SELF-SIMILAR SOLUTIONS FOR THE MASSLESS VLASOV-EINSTEIN SYSTEM.

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Problem: Are there solutions of Einstein equations with suitable matter models exhibiting singularity formation without black hole formation?.

(Without horizon formation$=$Naked singularities).

Cosmic censorship hypothesis (Weak). (R. Penrose).

Einstein equations:

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad , \quad R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$$

$G_{\alpha\beta} =$ Einstein tensor, $T_{\alpha\beta} =$ Energy-matter

Cosmic censorship hypothesis.

Scalar field:

Christodoulu: Existence of solutions forming naked singularities (nongeneric).

Choptuik: Oscillations in self-similar variables. (Numerical solution).

Critical collapse: Transition between black-hole formation and absence of singularities.

Singularity formation for the Vlasov-Einstein system (joint work with A. Rendall).

Vlasov-Einstein system with spherical symmetry for a collisionless distribution of matter. (Schwarzschild's coordinates).

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2[d\theta^2 + \sin^2(\theta)d\varphi^2]$$

$$r = |x| \ , \ w = \frac{x \bullet v}{r} \ , \ F = |x \wedge v|^2$$

$$f = f(r, w, F, t)$$

Vlasov-Einstein system (spherical symmetry):

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - \left(\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{F}{r^3 E}\right)\partial_w f = 0$$

$$E = \sqrt{1 + w^2 + \frac{F}{r^2}}$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p$$

$$\mu(0) = \lambda(0) = \lambda(\infty) = 0$$

$$\rho = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} E f dF dw$$

$$p = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{w^2}{E} f dF dw$$

Goal: To construct self-similar solutions for the Vlasov-Einstein system generating singularities without formation on a horizon.

Absence of singularities for the Vlasov-Poisson system in dimension 3 (Pffaffelmoser, Lions-Perthame).

Vlasov-Poisson. Spatial dimension $4.$ (Lemou-Mehats-Raphael).

Vlasov-Einstein. (Gundlach-Martín-García).

Self-similar solutions $\Leftrightarrow$ Symmetry group.

Near the singularity high velocities expected:

$$w >> 1$$

$$E = \sqrt{1 + w^2 + \frac{F}{r^2}} \approx \sqrt{w^2 + \frac{F}{r^2}}$$

This motivates to study the massless Vlasov-Einstein system.

$$E = \sqrt{w^2 + \frac{F}{r^2}}$$

Symmetry group for the massless Vlasov-Einstein system:

$$r \to \theta r \ , \quad t \to \theta t \ , \quad w \to \frac{1}{\sqrt{\theta}} w \ , \quad F \to \theta F \ , \quad f \to f$$

(Self-similar solutions would be invariant under this symmetry group).

A more convenient system of variables:

$$w \to v = \frac{w}{\sqrt{F}}$$

Massless Vlasov-Einstein system:

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\tilde{E}}\partial_r f - \left(\lambda_t v + e^{\mu-\lambda}\mu_r \tilde{E} - e^{\mu-\lambda}\frac{F}{r^3\tilde{E}}\right)\partial_w f = 0$$

$$\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p$$

$$\mu(0) = \lambda(0) = \lambda(\infty) = 0$$

$$\rho = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\tilde{E} f F dF dv \ , \ p = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\tilde{E}} f F dF dv$$

In these new variables the gravitational fields depend on f only through the quantity:

$$\zeta(r,v,t) = \int_0^\infty fFdF$$

The system of equations for (f,λ,μ) can be replaced by a system of equations for (ζ,λ,μ).

Self-similar solutions:

$$f(r,v,F,t) = G(y,V,\Phi) \ , \ \mu(r,t) = U(y) \ \ , \ \ \lambda(r,t) = \Lambda(y)$$

$$y = \frac{r}{(-t)} \ \ , \ \ V = (-t)v \ \ , \ \ \Phi = \frac{F}{(-t)}$$

$$
yG_y - VG_V + \Phi G_\Phi + e^{U-\Lambda}\frac{V}{\hat{E}}G_y -
$$

$$
-\left(y\Lambda_y V + e^{U-\Lambda}U_y\hat{E} - e^{U-\Lambda}\frac{1}{y^3\hat{E}}\right)G_V
$$

$$
= 0
$$

$$
\hat{E} = \sqrt{V^2 + \frac{1}{y^2}}
$$

$$
e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2\tilde{\rho}
$$

$$
e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2\tilde{p}
$$

$$
\tilde{\rho} = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\hat{E}G\Phi d\Phi dV \ , \ \tilde{p} = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{V^2}{\hat{E}}G\Phi d\Phi dV
$$

Integration of the system:

$$G = \text{constant along characteristics}$$

$$\frac{dy}{d\sigma} = e^{-\Lambda}\frac{\partial H}{\partial V} \;,\; \frac{dy}{d\sigma} = e^{-\Lambda}\frac{\partial H}{\partial V}$$

$$H = \frac{e^U}{y}\sqrt{V^2y^2+1} + yVe^{\Lambda}$$

Trajectories contained in $\{H = h\}$.

Other auxiliary function:

$$\zeta(r,v,t) = (-t)^2\Theta(y,V)$$

Self-similar solutions:

$$\Theta = \Theta(y, V) \ , \ \tilde{\rho} = \frac{\pi}{r^2}\int_{-\infty}^{\infty} \hat{E}\Theta dV \ , \ \tilde{p} = \frac{\pi}{r^2}\int_{-\infty}^{\infty} \frac{V^2}{\hat{E}}\Theta dV$$

Along characteristics:

$$\frac{d\Theta}{d\sigma} = 2\Theta$$

$$H = \frac{e^U}{y}\sqrt{V^2y^2 + 1} + yVe^{\Lambda} = h$$

$$e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2\tilde{\rho}$$

$$e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2\tilde{p}$$

$$\hat{E} = \sqrt{V^2 + \frac{1}{y^2}}$$

Singular self-similar solutions. (Dust-like solutions).

Solutions supported in one characteristic curve:

$$G(y,V,\Phi) = A(y,V,\Phi)\delta(H(y,V) - h)$$

$$\Theta(y,V) = \beta(\sigma)\delta(H(y,V) - h) \ , \ \beta(\sigma) = \beta_0 e^{2\sigma} \ , \ \beta_0 \geq 0$$

$$\{H(y,V) = h\} = \{V = V_1(y), V = V_2(y) \ , \ y \geq y_0\} \ , \ V_1(y) \leq V_2(y)$$

Fully dispersive solution: Support in three-dimensional sets of the space (y,V,Φ).

Dust solutions: Support in manifolds with smaller dimension than the dimension of the phase space.

These solutions are supported in two-dimensional surfaces of the space (y,V,Φ).

The problem of the considered dust-like solutions can be reduced to the (4-dimensional) system of ODEs:

$$e^{-2\Lambda}(2y\Lambda_y - 1) + 1 = 8\pi y^2 \tilde{\rho} \ , \ e^{-2\Lambda}(2yU_y + 1) - 1 = 8\pi y^2 \tilde{p}$$

$$\frac{d\sigma_i}{dy} = \frac{1}{y + e^{U-\Lambda} \frac{V_i y}{\sqrt{V_i^2 y^2 \text{‘} 1}}} \ , \ i = 1, 2$$

$$\tilde{\rho}(y) = \frac{\pi \beta_0 \chi_{\{y>y_0\}}}{y^3} \sum_{i=1}^{2} \frac{(-1)^i e^{2\sigma_i(y)} \left[(V_i(y))^2 y^2 + 1 \right]}{\left[V_i e^U y + y e^{\Lambda} \sqrt{V_i^2 y^2 + 1} \right]}$$

$$\tilde{p}(y) = \frac{\pi \beta_0 \chi_{\{y>y_0\}}}{y} \left[\sum_{i=1}^{2} \frac{(-1)^i e^{2\sigma_i(y)} (V_i(y))^2}{\left[V_i e^U y + y e^{\Lambda} \sqrt{V_i^2 y^2 + 1} \right]} \right]$$

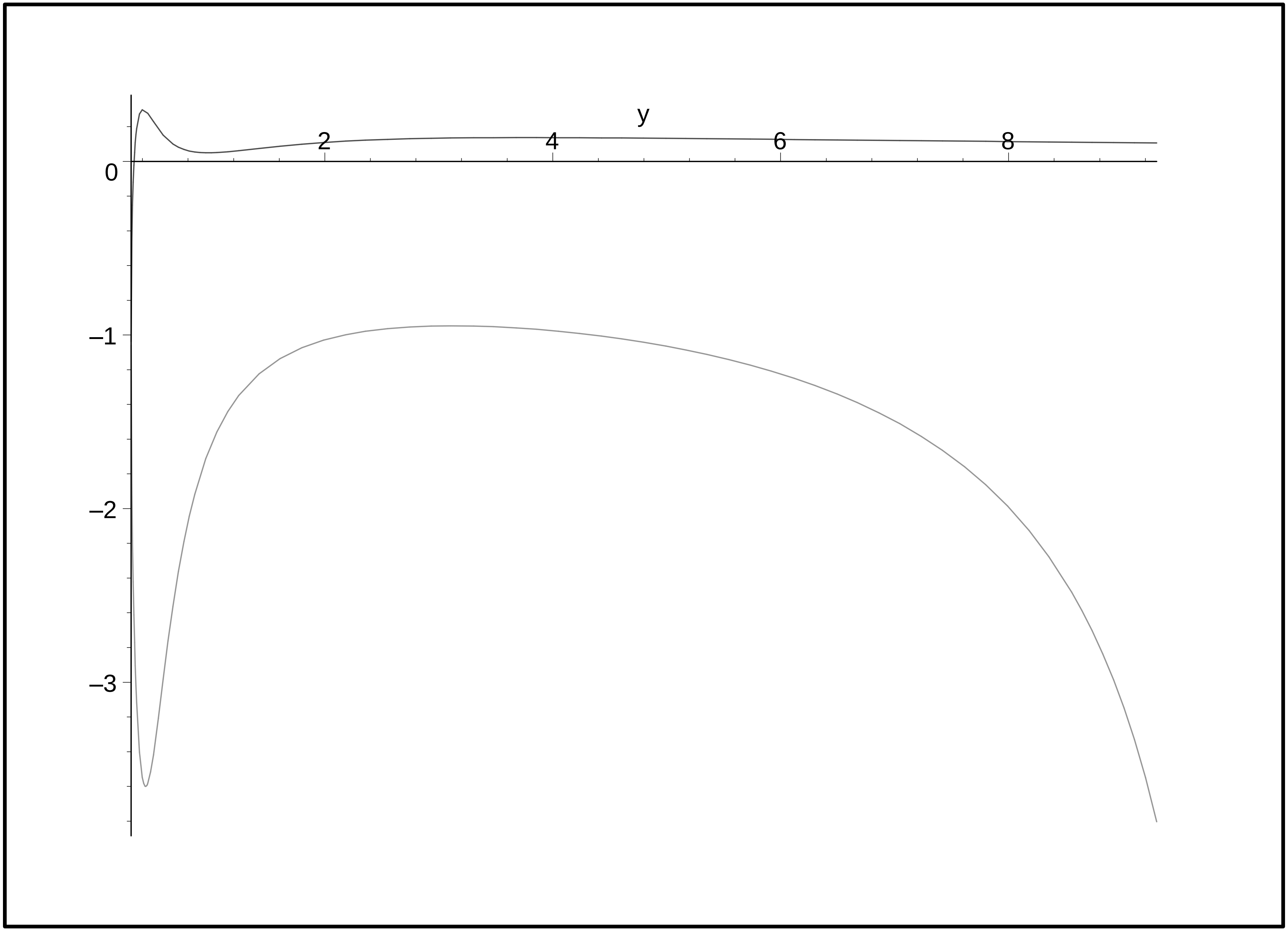
Behaviour of the support and the gravitational fields near the minimum radius:

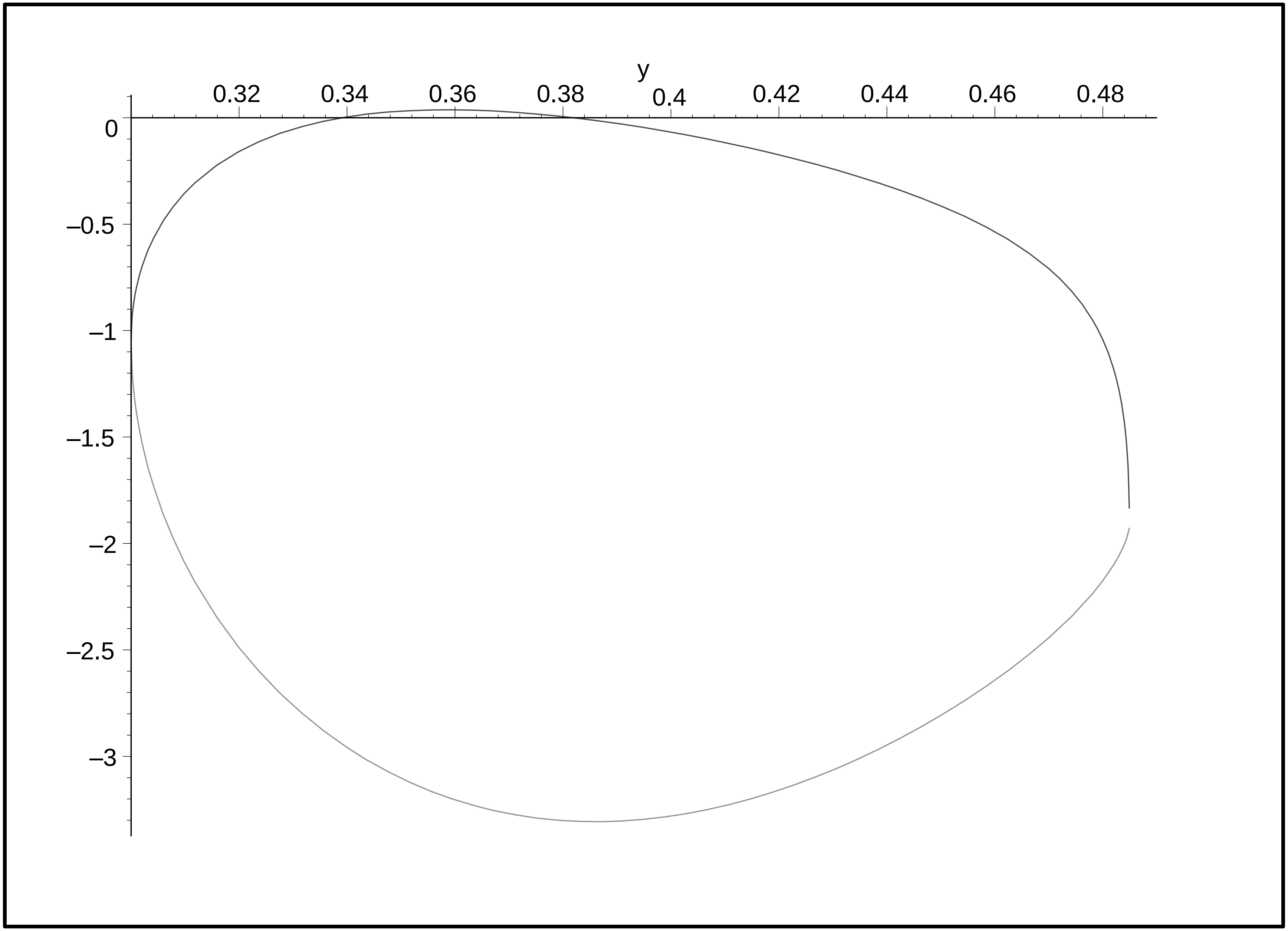
$$V_i(y_0) = V_0 = -\frac{1}{\sqrt{1-y_0^2}} \quad , \quad i = 1,2$$

$$V_i(y) - V_0 \sim \frac{K_i}{\sqrt{y-y_0}} \quad , \quad y \to y_0^+ \quad , \quad i = 1,2$$

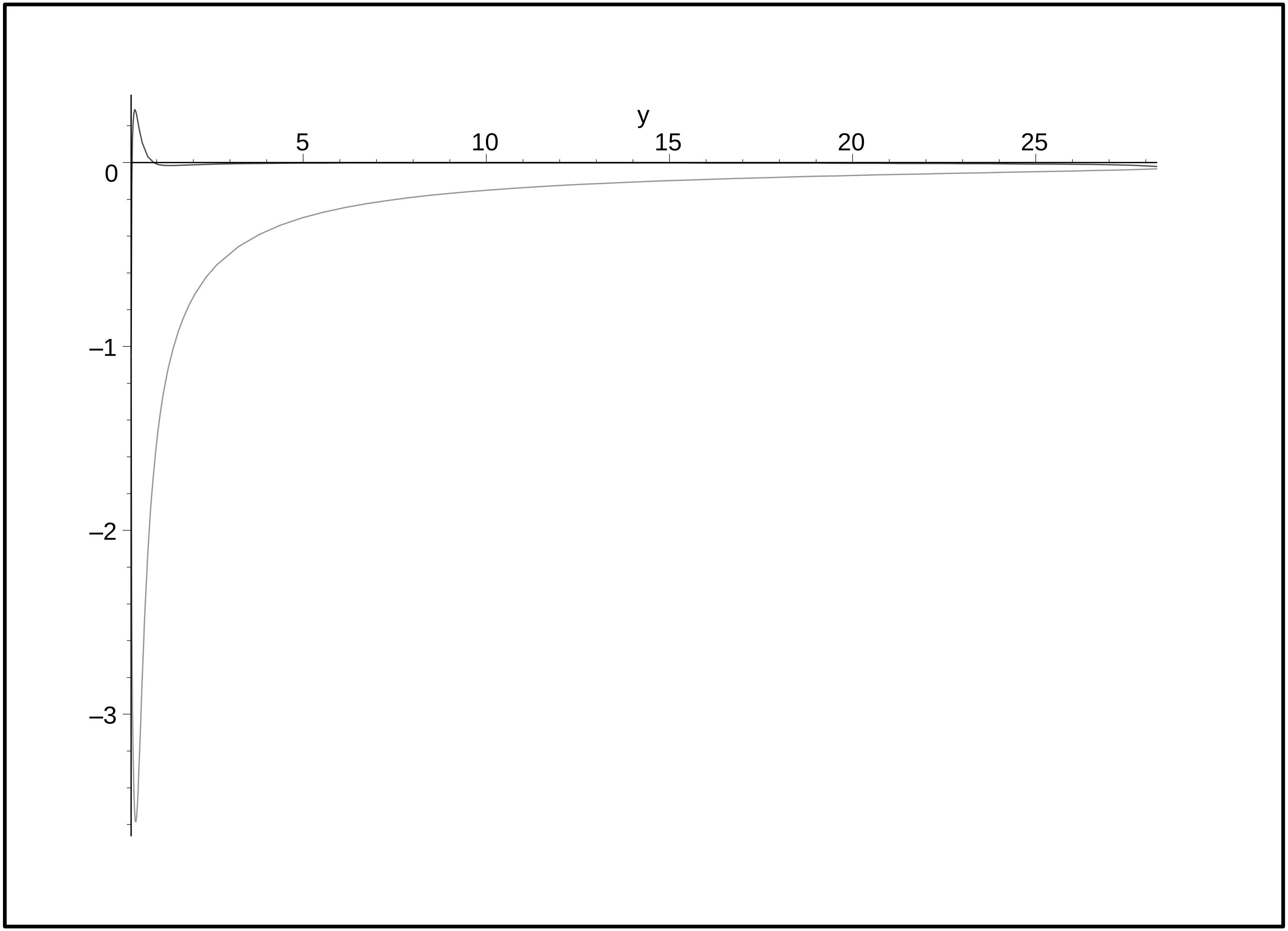
$$\Lambda(y) \sim \theta_1 \sqrt{y-y_0} \quad , \quad y \to y_0^+$$

$$U(y) \sim \theta_2 \sqrt{y-y_0} \quad , \quad y \to y_0^+$$


y
2
4
6
8
0
–1
–2
–3



y
0.32
0.34
0.36
0.38
0.4
0.42
0.44
0.46
0.48
0
–0.5
–1
–1.5
–2
–2.5
–3


y
5
10
15
20
25
0
–1
–2
–3

Dust-like solutions: Main result.

Theorem: For y_0 small enough there exist β_0 such that the solution of the previous ODE system with the prescribed initial conditions at $y \to y_0^+$ and defined for all $y \geq y_0$. Moreover:

$$U \sim \log\left(\frac{y}{y_0}\right) + \log\left(\sqrt{1-y_0^2}\right) + o(1)\ ,\ \Lambda \to \log\left(\sqrt{3}\right) \text{ as } y \to \infty$$

$$V_1 \sim -\frac{2y_0\sqrt{3(1-y_0^2)}}{(1-4y_0^2)y}\ ,\ V_2 \sim -\frac{\sqrt{1-y_0^2}}{\sqrt{3}\,y_0}\frac{C_1}{y}\left(\frac{y_0}{y}\right)^2 \text{ as } y \to \infty$$

Idea of the proof:

Reduction to an autonomous system:

$$s = \log\left(\frac{y}{y_0}\right) \ , \ U = \log\left(\frac{y}{y_0}\right) + u \ , \ \zeta_i = yV_i \ , \ Q_i = \frac{y_0}{y} e^{\sigma_i} \ , \ i = 1,2$$

Then the differential equations become:

$$e^{u}\sqrt{\zeta_i^2+1}+y_0\zeta_i e^{\Lambda}=\sqrt{1-y_0^2} \quad , \quad i=1,2 \quad , \quad \zeta_1<\zeta_2$$

$$\frac{dQ_i}{ds}=-\frac{e^{u}Q_i\zeta_i}{\left[y_0e^{\Lambda}\sqrt{(\zeta_i)^2+1}+\zeta_i e^{u}\right]} \quad , \quad i=1,2$$

$$e^{-2\Lambda}(2\Lambda_s-1)+1=\frac{\theta}{2}\sum_{i=1}^{2}\frac{Q_i^2[\zeta_i^2+1]}{\left|\zeta_i e^{u}+y_0e^{\Lambda}\sqrt{\zeta_i^2+1}\right|} \quad , \quad \theta=\frac{16\pi^2\beta_0}{y_0}$$

$$e^{-2\Lambda}(2u_s+3)-1=\frac{\theta}{2}\sum_{i=1}^{2}\frac{Q_i^2(\zeta_i)^2}{\left|\zeta_i e^{u}+y_0e^{\Lambda}\sqrt{\zeta_i^2+1}\right|}$$

with initial conditions:

$$u=0,\ \Lambda=0\ ,\ Q_i=1\ ,\ i=1,2\ \text{ at }\ s=0^{+} \qquad (*)$$

- There exists a unique trajectory satisfying (*).

(The square root singularity near y_0 can be removed):

$$Z = \sqrt{(e^{-2u}(1-y_0^2)-1)(1-y_0^2 e^{2(\Lambda-u)}) + y_0^2(1-y_0^2)e^{2(\Lambda-2u)}}$$

$$G = e^{-2\Lambda} \ , \ ds = 2GZd\chi \ , \ \chi = 0 \text{ at } s = 0$$

$$\frac{dQ_1}{d\chi} = 2GQ_1\zeta_1 \ , \ \frac{dQ_2}{d\chi} = -2GQ_1\zeta_1$$

$$\frac{dG}{d\chi} = 2G\left[Z(1-G) - \frac{\theta e^{-u}}{2}[Q_1^2(\zeta_1^2+1) + Q_2^2(\zeta_2^2+1)]\right]$$

$$\frac{dZ}{d\chi} = (3G - 1 - 2G\Delta)(Z^2+1)$$

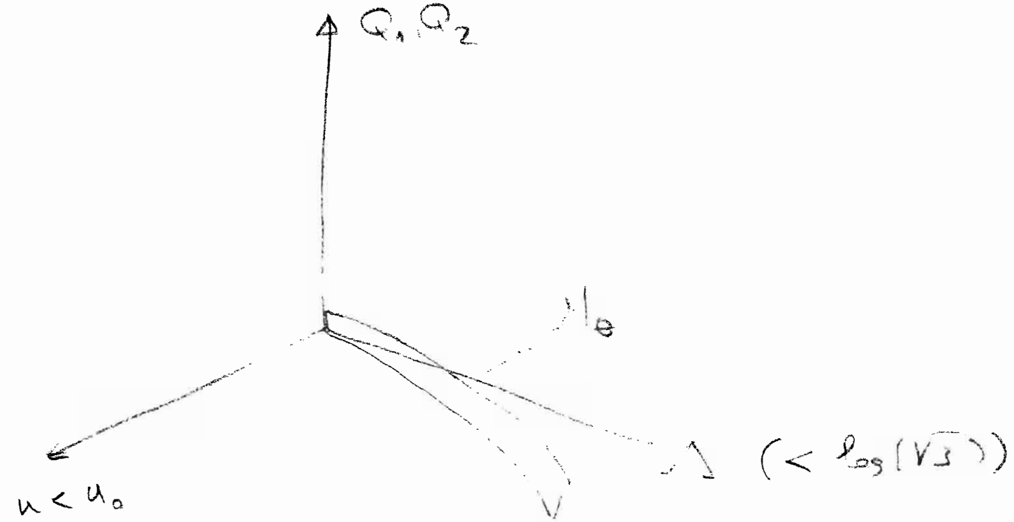
$\Delta = \Delta(G, Z, Q_1, Q_2)$ is analytic near $(G, Z, Q_1, Q_2) = (1, 0, 1, 1)$.

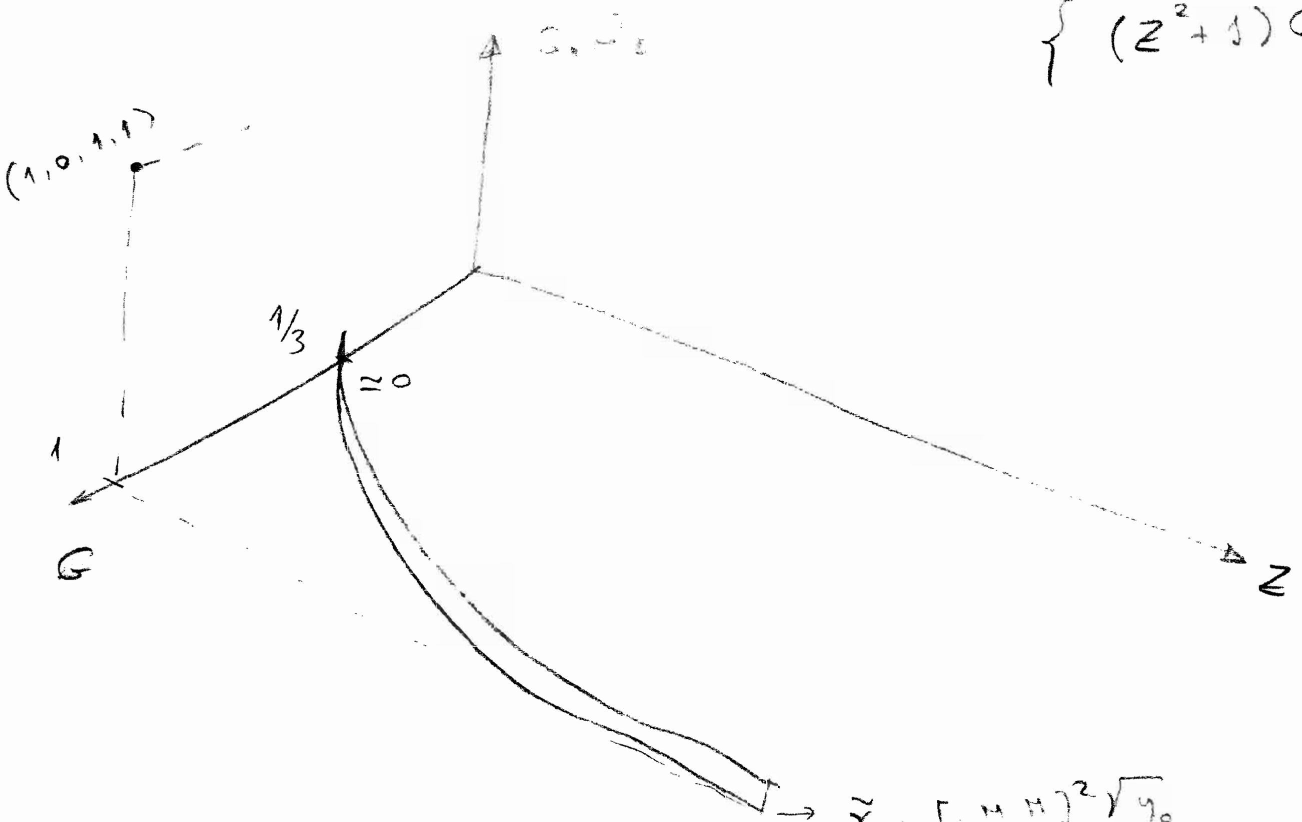
- A trajectory globally defined approaches the equilibrium point:

$$Q_1 = Q_{1,\infty} = 0 \ , \ Q_2 = Q_{2,\infty} = \frac{2\sqrt{y_0}}{3^{\frac{1}{4}}\sqrt{\theta}}$$

$$\Lambda = \Lambda_\infty = \frac{\log(3)}{2} \ , \ u = u_\infty = \log\left(\sqrt{1 - y_0^2}\right)$$

- For y_0 small it is possible to approximate the three-dimensional stable manifold of the equilibrium point $(Q_{1,\infty}, Q_{2,\infty}, \Lambda_\infty, u_\infty)$.
- Shooting argument: Changing θ is it possible to have the point $(1, 1, 0, 0)$ in this stable manifold.





In the next region it is convenient to use the variables G, Z

$$G = e^{-2\Lambda}, \quad Z \text{ in } [illegible]$$

Then the curve γ becomes

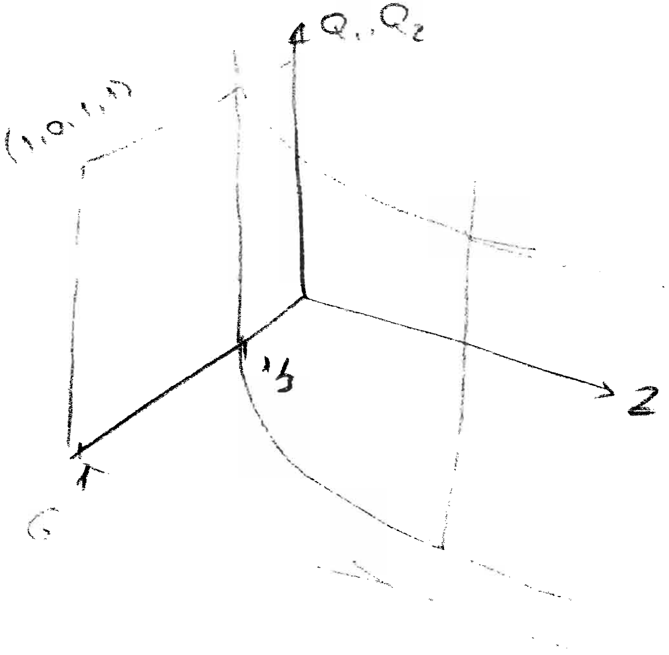
$$\left\{ (Z^2+1)\,G\,(1-G)^2 = \frac{4}{3^3} \right\} \equiv \tilde{\gamma}$$

$\rightarrow \tilde{\gamma} \times$ [illegible] $]^2 \sqrt{y_0}$

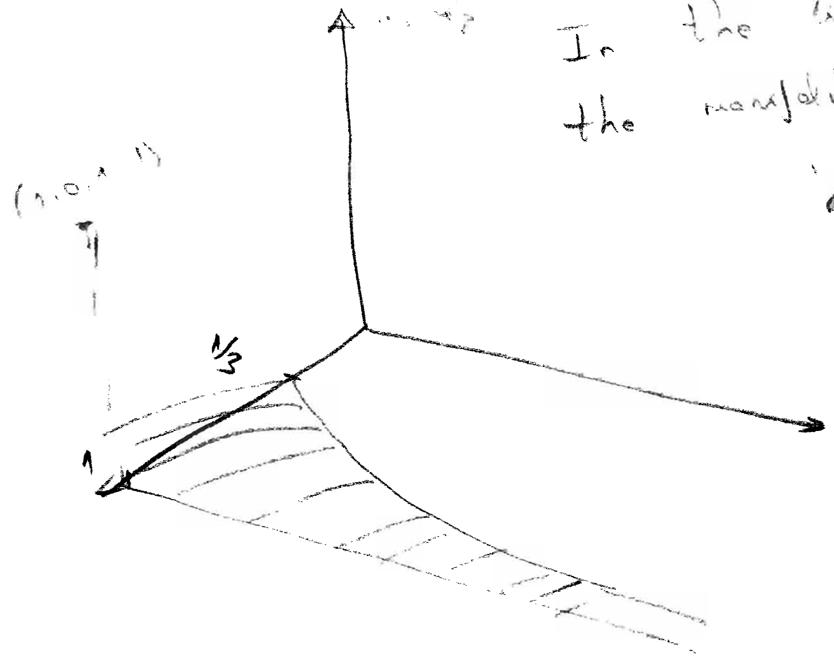
We now extend this portion of the manifold to values with Q_1, Q_2 of order one. The goal is to show that the manifold contains the point $(G, Z, Q_1, Q_2) = (1, 0, 1, 1)$ for some θ. We study the asymptotics of the manifold for $\theta \to 0, \infty$.

In the limit $\theta \to 0^+$ the manifold θ becomes vertical.

In the limit $\theta \to \infty$ the manifold becomes 'flat'.



$\theta \to 0^+$



$\theta \to \infty$

SOME PHYSICAL PROPERTIES OF THE SOLUTION (I).

The solution obtained in this way produces a true singularity for the curvature. (Blow-up of Kretschmann scalar):

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \geq \frac{16m^2}{r^6} \;\;,\;\; m = \frac{r}{2}\left(1 - e^{-2\Lambda\left(\frac{r}{(-t)}\right)}\right) \sim \frac{r}{3} \;\text{ as }\; \frac{r}{(-t)} \to \infty$$

(The singularity is not a coordinate singularity).

SOME PHYSICAL PROPERTIES OF THE SOLUTION (II).

The obtained singularity describes the portion of space-time contained inside the light-cone arriving to the point $r = 0,\ t = 0^-$. Such a light-cone is reached as $r \to \infty$. Consequence: The solution cannot be continued by means of a direct "glueing" with a planar space. To obtain a solution that behaves asymptotically as Minkowsky space one should move away from the self-similar regime.

Gundlach-Martin-García self-similar solution for the VE system.

- Fully dispersive.
- Different rescaling group.
- Singular along the light-cone.
- Small perturbation of Minkowsky space.
- Numerical solution of some equations.

WORK IN PROGRESS. FURTHER DEVELOPMENTS.

(1) Thickening of the solution. To derive a fully-dispersive self-similar solution. (Asymptotics).

(2) It is possible to "glue" ("match") this solution with an asymptotically flat Minkowsky space?. (Away from the self-similar setting).

(3) Effect of the mass of the particles. (Away from the self-similar setting).

CONCLUDING REMARKS.

- Existence of self-similar "dust-like" solutions for the massless Vlasov-Einstein model.
- The key idea is to reduce the problem to an autonomous four-dimensional dynamical system.
- Asymptotic expansions for fully dispersive radially symmetric self-similar solutions of the VE system generating singularities in finite time.
- Rigorous construction of the fully dispersive self-similar solutions?.
- Solutions of the full VE system (with mass): How to modify the solutions away from the singularity to connect then with a flat Minkowsky space-time?.