next time-slice $\Sigma_{t+dt}$ ( + hydro)

\[ \Downarrow \]

Deduce $h^{ij}(t + dt)$ from Dirac and trace-free conditions

\[ \Downarrow \]

Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t + dt)$ and $\tilde{\gamma}^{ij}(t + dt)$.

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Iterate on the system of elliptic equations for $\mathcal{N}, \Psi^2 \mathcal{N}$ and $\beta^i$ on $\Sigma_{t+dt}$.
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Non-uniqueness problem

Cordero-Carrión et al. (2009)
**Conformal flatness condition**

Within 3+1 formalism, one imposes that:

\[ \gamma_{ij} = \psi^4 f_{ij} \]

with \( f_{ij} \) the flat metric and \( \psi(t, x^1, x^2, x^3) \) the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data,

- discards all dynamical degrees of freedom of the gravitational field (\( A \) and \( \tilde{B} \) are zero by construction)
- exact in spherical symmetry: e.g. the Schwarzschild metric can be described within CFC

\[ \Rightarrow \] captures many non-linear effects.

- The Kerr solution cannot be exactly described in CFC, but rotation can be included in BH solution.
Einstein equations in CFC

Set of 5 non-linear elliptic PDEs ($K = 0$)

\[ \Delta \psi = -2\pi \psi^{-1} \left( E^* + \frac{\psi^6 K_{ij} K^{ij}}{16\pi} \right), \]
\[ \Delta (N \psi) = 2\pi N \psi^{-1} \left( E^* + 2S^* + \frac{7\psi^6 K_{ij} K^{ij}}{16\pi} \right), \]
\[ \Delta \beta^i + \frac{1}{3} D^i D_j \beta^j = 16\pi N \psi^{-2} (S^*)^i + 2\psi^{10} K^{ij} D_j \frac{N}{\psi^6}. \]

\[ E^* = \psi^6 E, \quad (S^*)^i = \psi^6 S^i, \ldots \]

are conformally-rescaled projections of the stress-energy tensor.
**Spherical collapse of matter**

We consider the case of the collapse of an **unstable** relativistic star, governed by the equations for the hydrodynamics

\[
\frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} U}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = Q,
\]

with \( U = (\rho W, \rho h W^2 v_i, \rho h W^2 - P - D) \).

At every time-step, we solve the equations of the CFC system

(elliptic) \[ \Rightarrow \text{exact in spherical symmetry! (isotropic gauge)} \]

- During the collapse, when the star becomes very compact, the elliptic system would no longer converge, or give a wrong solution (wrong ADM mass).
- Even for equilibrium configurations, if the iteration is done only on the metric system, it may converge to a wrong solution.
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**Collapse of gravitational waves**

Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.

- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a **wrong** solution.

⇒ effect on the ADM mass computed from $\psi$ at $r = \infty$. 
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In the *extended conformal thin sandwich* approach for initial data, the system of PDEs is the same as in CFC.

Pfeiffer & York (2005) have numerically observed a parabolic branching in the solutions of this system for perturbation of Minkowski spacetime.

Some analytical studies have been performed by Baumgarte *et al.* (2007), which have shown the genericity of the non-uniqueness behavior.

*from Pfeiffer & York (2005)*
A cure in the CFC case
Origin of the problem

In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$  

Local uniqueness of solutions can be proven using a maximum principle:

If $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8} \psi^5 K_{ij} K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.
ORIGIN OF THE PROBLEM

In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$ 

Local uniqueness of solutions can be proven using a maximum principle:

if $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8} \psi^5 K_{ij} K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.
Approximate CFC

Let $L, V^i \mapsto (LV)^{ij} = D^i V^j + D^j V^i - \frac{2}{3} f^{ij} D_k V^k$.

In CFC, $K^{ij} = \psi^{-4} \tilde{A}^{ij}$, with $\tilde{A}^{ij} = \frac{1}{2N} (L\beta)^{ij}$,

here $K^{ij} = \psi^{-10} \hat{A}^{ij}$, with $\hat{A}^{ij} = (LX)^{ij} + \hat{A}^{ij}_{TT}$.

Neglecting $\hat{A}^{ij}_{TT}$, we can solve in a hierarchical way:

1. Momentum constraints $\Rightarrow$ linear equation for $X^i$ from the actually computed hydrodynamic quantity $S^*_j = \psi^6 S_j$,
2. Hamiltonian constraint $\Rightarrow \Delta \psi = -2\pi \psi^{-1} E^* - \psi^{-7} \hat{A}^{ij} \hat{A}_{ij}/8$,
3. linear equation for $N\psi$,
4. linear equation for $\beta$, from the definitions of $\hat{A}^{ij}$.

It can be shown that the error made neglecting $\hat{A}^{ij}_{TT}$ falls within the error of CFC approximation.
Approximate CFC

Let $L, V^i \mapsto (LV)^{ij} = \mathcal{D}^i V^j + \mathcal{D}^j V^i - \frac{2}{3} f^{ij} \mathcal{D}_k V^k$.

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**NEW EQUATIONS IN CFC**

The conformally-rescaled projections of the stress-energy tensor $E^* = \psi^6 E$, $(S^*)^i = \psi^6 S^i$, ... are supposed to be known from hydrodynamics evolution.

\[
\Delta X + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j X^j = 8\pi (S^*)^i,
\]

\[
\hat{A}^{ij} \simeq \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3} f^{ij} \mathcal{D}_k X^k,
\]

\[
\Delta \psi = -2\pi \psi^{-1} E^* - \frac{\psi^{-7}}{8} \hat{A}^{ij}\hat{A}_{ij},
\]

\[
\Delta (N\psi) = 2\pi N \psi^{-1} (E^* + 2S^*) + N \psi^{-7} \frac{7\hat{A}^{ij}\hat{A}_{ij}}{8},
\]

\[
\Delta \beta^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j \beta^j = \mathcal{D}_j \left(2N \psi^{-6} \hat{A}^{ij}\right).
\]
Application

Axisymmetric collapse to a black hole

Using the code CoCoNuT combining Godunov-type methods for the solution of hydrodynamic equations and spectral methods for the gravitational fields.

- Unstable rotating neutron star initial data, with polytropic equation of state,
- approximate CFC equations are solved every time-step.
- Collapse proceeds beyond the formation of an apparent horizon;
- Results compare well with those of Baoitthi et al. (2005) in GR, although in approximate CFC.

Other test: migration of unstable neutron star toward the stable branch.

Cordero-Carrión et al. (2009)
New constrained formulation
NEW CONSTRAINED FORMULATION

Evolution equations

In the general case, one cannot neglect the TT-part of $\hat{A}^{ij}$ and one must therefore evolve it numerically.

<table>
<thead>
<tr>
<th>sym. tensor</th>
<th>longitudinal part</th>
<th>transverse part</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{A}^{ij} =$</td>
<td>$(LX)^{ij}$</td>
<td>$+\hat{A}_{TT}^{ij}$</td>
</tr>
<tr>
<td>$h^{ij} =$</td>
<td>0 (gauge)</td>
<td>$+h^{ij}$</td>
</tr>
</tbody>
</table>

The evolution equations are written only for the transverse parts:

$$\frac{\partial \hat{A}_{TT}^{ij}}{\partial t} = \left[ \mathcal{L}_\beta \hat{A}^{ij} + N\psi^2 \Delta h^{ij} + S^{ij} \right]^{TT},$$

$$\frac{\partial h^{ij}}{\partial t} = \left[ \mathcal{L}_\beta h^{ij} + 2N\psi^{-6} \hat{A}^{ij} - (L_\beta)^{ij} \right]^{TT}.$$
NEW CONSTRAINED FORMULATION

If all metric and matter quantities are supposed known at a given time-step.

1. Advance hydrodynamic quantities to new time-step,
2. advance the TT-parts of $\hat{A}^{ij}$ and $h^{ij}$,
3. obtain the logitudinal part of $\hat{A}^{ij}$ from the momentum constraint, solving a vector Poisson-like equation for $X^i$ (the $\Delta_{jk}^i$’s are obtained from $h^{ij}$):

$$\Delta X^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j X^j = 8\pi (S^*)^i - \Delta_{jk}^i \hat{A}^{jk},$$

4. recover $\hat{A}^{ij}$ and solve the Hamiltonian constraint to obtain $\psi$ at new time-step,
5. solve for $N\psi$ and recover $\beta^i$. 
Summary - Perspectives

- A fully-constrained formalism of Einstein equations, aimed at obtaining stable solutions in astrophysical scenarios (with matter) has been presented, implemented and tested;
- A way to cure the uniqueness problem in the elliptic part of Einstein equations has been devised;
  ⇒ the accuracy has been checked: the additional approximation in CFC does not introduce any new errors.

The numerical codes are present in the Lorene library: http://lorene.obspm.fr, publicly available under GPL.

Future directions:
- Implementation of the new FCF and tests in the case of gravitational wave collapse;
- Use of the CFC approach together with excision methods in the collapse code to simulate the formation of a black hole (work by N. Vasset);
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