# A constrained scheme for Einstein equations in numerical relativity

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*based on collaboration with*
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É. Gourgoulhon & J.L. Jaramillo.

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# Plan

1. Introduction
   - Constraints issues in 3+1 formalism
   - Motivation for a fully-constrained scheme
2. Description of the formulation and strategy
   - Covariant 3+1 conformal decomposition
   - Einstein equations in Dirac gauge and maximal slicing
   - Integration strategy
3. Non-uniqueness problem
   - CFC and FCF
   - A cure in CFC
   - New constrained formulation

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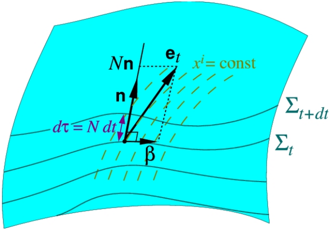
# 3+1 formalism

Decomposition of spacetime and of Einstein equations




**Evolution equations:**

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k_{\ j} +$$
$$N \left[ K K_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left( \frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

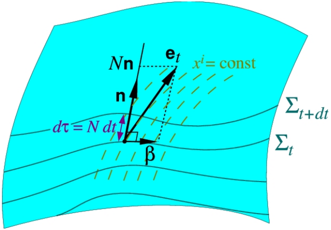
**Constraint equations:**

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\, (dx^i + \beta^i dt)\, (dx^j + \beta^j dt)$$

# 3+1 FORMALISM

Decomposition of spacetime and of Einstein equations

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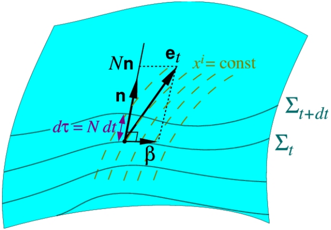


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# Constraint violation

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

**Free evolution**

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

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**Appearance of constraint violating modes**

Some cures have been investigated (and work):

- constraint-preserving boundary conditions (Lindblom *et al.* 2004)
- constraint projection (Holst *et al.* 2004)
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# Some reasons not to solve constraints

computational cost of usual elliptic solvers ...

few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems

definition of boundary conditions at finite distance and at black hole excision boundary

# Motivations for a fully-constrained scheme

"Alternate" approach (although most straightforward)

- partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
- fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams *et al.* (1994), Choptuik *et al.* (2003)

⇒Rather popular for 2D applications, but disregarded in 3D
Still, many advantages:

- constraints are verified!
- elliptic systems have good stability properties
- easy to make link with initial data
- evolution of only two scalar-like fields ...

# Description of the formulation and strategy

*Bonazzola et al. (2004)*

# Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura formalism)

**Dynamical degrees of freedom of the gravitational field:**

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

**Problem**

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# Introduction of a flat metric

We introduce $f_{ij}$ (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of $\gamma_{ij}$, and $\mathcal{D}_i$ the associated covariant derivative.

**Define:**

$$\tilde{\gamma}_{ij} := \Psi^{-4}\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4 \tilde{\gamma}_{ij}$$

with

$$\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$$f := \det f_{ij}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of $\gamma_{ij}$ and verifies $\det \tilde{\gamma}_{ij} = f$

⇒*no more tensor densities: only tensors.*

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

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# Generalized Dirac gauge

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

**Divergence-free condition on $\tilde{\gamma}^{ij}$**

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric $f_{ij}$.

Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t\right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}_{ij} / \partial t\right) = 0$

*Notice:* Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

# Generalized Dirac gauge properties

- $h^{ij}$ is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, $h^{ij}$ is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of $h^{ij}$
- ${}^3R$ does not contain any second-order derivative of $h^{ij}$
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[ (\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the Conformal Flatness Condition (CFC) verifies the Dirac gauge $\Rightarrow$possibility to easily use initial data for binaries now available

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# Einstein equations

## Dirac gauge and maximal slicing ($K = 0$)

### Hamiltonian constraint

$$\begin{aligned}\Delta(\Psi^2 N) \quad &= \quad \Psi^6 N \left(4\pi S + \frac{3}{4}\tilde{A}_{kl} A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2 \Bigg[ N \Big( \frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} \\ &\quad - \frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k \ln\Psi \Big) + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N \Bigg]\end{aligned}$$

### Momentum constraint

$$\begin{aligned}\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) \quad &= \quad 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12 N A^{ij}\mathcal{D}_j \ln\Psi - 2\Delta^i{}_{kl} N A^{kl} \\ &\quad - h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l\end{aligned}$$

### Trace of dynamical equations

$$\Delta N = \Psi^4 N\left[4\pi(E+S) + \tilde{A}_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N$$

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6 components - 3 Dirac gauge conditions - $(\det \tilde{\gamma}^{ij} = 1)$

### 2 degrees of freedom

$$-\frac{\partial^2 A}{\partial t^2} + \Delta A = S_A$$

$$-\frac{\partial^2 \tilde{B}}{\partial t^2} + \Delta \tilde{B} = S_{\tilde{B}}$$

with $A$ and $\tilde{B}$ two scalar potentials representing the degrees of freedom.

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# Integration procedure

Everything is know on slice $\Sigma_t$

$\Downarrow$

Evolution of $A$ and $\tilde{B}$ to next time-slice $\Sigma_{t+dt}$ (+ hydro)

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Deduce $h^{ij}(t+dt)$ from Dirac and trace-free conditions

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Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.

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Iterate on the system of elliptic equations for $N, \Psi^2 N$ and $\beta^i$ on $\Sigma_{t+dt}$

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# Non-uniqueness problem

*Cordero-Carrión et al. (2009)*

# Conformal flatness condition

Within 3+1 formalism, one imposes that :

$$\gamma_{ij} = \psi^4 f_{ij}$$

with $f_{ij}$ the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data,

- discards all dynamical degrees of freedom of the gravitational field ($A$ and $\tilde{B}$ are zero by construction)
- exact in spherical symmetry: e.g. the Schwarzschild metric can be described within CFC

$\Rightarrow$ captures many non-linear effects.

- The Kerr solution cannot be exactly described in CFC, but rotation can be included in BH solution.

# Einstein equations in CFC

**set of 5 non-linear elliptic PDEs ($K = 0$)**

$$\Delta \psi = -2\pi\psi^{-1}\left(E^* + \frac{\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta(N\psi) = 2\pi N\psi^{-1}\left(E^* + 2S^* + \frac{7\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j = 16\pi N\psi^{-2}(S^*)^i + 2\psi^{10}K^{ij}\mathcal{D}_j\frac{N}{\psi^6}.$$

$$E^* = \psi^6 E, \quad (S^*)^i = \psi^6 S^i, \ldots$$

are conformally-rescaled projections of the stress-energy tensor.

# Spherical collapse of matter

We consider the case of the collapse of an unstable relativistic star, governed by the equations for the hydrodynamics

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\boldsymbol{U}}{\partial t}+\frac{\partial\sqrt{-g}\boldsymbol{F}^i}{\partial x^i}\right]=\boldsymbol{Q},$$

with $\boldsymbol{U} = (\rho W, \rho h W^2 v_i, \rho h W^2 - P - D)$.
At every time-step, we solve the equations of the CFC system (elliptic)
$\Rightarrow$exact in spherical symmetry! (isotropic gauge)

- During the collapse, when the star becomes very compact, the elliptic system would no longer converge, or give a wrong solution (wrong ADM mass).
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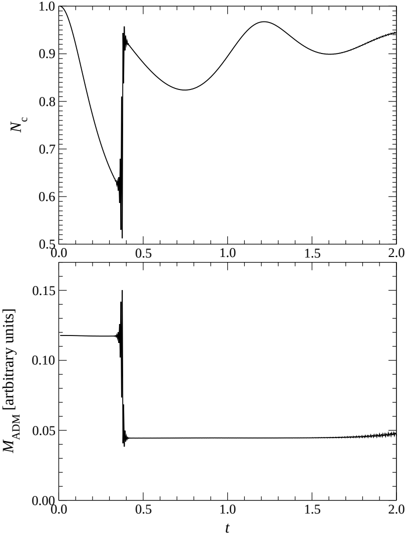
# Collapse of gravitational waves

Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.




- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a wrong solution.

$\Rightarrow$effect on the ADM mass computed from $\psi$ at $r = \infty$

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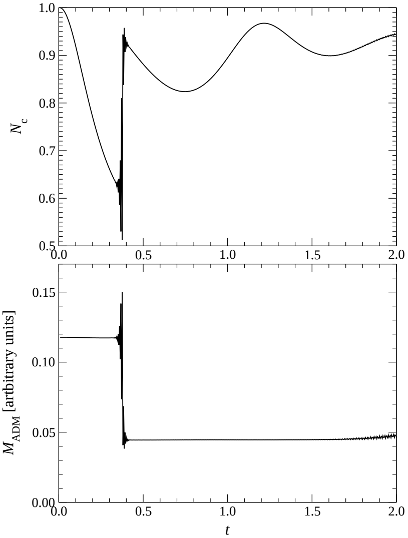
Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.

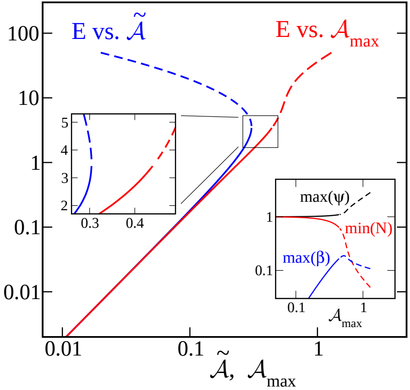



- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a wrong solution.

$\Rightarrow$effect on the ADM mass computed from $\psi$ at $r = \infty$.

# Other studies

- In the *extended conformal thin sandwich* approach for initial data, the system of PDEs is the same as in CFC.
- Pfeiffer & York (2005) have numerically oberved a parabolic branching in the solutions of this system for perturbation of Minkowski spacetime.
- Some analytical studies have been performed by Baumgarte *et al.* (2007), which have shown the genericity of the non-uniqueness behavior.




*from* Pfeiffer & York *(2005)*

# A cure in the CFC case

# Origin of the problem

In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$

Local uniqueness of solutions can be proven using a maximum principle:

if $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi\psi^5 E - \frac{1}{8}\psi^5 K_{ij}K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.

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# Approximate CFC

$$\text{Let } L,\ V^i \mapsto (LV)^{ij} = \mathcal{D}^i V^j + \mathcal{D}^j V^i - \frac{2}{3} f^{ij} \mathcal{D}_k V^k.$$

$$\text{In CFC, } K^{ij} = \psi^{-4} \tilde{A}^{ij},\ \text{with } \tilde{A}^{ij} = \frac{1}{2N} (L\beta)^{ij},$$

$$\text{here } K^{ij} = \psi^{-10} \hat{A}^{ij},\ \text{with } \hat{A}^{ij} = (LX)^{ij} + \hat{A}^{ij}_{\mathrm{TT}}.$$

Neglecting $\hat{A}^{ij}_{\mathrm{TT}}$, we can solve in a hierarchical way:

1. Momentum constraints ⇒linear equation for $X^i$ from the actually computed hydrodynamic quantity $S_j^* = \psi^6 S_j$,
2. Hamiltonian constraint ⇒$\Delta\psi = -2\pi\psi^{-1}E^* - \psi^{-7}\hat{A}^{ij}\hat{A}_{ij}/8$,
3. linear equation for $N\psi$,
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# New equations in CFC

The conformally-rescaled projections of the stress-energy tensor $E^* = \psi^6 E, \quad (S^*)^i = \psi^6 S^i, \ldots$ are supposed to be known from hydrodynamics evolution.

$$\Delta X + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 8\pi \left(S^*\right)^i ,$$

$$\hat{A}^{ij} \simeq \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3} f^{ij} \mathcal{D}_k X^k,$$

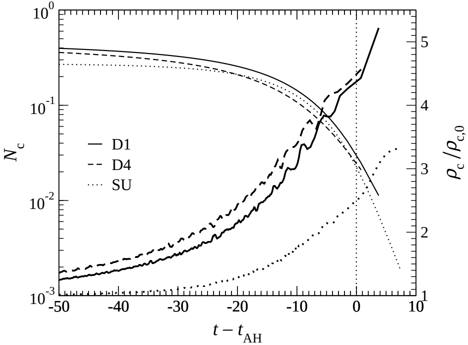
$$\Delta\psi = -2\pi\psi^{-1} E^* - \frac{\psi^{-7}}{8} \hat{A}^{ij} \hat{A}_{ij},$$

$$\Delta(N\psi) = 2\pi N \psi^{-1} \left(E^* + 2S^*\right) + N\psi^{-7} \frac{7\hat{A}_{ij}\hat{A}^{ij}}{8},$$

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j = \mathcal{D}_j \left(2N\psi^{-6}\hat{A}^{ij}\right).$$

Using the code CoCoNuT combining Godunov-type methods for the solution of hydrodynamic equations and spectral methods for the gravitational fields.

- Unstable rotating neutron star initial data, with polytropic equation of state,
- approximate CFC equations are solved every time-step.
- Collapse proceeds beyond the formation of an apparent horizon;
- Results compare well with those of Baoitti *et al.* (2005) in GR, although in approximate CFC.




Cordero-Carrión *et al.* (2009)

Other test: migration of unstable neutron star toward the stable branch.

# New constrained formulation

In the general case, one cannot neglect the TT-part of $\hat{A}^{ij}$ and one must therefore evolve it numerically.

| sym. tensor | longitudinal part | transverse part |
|---|---|---|
| $\hat{A}^{ij} =$ | $(LX)^{ij}$ | $+\hat{A}^{ij}_{\rm TT}$ |
| $h^{ij} =$ | 0 (gauge) | $+h^{ij}$ |

The evolution equations are written only for the transverse parts:

$$\frac{\partial \hat{A}^{ij}_{\rm TT}}{\partial t} = \left[\mathcal{L}_\beta \hat{A}^{ij} + N\psi^2 \Delta h^{ij} + \mathcal{S}^{ij}\right]^{\rm TT},$$
$$\frac{\partial h^{ij}}{\partial t} = \left[\mathcal{L}_\beta h^{ij} + 2N\psi^{-6}\hat{A}^{ij} - (L\beta)^{ij}\right]^{\rm TT}.$$

# New constrained formulation

If all metric and matter quantities are supposed known at a given time-step.

1. Advance hydrodynamic quantities to new time-step,
2. advance the TT-parts of $\hat{A}^{ij}$ and $h^{ij}$,
3. obtain the logitudinal part of $\hat{A}^{ij}$ from the momentum constraint, solving a vector Poisson-like equation for $X^i$ (the $\Delta^i_{jk}$'s are obtained from $h^{ij}$):

$$\Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 8\pi (S^*)^i - \Delta^i_{jk}\hat{A}^{jk},$$

4. recover $\hat{A}^{ij}$ and solve the Hamiltonian constraint to obtain $\psi$ at new time-step,
5. solve for $N\psi$ and recover $\beta^i$.

## Summary - Perspectives

- A fully-constrained formalism of Einstein equations, aimed at obtaining stable solutions in astrophysical scenarios (with matter) has been presented, implemented and tested ;
- A way to cure the uniqueness problem in the elliptic part of Einstein equations has been devised ;

⇒ the accuracy has been checked: the additional approximation in CFC does not introduce any new errors.

The numerical codes are present in the LORENE library: `http://lorene.obspm.fr`, publicly available under GPL.

Future directions:

- Implementation of the new FCF and tests in the case of gravitational wave collapse;
- Use of the CFC approach together with excision methods in the collapse code to simulate the formation of a black hole (work by N. Vasset);

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