Post-Newtonian mathematical methods: asymptotic expansion of retarded integrals

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Perturbative methods for isolated systems in GR

Approximate modelling of isolated systems cardinal in GR

- numerical
- perturbative

approximations

Two main methods for the analytic perturbative schemes

- post-Minkowskian (PM) expansion: usual perturbative approach with a Minkowskian background $\eta_{\mu\nu}$

formal expansion parameter G

- post-Newtonian (PN) approximation: may be defined in principle as a perturbative approach in a class of frame theories depending on a parameter λ [see e.g. Ehlers 1986]

$\lambda = 1 \rightarrow$ general relativity
$\lambda = 0 \rightarrow$ to Newton-Cartan theory

formal expansion parameter $1/c$

PN scheme for isolated systems: a pragmatic approach

Hypothesis on the physical nature of the system:

- matter source with compact support described by $T^{\mu\nu}$
- no-incoming wave condition

PN expansion for practical computations:

- introduction of a (non-unique) "time" field $t(x)$
  slices $t =$cst. endowed with an Euclidean metric δ_{ij} and of a PN-type gauge in which $\eta_{00} = -1$, $\eta_{ij} = \delta_{ij}$ is the flat leading metric
- choice of a fluid variables depending on the PN parameter $1/c^2$ with finite limit as $1/c \to 0$
- iterative search based on the Einstein equation of the 4-metric under the form $g^{(m)}_{\mu\nu}(\mathbf{x}, t) = \eta_{\mu\nu} + \sum_{m=1}^{+\infty} \frac{1}{c^m} g^{(m)}_{\mu\nu}(\mathbf{x}, t)$
  where $g^{(m)}_{\mu\nu}$ may depend on c but must be $o(c^\alpha)$ as $1/c$ goes to 0
- formal asymptotic expansion in powers of $1/c$ of quantities of interest

Validity of the PN asymptotic description

for fully dynamical systems, little knowledge about the asymptotic behavior
$\hookrightarrow$ e.g. compact binaries

naively, reasonable convergence expected when $1/c \ll 1$ in a system of unit where matter quantities (e.g. ρ, v^i) and G are of order ~ 1 (at most)

Necessary condition 1:

$$[Gm/(Lc^2)]^{1/2} \ll 1\,, \qquad\qquad v/c \ll 1$$

m typical mass in balls of radius L

L typical length of variation of gravitational field (in the most unfavorable case)

for a quantity Q, typical time variation scale of Q supposed implicitly to introduce factor of order $\lesssim \partial_i Q$ i.e. $\partial_t Q \lesssim \partial_i Q$

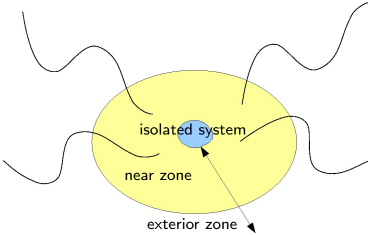
$\hookrightarrow$ true only if the interaction propagation effects are neglected since propagation of Q means $\partial_t Q/c \sim \partial_i Q$

Necessary condition 2:

propagation inside system outer radius D negligible $\Rightarrow$ $D \ll \lambda$

Dynamical characteristics of the matter system

Extension condition:
domain of validity of the asymptotic expansion: $|\mathbf{x}| \ll \lambda \Rightarrow D \ll \lambda$



Stress-energy tensor:

- assumed to be smooth with compact support
- assumed to scale the same as for a matter (perfect) fluid

$$T^{00} \sim \rho c^2\,, \qquad T^{0i} \sim \rho v^i c\,, \qquad T^{ij} \sim c^0(\rho v^i v^j + p\delta^{ij})$$

Matter variables in the present formalism chosen to be

$$\sigma = \frac{T^{00} + T^{ij}\delta_{ij}}{c^2}\,, \qquad \sigma_i = \frac{T^{0j}\delta_{ij}}{c}\,, \qquad \sigma_{ij} = T^{kl}\delta_{ik}\delta_{jl}$$

Einstein equations in harmonic gauge

Explicit form of the Einstein equations

$$\partial_{\rho\sigma}[(-g)(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] = \frac{16\pi G}{c^4}(T^{\mu\nu} + t^{\mu\nu})$$

where $t^{\mu\nu}$ = Landau-Lichitz pseudo-tensor

$\hookrightarrow$ $(-g)t^{\mu\nu}$ combination of contraction $\partial(\sqrt{-g}\, g^{\mu\nu})\partial(\sqrt{-g}\, g^{\rho\sigma})$
expandable in powers of 1PM quantity $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$

Harmonic gauge condition

$$\nabla^\nu \nabla_\nu x^\mu = 0 \quad \Leftrightarrow \quad \partial_\nu h^{\mu\nu} = 0$$

Relaxed Einstein equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \equiv \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(\partial h, \partial h)$$

$\longleftarrow$ gauge condition implies equations for σ, σ_i i.e. the equations of motion

Leading order PN retarded solution:

generalized Newtonian potential

$$\begin{cases} \Box(h^{00}+h^{ii}) \approx \dfrac{16\pi G}{c^2}\sigma \\ \Box h^{0i} \approx \dfrac{16\pi G}{c^3}\sigma_i \\ \Box h^{ij} = \mathcal{O}\left(\dfrac{1}{c^4}\right) \end{cases} \Rightarrow \begin{cases} g_{00} = -1 + \dfrac{2}{c^2}V + \mathcal{O}\left(\dfrac{1}{c^4}\right) \\ g_{0i} = -\dfrac{4}{c^3}V_i + \mathcal{O}\left(\dfrac{1}{c^5}\right) \\ g_{ij} = \left(1+\dfrac{2}{c^2}V\right)\delta_{ij} + \mathcal{O}\left(\dfrac{1}{c^4}\right) \end{cases}$$

with $V = \Box_{\mathrm{R}}^{-1}(-4\pi G\sigma) \equiv \int \frac{d^3\mathbf{x}'}{-4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}(-4\pi\sigma)[\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c]$ and

$V_i = \Box_{\mathrm{R}}^{-1}(-4\pi G\sigma_i)$

Expansion of the retardations:

$$V = \sum_{m=0}^{+\infty}\frac{(-1)^m}{m!c^m}\partial_t^m \int \frac{d^3\mathbf{x}'}{-4\pi}|\mathbf{x}-\mathbf{x}'|^{m-1}(-4\pi\sigma)[\mathbf{x}', t]$$

Issues of the PN scheme at higher order

Iterative computation of $\mathbf{h}^{\mu\nu}$ at higher order

- insertion of $h^{\mu\nu}$ already computed up to current order $1/c^m$ into $\tau^{\mu\nu}$
- solution for $h^{\mu\nu}$ at order $m+2$ formally given by

$$h^{\mu\nu}_{[\leq m+2]} \equiv \sum_{k=1}^{m+2} \frac{1}{c^k} h^{\mu\nu}_{(k)} = \frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{R}} \left[\tau^{\mu\nu}_{[\leq m-2]}\big(h^{\rho\sigma}_{(\leq m)}\big) + o(1/c^m) \right]$$

Serious problem in a naive expansion procedure

non-compact support terms $\Lambda^{\mu\nu}(h,h)$ entering $\tau^{\mu\nu}$

$\hookrightarrow$ no good accuracy of the PN source outside the near zone

$\hookrightarrow$ asymptotic series expansion of the integral not granted

extension of the effective source over the exterior zone

$\Rightarrow$ diverging integrals in the PN expansion

$$\Box^{-1}_{\mathrm{R}}(\partial_i V \partial_j V) = \sum_{k=1}^{m} \frac{(-1)^k}{k! c^k} \partial_t^k \int \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} (\partial_i V \partial_j V)[\mathbf{x}', t] + o\left(\frac{1}{c^m}\right)$$

Problem to be addressed

What is the iterative PN expansion at order $m-2$ of

$$\Box_{\mathrm{R}}^{-1}\tau = \int \frac{d^3\mathbf{x}'}{-4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\tau_{[\leq m-2]}(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c) \ ?$$

information on the field behavior far from the system required...

Basic idea to study the field structure outside the near zone

$h^{\mu\nu}$ in the exterior zone solution of the vacuum Einstein equations
$\hookrightarrow$ contained in the most general PM asymptotic solution

Principle of the algorithm:

- decompose $h^{\mu\nu}$ as $\sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}$
- find iteratively the most general solution of $\Box h^{\mu\nu}_{(n+1)} = \Lambda^{\mu\nu}_{(n)}(\partial h_{(\leq n)}, \partial h_{(\leq n)})$ (outside near-zone isolated points)
- absorb homogeneous solution in a redefinition of the moments

Solutions expressed in terms of FP integrals:

FP $\int d^3\mathbf{x}' F(\mathbf{x}', t)$ for a function F smooth on $\mathbb{R}^{*3}$ defined in 3 steps

- computation of $I[F](B) \equiv \int d^3\mathbf{x}' |x|'^B F(\mathbf{x}')$
- expansion of $I[F](B)$ in a Laurent series of the form $\sum_{k=k_0}^{+\infty} I_k[F] B^k$
- FP $\displaystyle\int d^3\mathbf{x}' F(\mathbf{x}', t) = I_0[F]$

Linearized exterior field I

Linearized Einstein equations in vacuum:

$$\Box h^{\mu\nu}_{(1)} = 0 \qquad \partial_\nu h^{\mu\nu}_{(1)} = 0$$

with the no-incoming wave condition (absence of advanced integral)

$$\lim_{\substack{r\to+\infty\\ t+r/c\to\mathrm{cst}}} h^{\mu\nu}_{(m)} = 0 \qquad \lim_{\substack{r\to+\infty\\ t+r/c\to\mathrm{cst}}} \left[\left(\partial_r + \frac{1}{c}\partial_t\right)\left(r h^{\mu\nu}_{(m)}\right)\right] = 0$$

distance to the origin $|\mathbf{x}|$

Form of the most general solutions in Minkowskian-like coordinates:

- in spherical symmetry $\dfrac{I(t-r/c)}{r}$
- in general $\displaystyle\sum_{\ell\geq 0} \partial_{i_1 i_2 \dots k_1 k_2 \dots k_\ell}\left(\frac{I_{j_1 j_2 \dots k_1 k_2 \dots k_\ell}(t-r/c)}{r}\right)$

  (with possible contraction to ε_{abc})

Linearized exterior field II

Useful Notation

multi-index $i_1 i_2 ... i_\ell$ denoted by L

Most general exterior linear solution

$$h^{\mu\nu}_{(1)} = h^{\mu\nu}_{\text{can}}(I_L, J_L) + \text{linear gauge transformation term in } \phi^\mu_{(1)}$$

$$\text{with} \quad \Box\phi^\mu_{(1)} = 0 \quad \text{and} \quad \phi^\mu_{(1)} = \phi^\mu_{(1)}[W_L, X_L, Y_L, Z_L]$$

$$\Rightarrow$$

$$h^{\mu\nu} \text{ entirely parameterized by } \{I_L, ..., Z_L\}$$

- I_L= source mass-type moment of order ℓ
  J_L= source current-type moment of order ℓ
- $\{W_L, X_L, Y_L, Z_L\}$ = gauge moments

unicity of the multipole parameterization iff the moments are STF
e.g. $I_L = \text{STF}_L I_L$

Post-Minkowskian iteration

Search of a particular solution of $\Box \mathbf{h}^{\mu\nu}_{(n+1)} = \Lambda^{\mu\nu}_{(n+1)}(\mathbf{h}_{(\leq n)}, \mathbf{h}_{(\leq n)})$

$\Box_{\mathrm{R}}^{-1} \Lambda^{\mu\nu}_{(n+1)}$ ill-defined

$\hookrightarrow$ idea: construct the particular solution by regularization

$\rightarrow$ use of FP regularization due to the fundamental property

$$\Box(\mathrm{FP}\Box_{\mathrm{R}}^{-1} F) = F$$

solution under assumption of past stationarity : $p^{\mu\nu}_{(n+1)} = \mathrm{FP}\Box_{\mathrm{R}}^{-1} \Lambda^{\mu\nu}_{(n+1)}$

Determination of the homogeneous solution $\mathbf{q}^{\mu\nu}_{(n+1)}$

$$\partial_\nu h^{\mu\nu}_{(n+1)} = \partial_\nu p^{\mu\nu}_{(n+1)} + \partial_\nu q^{\mu\nu}_{(n+1)} = 0 \quad \text{and} \quad \Box q^{\mu\nu}_{(n+1)} = 0 \quad \Rightarrow \quad q^{\mu\nu}_{(n+1)}$$

General solution

$$h^{\mu\nu}_{(n+1)} = p^{\mu\nu}_{(n+1)} + q^{\mu\nu}_{(n+1)}$$

(homogeneous solution absorbed in a moment redefinition)

Structure of the exterior field

for systems that are in particular

- isolated with $T^{\mu\nu}$ of compact support
- stationary in the past i.e. $\partial_t h^{\mu\nu}(\mathbf{x}, t) = 0$ for $t \leq \mathcal{T}$

then the large r behavior of $h^{\mu\nu}_{(n)}$ reads

$$h^{\mu\nu}_{(n)}(\mathbf{x}, t) = \sum_{\ell} \hat{n}^L \Big\{ \sum_{0\leq p\leq n-1\,,1\leq k\leq N} \frac{\ln^p r}{r^k} F_{Lkp}(t - r/c) + R^L_N(r, t - r/c) \Big\}$$

with $F_{Lkp}(u)$ being $C^\infty(\mathbb{R})$ and constant at the stationary epoch
$R^L_N(u)$ being $\mathcal{O}(1/r^N)$
$n^i = x^i/r$, $n^L = n^{i_1} n^{i_2} \ldots n^{i_\ell}$ and $\hat{n}^L = \mathrm{STF}_L n^L$

resulting solution with the above structure represented by $\mathcal{M}_{[\leq N]}(h^{\mu\nu}_{(n)})$

About the present version of the PN scheme

Features expected from the present PN scheme:

- simple hypothesis including
  - existence of an exterior zone D_e in which $h^{\mu\nu} = \mathcal{M}(h^{\mu\nu})$
  - existence of an near zone D_n in which $h^{\mu\nu}$ given by the searched PN expression $\overline{h}^{\mu\nu}_{[\leq M]}$
  - existence of an intermediate zone $D_n \cap D_e$ with typical radius R_i
  - stationarity in the remote past
- convenient formulation:
  - with a simple and speaking final form
  - useful in practical calculations

Notation:

- omission of indices in the field and the source, e.g. $\Box \left[\frac{c^4}{16\pi G} h\right] = \tau$
- truncated retarded operator

$$(\Box_{\mathrm{R}}^{-1})_{(R_i>)}[\tau] = \int_{|\mathbf{x}'|<R_i} \frac{d^3\mathbf{x}'}{-4\pi} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tau(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)$$

Statement of the main result

under the preceding hypothesis, one has $\overline{\Box_{\mathrm{R}}^{-1}}[\tau] = \overline{\Box_{\mathrm{R}}^{-1}}[\tau] + \mathcal{H}[\tau]$

with $$\overline{\Box_{\mathrm{R}}^{-1}}[\tau] = \sum_{k\geq 0, n} \frac{(-1)^k}{k!} \frac{\partial_t^k}{c^k} \mathrm{FP} \int \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\overline{\tau}_{(n)}(\mathbf{x}', t)}{c^n}$$

$$\mathcal{H}[\tau] = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!} \hat{\partial}_L \left\{ \frac{\overline{\mathcal{R}[\tau]_L(t-r/c) - \mathcal{R}_L[\tau](t+r/c)}}{2r} \right\} \Bigg]$$

$$\frac{\mathcal{R}_L[\tau](t)}{c^{2\ell+1}(2\ell+1)!!} = -\sum_n \mathrm{FP} \int \frac{d^3\mathbf{x}'}{-4\pi} \hat{\partial}'_L \left(\frac{1}{|\mathbf{x}'|} \overset{(-2\ell-1)}{\mathcal{M}}(\tau_{\mathrm{ns}})_{(n)}(t - |\mathbf{x}'|/c) \right)$$

$$\hat{\partial}_L \left(\frac{f\,(t-r/c)}{r} \right) = (-1)^\ell \hat{n}_L \sum_{i=0}^{\ell} \frac{(\ell+i)!}{2^i i! (\ell-i)!} \frac{f^{(\ell-i)}(t-r/c)}{c^{\ell-i} r^{i+1}}$$

$\tau_{\mathrm{ns}}(\mathbf{x}, t) = \tau(\mathbf{x}, t) - \tau(\mathbf{x}, -\infty)$ if τ stationary in the remote past

Equivalent expression of $\mathcal{H}[\tau]$

Transformation of the sum over ℓ:

- expansion of the retardations $\longrightarrow$ sum on p'
- application of the multiple-space derivative
- change of variable $p = p' - \ell$ (for $\ell \geq 0$)

$\overline{\Box_{\mathrm{R}}^{-1}}[\tau]$ depending on $\mathcal{R}_L$ through time derivative $\dfrac{\mathcal{R}_L^{(2\ell+1)}[\tau](t)}{c^{2\ell+1}(2\ell+1)!!}$

$\hookrightarrow$ contains terms in $\frac{1}{c^{\ell-s}}\mathrm{FP}\int d^3\mathbf{x}|\mathbf{x}|^{-\ell-s-1}\partial_t^{\ell-s}\mathcal{M}_{(n)}(\tau_{\mathrm{ns}})[\mathbf{x}, t-|\mathbf{x}|/c]$

$0 \leq s \leq \ell$

Form of individual terms resulting from the large r structure:

$$\hat{n}^L FP_{B=0}\int_0^{+\infty} dr\ r^{B+1-\ell-s}\frac{\ln^p r}{r^k}F_{Lkp}(t-r/c) =$$

≥ 2

$$\frac{2^{1+\ell+s+k-B}c^B}{c^{1+\ell+s+k}}\hat{n}^L FP_{B=0}\int_0^{+\infty} dt'\, t'^{B+1-\ell-s-k}\ln^p(t'c/2)F_{Lkp}(t-t')$$

Actual meaning of the formal series

terms composing $\mathcal{H}[\tau]$ of order $o\left(\frac{1}{c^m}\right)$ if ℓ large enough or k large enough

then $\overline{\Box_{\mathrm{R}}^{-1}[\tau]}$ truncated as follows:

Truncation of the PN solution

- formal expansion of the retarded integral assumed to be truncated at arbitrary high order m, so that it only depends on $\overline{\tau}_{\mathrm{ns}(\leq m)}$
- formal truncation of the sum at order m which means both
  - truncation in ℓ in the sum composing $\mathcal{H}[\tau_{\mathrm{ns}}]$
  - truncation in the multipolar-like order N in $\mathcal{M}(\tau_{\mathrm{ns}})$ (sum over n)

$\hookrightarrow$ truncated version $\overline{\Box_{\mathrm{R}}^{-1}[\tau]}$ contains a finite number of lower order terms

Notation for a truncated quantity:

$\mathcal{H}[\tau]|_{\leq m} = \mathcal{H}[\tau]$ truncated at order $1/c^m$ included (with possible logs)

no expansion assumed

Lemma on the structure of the PN field

(main result for the iteration procedure temporarily accepted here)

Structure of $\overline{h}_{(m)}$

asymptotic expansion as $r \to \infty$ noted $\mathscr{M}(\overline{h}_{(m)})$ of the form

$$\sum\nolimits_{\text{finite sum}} G_{(m),L,a,p}(t)\hat{n}^L r^a \ln^p r + \mathcal{O}\left(\frac{1}{r^N}\right)$$

Proof by recurrence:

- first two ranks:

$$h^{\mu\nu}_{\text{lowest}} = \frac{16\pi G}{c^4}\Box^{-1}_{\mathrm{R}} T^{\mu\nu}|_{\text{lowest}} = -\frac{4G}{c^4}\int d^3\mathbf{x}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T^{\mu\nu}_{\text{lowest}}(\mathbf{x}', t)$$

$$\Rightarrow \mathscr{M}(h_{\text{lowest}}) = -\frac{4G}{c^4}\sum_{\ell=0} \frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\right)\int d^3\mathbf{x}'\hat{x}'^L T_{\text{lowest}}(\mathbf{x}', t)$$

General term of the lemma recurrence I

- hypothesis assumed to be true at rank lower than m
  $\hookrightarrow$ similar structure for $\tau_{(m-2)}$

  $\mathcal{H}[\tau_{[\leq m-2]}]|_{\leq m-2}$ seen to be of the required form since it has a finite number of terms

  amounts to show that $\mathrm{FP}\int \frac{d^3\mathbf{x}'}{-4\pi}|\mathbf{x}-\mathbf{x}'|^{k-1}\tau_{(m-2-k)}(\mathbf{x}',t)$ has the required structure

  step 1: subtract & add $\mathscr{M}_{\leq N}(\tau_{(m-k)})$ to the source $\rightarrow$
  $(\tau_{(m-2-k)} - \mathscr{M}_{[\leq N]}(\tau_{(m-2-k)})) + \mathscr{M}_{[\leq N]}(\tau_{(m-2-k)})$ (N high enough)

  step 2: consider the integral under investigation over
  $[\tau_{(m-2-k)} - \mathscr{M}_{[\leq N]}(\tau_{(m-2-k)})]$
  $\hookrightarrow$ has compact support so that usual multipole expansion applies

General term of the lemma recurrence II

multipole-like integral $\int \frac{d^3\mathbf{x}'}{-4\pi} \hat{x}'^L \mathscr{M}_{[\leq N]}(\tau_{(m-2-k)})$ is a sum of terms

$$\mathrm{FP} \int \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}'|^B \hat{x}'^L \hat{n}'^J r'^a \ln^p r' = \int d\Omega \hat{n}'^L \hat{n}'^J \int_0^{+\infty} dr' \, r'^{B+a+l+2} \ln^p r'$$

important result

$$\mathrm{FB}_{B=0} \int_0^{+\infty} dr \;\; r^{B+a} \ln^p r = \mathrm{FP}_{B=0} \left(\int_0^R + \int_R^{+\infty} \right) dr \;\; r^{B+a} \ln^p r = 0$$

$\Rightarrow$ resulting expansion for the current step

$$-\frac{4G}{c^4} \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \left(\frac{1}{r}\right) \int d^3\mathbf{x}' \hat{x}'^L \tau_{(m-2-k)}(\mathbf{x}', t)$$

General term of the lemma recurrence III

step3: consider the integral under investigation over $\mathscr{M}_{\leq N}(\tau_{(m-2-k)})$

$\hookrightarrow$ does not have compact support!

is a sum of terms $\mathrm{FP}\int \frac{d^3\mathbf{x}'}{-4\pi}|\mathbf{x}-\mathbf{x}'|^{k-1}n'^L r'^a \ln^p r'$

integral computable

$\rightarrow$ using the fact that $\int d\Omega(\mathbf{n}')\hat{n}'^L F(\mathbf{n}.\mathbf{n}') = 2\pi\hat{n}^L \int_{-1}^{1} dz F(z) P_\ell(z)$

$\rightarrow$ performing the change of variable $u = |\mathbf{x}-\mathbf{x}'|/r$, $v = r'/r$

$$\mathrm{FP}_{B=0}\frac{\hat{n}^L}{2} r^{2+B+a+k-1}\int_0^{+\infty} dv v^{1+B+a}\ln^p(rv)\times \\ \times \int_{|1-u|}^{1+u} du\; u^k P_\ell([1+v^2-u^2]/2v)$$

generate terms of the form $\mathrm{FP}_{B=0} r^{B+i+q}/(q+B+1)^s \ln^j r$

General term of the lemma recurrence IV

two cases

1. if $q \neq -1$, B can be taken directly to zero above
2. if $q = -1$, $\mathrm{FP}_{B=0} \sum_k \frac{B^k}{k!} \frac{\ln^k r}{B^s} r^{i-1} \ln^j r = \frac{r^{i-1}}{s!} \ln^{j+s} r$

possible logarithms generated at this step

Remarks on the lemma proof

- may be inserted in the recurrence proof for the main result
- shows explicitly that the multipole expansion of the PN field has the same structure as the $r \to 0$, $c \to \infty$ expansion of the exterior field

$$\mathscr{M}(\overline{\tau}) = \mathcal{M}(\tau)$$

$\hookrightarrow \mathscr{M}(\overline{\tau})$ may denoted by $\mathcal{M}(\overline{\tau})$

Proof I

- $\Box_{\mathrm{R}}^{-1}[\tau] = (\Box_{\mathrm{R}}^{-1})_{(R_i>)}[\tau] + (\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau]$
- $(\Box_{\mathrm{R}}^{-1})_{(R_i>)}[\tau]$ over the near zone

  $\Rightarrow$ source expandable in powers of $1/c$: $\tau = \overline{\tau}$
  commutation integral/sum $\int_{|\mathbf{x}'|<R_i} d^3\mathbf{x}' \sum = \sum \int_{|\mathbf{x}'|<R_i} d^3\mathbf{x}'$

$$(\Box_{\mathrm{R}}^{-1})_{R_i>}[\tau] = \sum_{n\geq 0, k\geq 0} \frac{(-1)^k}{k!} \frac{\partial_t^k}{c^k} \mathrm{FP} \int_{|\mathbf{x}'|<R_i} \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\overline{\tau}_{(n)}}{c^n}$$

- convergent integral $\longrightarrow$ regularization added without any effect

  $\Rightarrow$ $\int_{|\mathbf{x}'|<R_i} d^3\mathbf{x}' \longrightarrow \mathrm{FP} \int_{|\mathbf{x}'|<R_i} d^3\mathbf{x}'$
- addition and removal of $\sum \int_{|\mathbf{x}'|>R_i} d^3\mathbf{x}'$

Proof II

- intermediate expression for the retarded integral

$$\Box_{\mathrm{R}}^{-1}[\tau] = \overline{\Box_{\mathrm{R}}^{-1}}[\tau] + \mathcal{H}_{\mathrm{hom}}[\tau]$$

$$\text{with} \quad \mathcal{H}_{\mathrm{hom}}[\tau] = (\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau] - \sum_{n\geq 0, k\geq 0} \frac{(-1)^k}{k!} \frac{\partial_t^k}{c^k} \mathrm{FP} \int_{|\mathbf{x}'|>R_i} \frac{d^3\mathbf{x}'}{-4\pi} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\overline{\tau}_{(n)}}{c^n}$$

- extension of the $\mathcal{H}_{\mathrm{hom}}[\tau] - (\Box_{\mathrm{R}}^{-1})_{R_i<}[\tau]$ over the exterior zone

$\Rightarrow \quad |\mathbf{x}-\mathbf{x}'|^{k-1}\overline{\tau}_{(n)}(\mathbf{x}',t) \to \sum_\ell \mathcal{M}(|\mathbf{x}-\mathbf{x}'|^{k-1}\overline{\tau}_{(n)}(\mathbf{x}',t))_{(\ell)}$

- commutation integral/sum (over ℓ)

$\Rightarrow$ elementary integrals over terms having the same structure as those entering $\mathcal{M}(\overline{\tau})_{(\ell)}$

$$\sum_{m,a,p} F_{(m)L,a,p}(t) \mathrm{FP} \int_{|\mathbf{x}'|>R_i} d^3\mathbf{x}' \hat{n}_L r'^a \ln^p r'$$

Proof III

- fundamental remark

$$\mathrm{FP}_{B=0} \int_{|\mathbf{x}'|>R_i} d^3\mathbf{x}'\, \hat{n}_L r'^{B+a} \ln^p r' = 4\pi\delta_{\ell 0} \mathrm{FP}_{B=0} \int_{R_i}^{+\infty} dr'\, r'^{B+a} \ln^p r'$$
$$= -4\pi\delta_{\ell 0} \mathrm{FP}_{B=0} \int_0^{R_i} dr'\, r'^{B+a} \ln^p r'$$

$\Rightarrow$

Consequence

$\int_{|\mathbf{x}'|>R_i} \mathcal{M}(^-)_{(\ell)} \to -\int_{|\mathbf{x}'|<R_i} \mathcal{M}(^-)_{(\ell)}$

passage exterior zone/near zone

- resummation of the series over k and n allowed

$$\mathcal{H}_{\mathrm{hom}}[\tau] - (\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau] = \sum_\ell \mathrm{FP} \int_{|\mathbf{x}'|<R_i} \frac{d^3\mathbf{x}'}{-4\pi} \times$$
$$\times\, \mathcal{M}_{|\mathbf{x}'|>|\mathbf{x}|} \left(\sum_{k\geq 0,n} \frac{(-1)^k}{k!} |\mathbf{x}-\mathbf{x}'|^{k-1} \frac{\partial_t^k \overline{\tau}_{(n)}}{c^{k+n}} \right)_\ell$$

Proof IV

- $(\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau]$ over the exterior zone

  $\Rightarrow$ source expandable in multipole moments: $\tau = \mathcal{M}(\tau)$
  commutation integral/sum $\int_{|\mathbf{x}'|>R_i} d^3\mathbf{x}' \sum = \sum \int_{|\mathbf{x}'|>R_i} d^3\mathbf{x}'$

$$(\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau] = \sum_{\ell} FP \int_{|\mathbf{x}'|>R_i} \frac{d^3\mathbf{x}'}{-4\pi} \mathcal{M}_{|\mathbf{x}'|>|\mathbf{x}|} \left(\frac{\tau(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \right)_{\ell}$$

- $\mathcal{M}\left(\overline{\frac{\tau(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}} \right) = \mathcal{M}\left(\frac{\tau(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \right)$

  in the near zone
  $\Rightarrow$ combination of $(\Box_{\mathrm{R}}^{-1})_{(R_i<)}[\tau]$ with the preceding integral

Proof V

- more explicit form of $\mathcal{M}_{|\mathbf{x}'|>|\mathbf{x}|}\left(\frac{\tau(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right)_\ell$

$$\mathcal{M}_{|\mathbf{x}'|>|\mathbf{x}|}\left(\frac{\tau(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) = \sum_{i\geq 0,j}\frac{(-1)^i}{i!}x_I\partial'_I\left(\frac{\mathcal{M}(\tau)_{(j)}(\mathbf{y},t-|\mathbf{x}'|/c)}{|\mathbf{x}'|}\right)_{\mathbf{y}=\mathbf{x}'}$$

$$\Rightarrow \quad \mathcal{M}_{|\mathbf{x}'|>|\mathbf{x}|}\left(\frac{\tau(\mathbf{x},t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right)_\ell = \sum_{m\leq\ell}\frac{(-1)^m}{m!}x_M\partial'_M\left(\frac{\mathcal{M}(\tau)_{(\ell-m)}(\mathbf{y},t-|\mathbf{x}'|/c)}{|\mathbf{x}'|}\right)_{\mathbf{y}=\mathbf{x}'}$$

- Taylor-like form for $\mathcal{H}[\tau]$: possible substitution $\tau\longrightarrow\tau_{\mathrm{ns}}$ here

$$\mathcal{H}[\tau] = \sum_{\ell\geq 0,m}\frac{(-1)^\ell}{\ell!}x_L\mathrm{FP}\int\frac{d^3\mathbf{x}'}{-4\pi}\partial'_L\left(\frac{\mathcal{M}(\tau)_{(m)}(\mathbf{y},t-|\mathbf{x}'|/c)}{|\mathbf{x}'|}\right)_{\mathbf{y}=\mathbf{x}'}$$

Proof VI

- STF decomposition of x_L

  $\mathcal{H}[\tau]$ of the form $\displaystyle\sum_{\ell\geq 0}\frac{(-1)^\ell}{\ell!}\sum_{p\geq 0}\frac{-\alpha_{\ell,p}}{(2\ell+1)!!}r^{2p}\hat{x}_L\frac{\partial_t^{2p+2\ell+1}\mathcal{R}_L(t)}{c^{2p+2\ell+1}}$

- equality with the expansion of an antisymmetric wave

$$\mathcal{H}[\tau]=\sum_{\ell=0}^{+\infty}\hat{\partial}_L\left\{\frac{\overline{\mathcal{R}_L(t-r/c)-\mathcal{R}_L(t+r/c)}}{2r}\right\}$$

Physical interpretation of $\mathcal{R}_L$

$\mathcal{H}[\tau]$ regular antisymmetric wave $\longrightarrow$ decomposable in plane waves

$$\mathcal{H}(\mathbf{x},t)=\int d^3\mathbf{k}\left[A^{\mu\nu}_{\text{out}}(\mathbf{k})\,e^{-2\pi ikct}+A^{\mu\nu}_{\text{in}}(\mathbf{k})\,e^{2\pi ikct}\right]e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\quad\text{with}$$

$$A^{\mu\nu}_{\substack{\text{out}\\ \text{in}}}(\mathbf{k})=\frac{\epsilon c}{2ki}\sum_{\ell=0}^{+\infty}\frac{(-2\pi ik)_{<L>}}{\ell!}F(\mathcal{R}_L^{\mu\nu})(-\epsilon ck)\ \ \text{with}\ \ \epsilon_{\text{out}}=-1,\ \epsilon_{\text{in}}=1$$

Properties of the resulting solution

- $\overline{\Box_{\mathrm{R}}^{-1}}[\tau]$ = particular solution
  $\longrightarrow$ contains conservative and dissipative instantaneous terms
- $\mathcal{H}[\tau]$ = homogeneous solution associated with the tail effects
  $\longrightarrow$ appears at 4PN ($1/c^8$)

$$\Downarrow$$

reaction force obtained by expanding the retarded integral up to 3.5PN

- commutators $[\partial_\mu, \mathcal{H}] = -[\partial_\mu, \overline{\Box_{\mathrm{R}}^{-1}}]$
  $\longrightarrow$ harmonicity condition automatically fulfilled if $\partial_\nu \tau^{\mu\nu} = 0$
- agreement with Blanchet-Poujade checked directly

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