SELF-SIMILAR SOLUTIONS FOR THE MASSLESS VLASOV-EINSTEIN SYSTEM.

J. J. L. Velázquez.

ICMAT (CSIC-UAM-UC3M-UCM), Madrid.
Problem: Are there solutions of Einstein equations with suitable matter models exhibiting singularity formation without black hole formation?.

(Without horizon formation=Naked singularities).
Cosmic censorship hypothesis (Weak). (R. Penrose).
Einstein equations:

\[ G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta} \]

\[ R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta} \]

\[ G_{\alpha\beta} = \text{Einstein tensor, } T_{\alpha\beta} = \text{Energy-matter} \]
Cosmic censorship hypothesis.

Scalar field:

Christodoulou: Existence of solutions forming naked singularities (nongeneric).

Choptuik: Oscillations in self-similar variables. (Numerical solution).

Critical collapse: Transition between black-hole formation and absence of singularities.
Singularity formation for the Vlasov-Einstein system (joint work with A. Rendall).

Vlasov-Einstein system with spherical symmetry for a collisionless distribution of matter. (Schwarzschild’s coordinates).

\[
ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 [d\theta^2 + \sin^2(\theta) d\phi^2]
\]

\[
r = |x| \ , \ w = \frac{x \cdot v}{r} \ , \ F = |x \wedge v|^2
\]

\[
f = f(r, w, F, t)
\]
Vlasov-Einstein system (spherical symmetry):

\[ \partial_t f + e^{\mu-\lambda} \frac{w}{E} \partial_r f - \left( \lambda_t w + e^{\mu-\lambda} \mu_r E - e^{\mu-\lambda} \frac{F}{r^3 E} \right) \partial_w f = 0 \]

\[ E = \sqrt{1 + w^2 + \frac{F}{r^2}} \]

\[ e^{-2\lambda} (2r\lambda_r - 1) + 1 = 8\pi r^2 \rho \]

\[ e^{-2\lambda} (2r\mu_r + 1) - 1 = 8\pi r^2 p \]

\[ \mu(0) = \lambda(0) = \lambda(\infty) = 0 \]

\[ \rho = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} Ef dF dw \]

\[ p = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{w^2}{E} f dF dw \]
Goal: To construct self-similar solutions for the Vlasov-Einstein system generating singularities without formation on a horizon.

Absence of singularities for the Vlasov-Poisson system in dimension 3 (Pffaffelmoser, Lions-Perthame).


Vlasov-Einstein. (Gundlach-Martín-García).
Self-similar solutions $\iff$ Symmetry group.
Near the singularity high velocities expected:

$$w >> 1$$

$$E = \sqrt{1 + w^2 + \frac{F}{r^2}} \approx \sqrt{w^2 + \frac{F}{r^2}}$$

This motivates to study the massless Vlasov-Einstein system.

$$E = \sqrt{w^2 + \frac{F}{r^2}}$$
Symmetry group for the massless Vlasov-Einstein system:

\[ r \rightarrow \theta r \ , \ t \rightarrow \theta t \ , \ w \rightarrow \frac{1}{\sqrt{\theta}} w \ , \ F \rightarrow \theta F \ , \ f \rightarrow f \]

(Self-similar solutions would be invariant under this symmetry group).

A more convenient system of variables:

\[ w \rightarrow v = \frac{w}{\sqrt{F}} \]
Massless Vlasov-Einstein system:

\[
\partial_t f + e^{\mu-\lambda} \frac{v}{\tilde{E}} \partial_r f - \left( \lambda_t v + e^{\mu-\lambda} \mu_r \tilde{E} - e^{\mu-\lambda} \frac{F}{r^3 \tilde{E}} \right) \partial_w f = 0
\]

\[
\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}
\]

\[
e^{-2\lambda}(2r \lambda_r - 1) + 1 = 8\pi r^2 \rho
\]

\[
e^{-2\lambda}(2r \mu_r + 1) - 1 = 8\pi r^2 p
\]

\[
\mu(0) = \lambda(0) = \lambda(\infty) = 0
\]

\[
\rho = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{E}fF dF dv , \quad p = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{w^2}{\tilde{E}} fF dF dv
\]
In these new variables the gravitational fields depend on $f$ only through the quantity:

$$\zeta(r, v, t) = \int_0^\infty fF dF$$

The system of equations for $(f, \lambda, \mu)$ can be replaced by a system of equations for $(\zeta, \lambda, \mu)$.

Self-similar solutions:

$$f(r, v, F, t) = G(y, V, \Phi), \quad \mu(r, t) = U(y), \quad \lambda(r, t) = \Lambda(y)$$

$$y = \frac{r}{(-t)}, \quad V = (-t)v, \quad \Phi = \frac{F}{(-t)}$$
\[ yG_y - VG_V + \Phi G_{\Phi} + e^{U-\Lambda} \frac{V}{\hat{E}} G_y - \]
\[ - \left( y\Lambda_y V + e^{U-\Lambda} U_y \hat{E} - e^{U-\Lambda} \frac{1}{y^3 \hat{E}} \right) G_V \]
\[ = 0 \]

\[ \hat{E} = \sqrt{V^2 + \frac{1}{y^2}} \]

\[ e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2 \tilde{\rho} \]

\[ e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2 \tilde{p} \]

\[ \tilde{\rho} = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{E} G\Phi d\Phi dV , \]
\[ \tilde{p} = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{V^2}{\hat{E}} G\Phi d\Phi dV \]
Integration of the system:

\[ G = \text{constant along characteristics} \]

\[
\frac{dy}{d\sigma} = e^{-\Lambda} \frac{\partial H}{\partial V}, \quad \frac{dy}{d\sigma} = e^{-\Lambda} \frac{\partial H}{\partial V}
\]

\[ H = \frac{e^U}{y} \sqrt{V^2 y^2 + 1} + yV e^\Lambda \]

Trajectories contained in \( \{ H = h \} \).

Other auxiliary function:

\[ \zeta(r, v, t) = (-t)^2 \Theta(y, V) \]
Self-similar solutions:

$$\Theta = \Theta(y, V) \ , \ \tilde{\rho} = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \hat{E}\Theta dV \ , \ \tilde{p} = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \frac{V^2}{\hat{E}} \Theta dV$$

Along characteristics:

$$\frac{d\Theta}{d\sigma} = 2\Theta$$

$$H = \frac{e^U}{y} \sqrt{V^2 y^2 + 1} + y V e^\Lambda = h$$

$$e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2 \tilde{\rho}$$

$$e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2 \tilde{p}$$

$$\hat{E} = \sqrt{V^2 + \frac{1}{y^2}}$$
Singular self-similar solutions. (Dust-like solutions).
Solutions supported in one characteristic curve:

\[ G(y, V, \Phi) = A(y, V, \Phi)\delta(H(y, V) - h) \]
\[ \Theta(y, V) = \beta(\sigma)\delta(H(y, V) - h) \], \quad \beta(\sigma) = \beta_0 e^{2\sigma}, \quad \beta_0 \geq 0 \]
\[ \{H(y, V) = h\} = \{V = V_1(y), V = V_2(y) \mid y \geq y_0\}, \quad V_1(y) \leq V_2(y) \]

Fully dispersive solution: Support in three-dimensional sets of the space \((y, V, \Phi)\).

Dust solutions: Support in manifolds with smaller dimension than the dimension of the phase space.

These solutions are supported in two-dimensional surfaces of the space \((y, V, \Phi)\).
The problem of the considered dust-like solutions can be reduced to the (4-dimensional) system of ODEs:

\[ e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2 \tilde{\rho} \quad , \quad e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2 \tilde{p} \]

\[ \frac{d\sigma_i}{dy} = \frac{1}{y + e^{U-\Lambda} \frac{V_i y}{\sqrt{V_i^2 y^2 + 1}}} , \quad i = 1, 2 \]

\[ \tilde{\rho}(y) = \frac{\pi \beta_0 \chi_{\{y>y_0\}}}{y^3} \sum_{i=1}^{2} \frac{(-1)^i e^{2\sigma_i(y)} [(V_i(y))^2 y^2 + 1]}{V_i e^{U} y + ye^\Lambda \sqrt{V_i^2 y^2 + 1}} \]

\[ \tilde{p}(y) = \frac{\pi \beta_0 \chi_{\{y>y_0\}}}{y} \left[ \sum_{i=1}^{2} \frac{(-1)^i e^{2\sigma_i(y)} (V_i(y))^2}{V_i e^{U} y + ye^\Lambda \sqrt{V_i^2 y^2 + 1}} \right] \]
Behaviour of the support and the gravitational fields near the minimum radius:

\[ V_i(y_0) = V_0 = -\frac{1}{\sqrt{1 - y_0^2}} , \quad i = 1, 2 \]

\[ V_i(y) - V_0 \sim \frac{K_i}{\sqrt{y - y_0}} , \quad y \to y_0^+ , \quad i = 1, 2 \]

\[ \Lambda(y) \sim \theta_1 \sqrt{y - y_0} , \quad y \to y_0^+ \]

\[ U(y) \sim \theta_2 \sqrt{y - y_0} , \quad y \to y_0^+ \]
Dust-like solutions: Main result.

Theorem: For \( y_0 \) small enough there exist \( \beta_0 \) such that the solution of the previous ODE system with the prescribed initial conditions at \( y \to y_0^+ \) and defined for all \( y \geq y_0 \). Moreover:

\[
U \sim \log\left(\frac{y}{y_0}\right) + \log\left(\sqrt{1 - y_0^2}\right) + o(1), \quad \Lambda \to \log\left(\sqrt{3}\right) \quad \text{as} \quad y \to \infty
\]

\[
V_1 \sim -\frac{2y_0\sqrt{3(1 - y_0^2)}}{(1 - 4y_0^2)y}, \quad V_2 \sim -\frac{\sqrt{1 - y_0^2}}{\sqrt{3}y_0} \frac{C_1}{y} \left(\frac{y_0}{y}\right)^2 \quad \text{as} \quad y \to \infty
\]
Idea of the proof:

Reduction to an autonomous system:

\[ s = \log\left(\frac{y}{y_0}\right), \quad U = \log\left(\frac{y}{y_0}\right) + u, \quad \zeta_i = yV_i, \quad Q_i = \frac{y_0}{y} e^{\sigma_i}, \quad i = 1, 2 \]
Then the differential equations become:

\[ e^u \sqrt{\xi_i^2 + 1} + y_0 \xi_i e^\Lambda = \sqrt{1 - y_0^2} \quad , \quad i = 1, 2 \quad , \quad \xi_1 < \xi_2 \]

\[ \frac{dQ_i}{ds} = -\frac{e^u Q_i \xi_i}{y_0 e^\Lambda \sqrt{(\xi_i)^2 + 1} + \xi_i e^u} \quad , \quad i = 1, 2 \]

\[ e^{-2\Lambda}(2\Lambda s - 1) + 1 = \frac{\theta}{2} \sum_{i=1}^{2} \frac{Q_i^2 [\xi_i^2 + 1]}{\xi_i e^u + y_0 e^\Lambda \sqrt{\xi_i^2 + 1}} \quad , \quad \theta = \frac{16\pi^2 \beta_0}{y_0} \]

\[ e^{-2\Lambda}(2u_s + 3) - 1 = \frac{\theta}{2} \sum_{i=1}^{2} \frac{Q_i^2 (\xi_i)^2}{\xi_i e^u + y_0 e^\Lambda \sqrt{\xi_i^2 + 1}} \]

with initial conditions:

\[ u = 0, \quad \Lambda = 0 \quad , \quad Q_i = 1 \quad , \quad i = 1, 2 \quad \text{at} \quad s = 0^+ \quad (*) \]
There exists a unique trajectory satisfying (*).
(The square root singularity near $y_0$ can be removed):

$$Z = \sqrt{(e^{-2u}(1 - y_0^2) - 1)(1 - y_0^2 e^{2(\Lambda - u)}) + y_0^2(1 - y_0^2)e^{2(\Lambda - 2u)}}$$

$$G = e^{-2\Lambda}, \quad ds = 2GZd\chi, \quad \chi = 0 \text{ at } s = 0$$

$$\frac{dQ_1}{d\chi} = 2GQ_1\zeta_1, \quad \frac{dQ_2}{d\chi} = -2GQ_1\zeta_1$$

$$\frac{dG}{d\chi} = 2G\left[Z(1 - G) - \frac{\theta e^{-u}}{2}[Q_1^2(\zeta_1^2 + 1) + Q_2^2(\zeta_2^2 + 1)]\right]$$

$$\frac{dZ}{d\chi} = (3G - 1 - 2G\Delta)(Z^2 + 1)$$

$\Delta = \Delta(G, Z, Q_1, Q_2)$ is analytic near $(G, Z, Q_1, Q_2) = (1, 0, 1, 1)$. 
A trajectory globally defined approaches the equilibrium point:

\[ Q_1 = Q_{1,\infty} = 0, \quad Q_2 = Q_{2,\infty} = \frac{2\sqrt{y_0}}{3^{\frac{1}{4}} \sqrt{\theta}} \]

\[ \Lambda = \Lambda_{\infty} = \frac{\log(3)}{2}, \quad u = u_{\infty} = \log\left(\sqrt{1 - y_0^2}\right) \]

For \( y_0 \) small it is possible to approximate the three-dimensional stable manifold of the equilibrium point \((Q_{1,\infty}, Q_{2,\infty}, \Lambda_{\infty}, u_{\infty})\).

Shooting argument: Changing \( \theta \) is it possible to have the point \((1, 1, 0, 0)\) in this stable manifold.
In the next region it is convenient to use the variable $G, Z$.

$G = e^{\lambda G}, \quad Z = e^Z$.

Then the curve $Y$ becomes

$$
\left\{ \left( Z^2 + 1 \right) G \left( 1 - G \right) = \frac{4}{3\pi} \right\} \Rightarrow \delta
$$
We now extend the portion of the manifold to values with $\Theta \geq 0$. We study the
behavior of the manifold for $\Theta \rightarrow \infty$. In the limit $\Theta \rightarrow 0^+$
the manifold becomes flat.
SOME PHYSICAL PROPERTIES OF THE SOLUTION (I).

The solution obtained in this way produces a true singularity for the curvature. (Blow-up of Kretschmann scalar):

$$R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq \frac{16m^2}{r^6}, \quad m = \frac{r}{2} \left(1 - e^{-2\Lambda\left(\frac{r}{(-t)}\right)}\right) \sim \frac{r}{3} \quad \text{as} \quad \frac{r}{(-t)} \to \infty$$

(The singularity is not a coordinate singularity).
SOME PHYSICAL PROPERTIES OF THE SOLUTION (II).

The obtained singularity describes the portion of space-time contained inside the light-cone arriving to the point \( r = 0, t = 0^- \). Such a light-cone is reached as \( r \to \infty \). Consequence: The solution cannot be continued by means of a direct "glueing" with a planar space. To obtain a solution that behaves asymptotically as Minkowsky space one should move away from the self-similar regime.
Gundlach-Martin-García self-similar solution for the VE system.

- Fully dispersive.
- Different rescaling group.
- Singular along the light-cone.
- Small perturbation of Minkowsky space.
- Numerical solution of some equations.
WORK IN PROGRESS. FURTHER DEVELOPMENTS.

(1) Thickening of the solution. To derive a fully-dispersive self-similar solution. (Asymptotics).

(2) It is possible to "glue" ("match") this solution with an asymptotically flat Minkowsky space?.. (Away from the self-similar setting).

(3) Effect of the mass of the particles. (Away from the self-similar setting).
CONCLUDING REMARKS.

- Existence of self-similar "dust-like" solutions for the massless Vlasov-Einstein model.
- The key idea is to reduce the problem to an autonomous four-dimensional dynamical system.
- Asymptotic expansions for fully dispersive radially symmetric self-similar solutions of the VE system generating singularities in finite time.
- Rigorous construction of the fully dispersive self-similar solutions?.
- Solutions of the full VE system (with mass): How to modify the solutions away from the singularity to connect then with a flat Minkowsky space-time?.