Numerical schemes for short wave long wave interaction equations

Paulo Amorim

Mário Figueira

CMAF - Université de Lisbonne

LJLL - Séminaire Fluides Compréssibles, 29 novembre 2010

Outline

- A **short wave long wave interaction** system: Convergence of semidiscrete finite difference schemes
- A **Schrödinger–Conservation law** system: Convergence of semidiscrete finite volume type schemes
- Numerical Results

Short wave long wave interaction equations

Systems of coupled nonlinear PDEs arising in gravity-capillary waves, plasma physics, etc... **(Benney, 1977)**

$$\begin{cases} i\partial_t u + ic_1\partial_x u + \partial_{xx} u = \alpha u\, v + \gamma |u|^2 u \\ \partial_t v + c_2\partial_x v + \mu\partial_x^3 v + \nu\partial_x v^2 = \beta\partial_x(|u|^2), \qquad t \in [0,\infty), \quad x \in \mathbb{R}. \end{cases}$$

Nonlinear Schrödinger equation coupled with:

- Dispersive equations (KdV)
- Hyperbolic equations (scalar conservation laws or systems)
- $\cdots$

A short wave long wave interaction system

$$\begin{cases} i\partial_t u + \partial_{xx} u = \alpha\, |u|^2 u + vu \\ \partial_t v = \partial_x(|u|^2). \end{cases}$$

Tsutsumi & Hatano (1994): Global well-posedness of the Cauchy problem in $H^{j+1/2}(\mathbb{R}) \times H^j(\mathbb{R})$, $j \geq 1$.

Bekiranov, Ogawa, Ponce (1998): Local well-posedness of the Cauchy problem in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$, $s \geq 0$.

Approximation of a short wave long wave interaction system

$$u^h(t) = (u_j(t))_{j\in\mathbb{Z}}, \quad v^h(t) = (v_j(t))_{j\in\mathbb{Z}}, \quad h = x_{j+1} - x_j$$
$$u_j(t), v_j(t) \approx u(x_j, t), v(x_j, t)$$

A semidiscrete finite difference scheme

$$\begin{cases} i\dfrac{du_j}{dt} + \Delta^h u_j = \alpha\,|u_j|^2 u_j + v_j u_j \\ \dfrac{dv_j}{dt} = D_0(|u_j|^2) \\ u_j(0) = u_{0j}, \quad v_j(0) = v_{0j}, \end{cases}$$

$$\Delta^h u_j := \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}), \qquad D_0 u_j := \frac{1}{2h}(u_{j+1} - u_{j-1}).$$

$$\mathbf{P}_1^h u^h := \text{ Piecewise linear, continuous interpolator}$$

Convergence result

Theorem (PA, M.Figueira, C.R.A.S. 2009)

Let $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$*. For each* $T > 0$*, the sequences* u^h, v^h *satisfy:*

$$\begin{aligned} \mathbf{P}_1^h u^h &\rightharpoonup u \quad \text{in} \quad L^\infty([-T,T]; H^1(\mathbb{R})) \quad \text{weak}* \\ \mathbf{P}_1^h u^h &\to u \quad \text{in} \quad L^\infty([-T,T]; L^2_{\mathrm{loc}}(\mathbb{R})) \\ \mathbf{P}_1^h v^h &\rightharpoonup v \quad \text{in} \quad L^\infty([-T,T]; L^2(\mathbb{R})) \quad \text{weak}* \end{aligned}$$

with (u, v) *the unique strong solution,*

$$(u,v) \in \big(C([-T,T];L^2) \cap L^\infty([-T,T];H^1)\big) \times C([-T,T];L^2).$$

Moreover, if $(u_0, v_0) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$*, then*

$$\begin{aligned} \mathbf{P}_1^h u^h &\to u \quad \text{in} \quad L^\infty([-T,T]; H^1_{\mathrm{loc}}(\mathbb{R})) \\ \mathbf{P}_1^h v^h &\to v \quad \text{in} \quad L^\infty([-T,T]; L^2_{\mathrm{loc}}(\mathbb{R})). \end{aligned}$$

Sketch of the proof

The theorem follows from the following estimates:

• **For all** $h > 0$ **there exist positive continuous functions** $a(t)$**,** $b(t)$**,** $c(t)$ **such that**

$$\|D_+ u^h(t)\|_{L^2} \le a(t) \quad \textbf{(bound on } H^1 \textbf{ norm of } u^h\textbf{)}$$
$$\|\Delta^h u^h(t)\|_{L^2} \le b(t) \quad \textbf{(bound on } H^2 \textbf{ norm of } u^h\textbf{)}$$
$$\|v^h\|_{L^2} \le c(t) \quad \textbf{(bound on } L^2 \textbf{ norm of } v^h\textbf{)}.$$

Sketch of the proof

- **Energy conservation:**

$$\frac{d}{dt}\left\{\frac{1}{2}\|D_+u^h\|_{L^2}^2+\frac{\alpha}{4}\|u^h\|_{L^4}^4+\frac{1}{2}\langle v,|u^h|^2\rangle_{L^2}\right\}=0$$

$+$ Conservation of $\|u^h\|_{L^2}$

$+$ Discrete Gagliardo–Niremberg–Sobolev inequalities:

$$\|\phi\|_{L^\infty}\leq C\|\phi\|_{L^2}^{1/2}\|D_+\phi\|_{L^2}^{1/2},\qquad \|\phi\|_{L^4}\leq C\|\phi\|_{L^2}^{3/4}\|D_+\phi\|_{L^2}^{1/4}$$

$$\Downarrow$$

$$\|D_+u^h\|_{L^2}^2\leq c+c\|v^h\|_{L^2}\|D_+u^h\|_{L^2}^{1/2}.$$

- $\|D_+u^h\|_{L^2}^2 \leq c + c\|v^h\|_{L^2}\|D_+u^h\|_{L^2}^{1/2}$.

But:

- Bound on $\|v^h\|_{L^2}$:

$$\frac{1}{2}\frac{d}{dt}\|v^h\|_{L^2}^2 = \langle D_0|u^h|^2, v^h\rangle_{L^2}$$

$$\Downarrow$$

$$\|v^h\|_{L^2} \leq c + c\int_0^t \|D_+u^h(s)\|_{L^2}^{3/2} ds$$

$$\Downarrow$$

$$\|D_+u^h\|_{L^2}^2 \leq c + c\|D_+u^h\|_{L^2}^{1/2} + c\left(\int_0^t \|D_+u^h(s)\|_{L^2}^{3/2} ds\right)\|D_+u^h\|_{L^2}^{1/2}.$$

$$\Downarrow$$

$$(1 + \|D_+u^h\|_{L^2})^{3/2} \leq c + c\int_0^t (1 + \|D_+u^h(s)\|_{L^2})^{3/2} ds$$

- **Gronwall lemma** $\Rightarrow \|D_+u^h(s)\|_{L^2} \leq a(t)$. □

A Schrödinger–Conservation law system

(Dias, Frid, Figueira, A.R.M.A. 2010)

Schrödinger–Conservation law system

$$\begin{cases} i\partial_t u + \partial_{xx} u = |u|^2 u + \alpha g(v) u \\ \partial_t v + \partial_x f(v) = \alpha \partial_x (g'(v)|u|^2) \qquad t \in [0,\infty), \quad x \in \mathbb{R}. \end{cases}$$

- **Nonlinear conservation law** with flux f
- **Coupling function** g
- g' has compact support (preserves physical domains)

Scalar conservation laws: entropy solutions

Background on scalar conservation laws

$$v(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$$

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = v_0(x) \in L^\infty(\mathbb{R})$$

- Smooth solutions develop **discontinuities in finite time**
- **Non-uniqueness** of weak solution
- Convex entropy $\eta(v)$, entropy flux: $q'(v) = \eta'(v) f'(v)$
- Entropy inequalities:

$$\partial_t \eta(v) + \partial_x q(v) \leq 0 \quad \text{in} \quad \mathcal{D}'.$$

There is a **unique** weak solution verifying the entropy inequalities: the **entropy solution**.

Entropy solutions

Entropy solution of

$$\begin{cases} i\partial_t u + \partial_{xx} u = |u|^2 u + g(v)u \\ \partial_t v + \partial_x f(v) = \partial_x (g'(v)|u|^2) \end{cases}$$

- $u(\cdot, t) \in H^1(\mathbb{R})$ solution of

$$i\partial_t u + \partial_{xx} u = |u|^2 u + g(v)u \quad \text{in} \quad \mathcal{D}'$$

- $v \in L^\infty(\mathbb{R} \times [0, \infty))$ verifies

$$\begin{aligned} &\partial_t \eta(v) + \partial_x (q_1(v) - q_2(v)|u|^2) \\ &\qquad \leq (\eta'(v) g'(v) - q_2(v)) \partial_x |u|^2 \quad \text{in} \quad \mathcal{D}'. \end{aligned}$$

A class of Semidiscrete schemes

We consider a class of semidiscrete schemes inspired by **finite volume schemes**:

$$u_j(t) \approx \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)dx, \quad v_j(t) \approx \int_{x_{j-1/2}}^{x_{j+1/2}} v(x,t)dx, \quad j \in \mathbb{Z}, \quad t > 0.$$

- $i\partial_t u_j + \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = |u_j|^2 u_j + g(v_j)u_j$
- $\partial_t v_j + \frac{1}{h}\big(H_{j,+}(v_j, v_{j+1}, |u_j|^2, |u_{j+1}|^2) + H_{j,-}(v_j, v_{j-1}, |u_j|^2, |u_{j-1}|^2)\big) = 0$

↑

Finite volume scheme

Finite volume schemes – motivation

- $h = x_{j+1/2} - x_{j-1/2}$
- $\partial_t v + \partial_x f(v) = 0 \quad \rightsquigarrow \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} dx$
- $\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \partial_t v \, dx \approx \partial_t v_j$
- $\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x f(v) \, dx = \frac{1}{h}\big(f(v(x_{j+1/2},t)) - f(v(x_{j-1/2},t))\big)$

$$\approx \frac{1}{h}\big(\mathbf{f}_+(v_j, v_{j+1}) + \mathbf{f}_-(v_j, v_{j-1})\big)$$

$$\uparrow$$

Numerical flux functions $\mathbf{f}_\pm$

Finite volume methods

The numerical flux functions $\mathbf{f}_{\pm}(v_1, v_2)$ must be

- **Monotone:** $\partial_1 \mathbf{f}_{\pm}(v_1, v_2) \geq 0, \quad \partial_2 \mathbf{f}_{\pm}(v_1, v_2) \leq 0.$
- **Conservative:** $\mathbf{f}_{+}(v_1, v_2) = -\mathbf{f}_{-}(v_2, v_1).$
- **Consistent** with the flux function: $\mathbf{f}_{\pm}(v, v) = \pm f(v).$

$H_{j,+}(v_1, v_2, a_1, a_2)$ are numerical flux functions consistent with $f(v) - g'(v)|u|^2$:

$$H_{j,+}(v, v, |u|^2, |u|^2) = f(v) - g'(v)|u|^2.$$

Examples

A Lax–Friedrichs scheme

$$H_{j,\pm}(v,w,a,b) = \pm\frac{1}{2}\big(f(v)+f(w)\big) - \frac{1}{2\lambda}(w-v)$$
$$\pm\big(-g'(v)|a|^2 - g'(w)|b|^2\big) - \frac{1}{2\gamma}(w-v)\frac{1}{2}(a+b)$$

A Godunov scheme

$$H_{j,\pm}(v_1,v_2,a_1,a_2) = \begin{cases} \min\limits_{v_1\le s\le v_2} \pm\Big(f(s) - g'(s)\frac{1}{2}(a_1+a_2)\Big), & v_1 \le v_2 \\ \max\limits_{v_2\le s\le v_1} \pm\Big(f(s) - g'(s)\frac{1}{2}(a_1+a_2)\Big), & v_2 \le v_1 \end{cases}$$

Main convergence result

- $i\partial_t u_j + \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = |u_j|^2 u_j + g(v_j)u_j$
- $\partial_t v_j + \frac{1}{h}\big(H_{j,+}(v_j, v_{j+1}, |u_j|^2, |u_{j+1}|^2) + H_{j,-}(v_j, v_{j-1}, |u_j|^2, |u_{j-1}|^2)\big) = 0$

Theorem (PA, M.Figueira, submitted.)

Let (u^h, v^h) be defined by the semidiscrete scheme above. Then there exist functions

$$u \in C([0,\infty); H^1(\mathbb{R})), \qquad v \in L^\infty(\mathbb{R} \times [0,\infty)),$$

entropy solutions of the Cauchy problem such that, up to a subsequence, (u^h, v^h) converge to (u, v) in $L^1_{\mathrm{loc}}(\mathbb{R} \times [0,\infty))$.

Convergence framework: **Compensated Compactness**

If

$$\partial_t \eta(v^h) + \partial_x\big(q_1(v^h) - |u^h|^2 q_2(v^h)\big) \in \{\text{ compact of } W^{-1,2}_{\mathrm{loc}}\},$$

then $(v^h)_{h>0}$ **is compact in** $L^1_{\mathrm{loc}}(\mathbb{R} \times [0, +\infty))$

and if

$(u^h)_{h>0}$ is **bounded** in (discrete) $H^1(\mathbb{R})$

$\Downarrow$

$(u^h, v^h) \to (u, v)$ solutions of the Schrödinger–Conservation law system.

Estimates

- $\|u^h(t)\|_2 = C$ (conservation of L^2 norm)
- $\|v^h(t)\|_\infty \leq M$ (uniform L^∞ bound)

From the conservation law we get the essential **viscosity estimate**:

$$\int_0^t \sum_{j\in\mathbb{Z}} \int_{v_j}^{v_{j+1}} \eta''(v)\big\{f(v) - |u_j|^2 g'(v) - \underbrace{H(v_j, v_{j+1}, |u_j|^2, |u_j|^2)}_{\text{numerical flux function}}\big\} dv\, ds$$

$$\leq c + c\int_0^t \|D_+ u^h(s)\|_2\, ds$$

Next: from the equations we get the **energy estimate**

- $$\frac{1}{2}\|D_+u^h\|_2^2 + \frac{1}{4}\|u^h\|_4^4 + \frac{1}{2}\langle g(v_j), |u_j|^2\rangle_{L^2} = c + \int_0^t \langle g'(v_j)\partial_t v_j, |u_j|^2\rangle_{L^2} ds.$$

Using the scheme and the **viscosity estimate** gives

$$\int_0^t \langle g'(v_j)\partial_t v_j, |u_j|^2\rangle_{L^2} ds \leq \sup_{(0,t)} \|D_+u^h\|_2\Big(c + c\int_0^t \|D_+u^h\|_2 ds\Big) + c\int_0^t \|D_+u^h\|_2 ds.$$

Gronwall $\Rightarrow \|D_+u^h(t)\|_2 \leq a(t)$ (H^1 estimate of u^h).

The **viscosity estimate** now becomes

$$\int_0^t \sum_{j\in\mathbb{Z}} \int_{v_j}^{v_{j+1}} \eta''(v)\Big\{ f(v) - |u_j|^2 g'(v) - \underbrace{H(v_j, v_{j+1}, |u_j|^2, |u_j|^2)}_{\text{numerical flux function}} \Big\} dv\, ds$$
$$\leq c(t).$$

• In the case of the **Lax–Friedrichs scheme**, one has the simpler **"quadratic total variation"** estimate: $\exists k > 0$ such that

$$k \int_0^t \sum_{j\in\mathbb{Z}} (1 + |u_j^h|^2)(v_{j+1}^h - v_j^h)^2\, ds \leq c(t)$$

which is **false** in general.

So: do we have

$$\partial_t \eta(v^h) + \partial_x\big(q_1(v^h) - |u^h|^2 q_2(v^h)\big) \in \{ \text{ compact of } W^{-1,2}_{\text{loc}} \}?$$

If ϕ is a test-function, we have

$$\begin{aligned}
&- \left\langle \partial_t \eta(v^h) + \partial_x(q_1(v^h) - q_2(v^h)|u^h|^2), \phi \right\rangle_{\mathcal{D}' \times \mathcal{D}} \\
&\quad = - \int_0^\infty h \sum_{j\in\mathbb{Z}} \overline{\phi}_j \eta'(v_j) \partial_t v_j \, dt - \int_0^\infty \sum_{j\in\mathbb{Z}} \phi_{j+1}\big(q_1(v_{j+1}) - q_1(v_j)\big) dt \\
&\qquad + \int_0^\infty \sum_{j\in\mathbb{Z}} \phi_{j+1}\big(q_2(v_{j+1})|u_{j+1}|^2 - q_2(v_j)|u_j|^2\big) dt
\end{aligned}$$

+ use the scheme to replace $\partial_t v_j$:

$$\partial_t v_j = -\frac{1}{h}\big(H_{j,+}(v_j, v_{j+1}, |u_j|^2, |u_{j+1}|^2) + H_{j,-}(v_j, v_{j-1}, |u_j|^2, |u_{j-1}|^2)\big)$$

The viscosity estimate

$$\int_0^t \sum_{j\in\mathbb{Z}} \int_{v_j}^{v_{j+1}} \eta''(v)\big\{f(v) - |u_j|^2 g'(v) - H(v_j, v_{j+1}, |u_j|^2, |u_j|^2)\big\} dv\, ds \\ \leq c(t)$$

allows us to bound the resulting terms. This gives

$$\partial_t \eta(v^h) + \partial_x\big(q_1(v^h) - |u^h|^2 q_2(v^h)\big) \in \{\text{ compact of } W^{-1,2}_{\mathrm{loc}}\}$$

and thus compactness of v^h.
Using similar techniques, one then proves that the resulting limit pair (u, v) is an **entropy solution** of the problem.

Numerical sheme

- Semi-implicit Cranck–Nicholson scheme for the Schrödinger equation

$$\bullet \quad i\frac{1}{\tau}(u_j^{n+1} - u_j^n) + \frac{1}{2h^2}\big(u_{j+1}^{n+1} + u_{j+1}^n - 2(u_j^{n+1} + u_j^n) + u_{j-1}^{n+1} + u_{j-1}^n\big)$$
$$= \big|\frac{1}{2}(u_j^{n+1} + u_j^n)\big|^2 \frac{1}{2}(u_j^{n+1} + u_j^n) + g(v_j^n)\frac{1}{2}(u_j^{n+1} + u_j^n),$$

- Semi-implicit Lax–Friedrichs scheme for the conservation law

$$\bullet \quad \frac{1}{\tau}(v_j^{n+1} - v_j^n) = -\frac{1}{2h}(f(v_{j+1}^n) - f(v_{j-1}^n))$$
$$+ \frac{1}{2h}(g'(v_{j+1}^n)|u_{j+1}^n|^2 - g'(v_{j-1}^n)|u_{j-1}^n|^2)$$
$$+ \frac{1}{2\lambda h}(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1})$$
$$+ \frac{1}{2\gamma h}\big[(v_{j+1}^{n+1} - v_j^{n+1})\frac{1}{2}(|u_j^n|^2 + |u_{j+1}^n|^2)$$
$$- (v_j^{n+1} - v_{j-1}^{n+1})\frac{1}{2}(|u_j^n|^2 + |u_{j-1}^n|^2)\big].$$

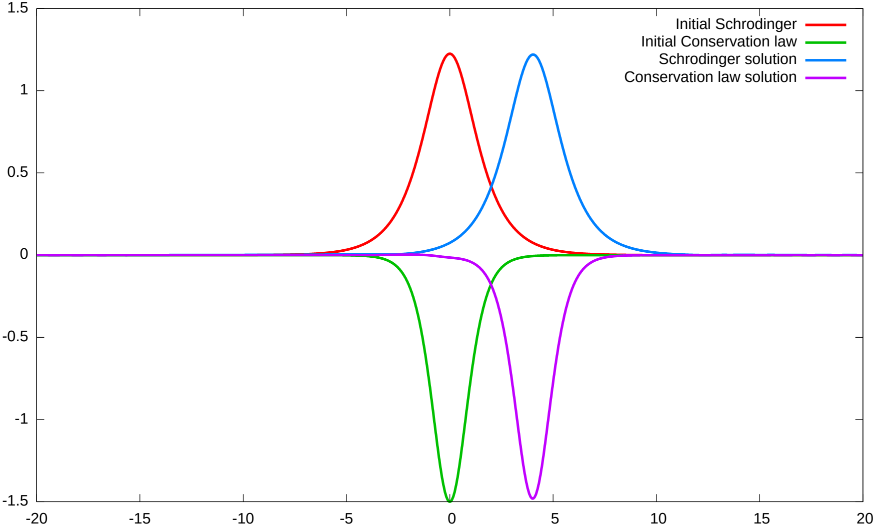
Numerical results (I): a test-case

$$\begin{cases} i\partial_t u + \partial_{xx} u = \alpha v u \\ \partial_t v = \partial_x(|u|^2) \end{cases}$$

Exact solutions: Traveling waves

$$u(x,t) = e^{it\lambda} e^{icx/2} C \operatorname{sech}^2(C(x-ct)),$$
$$v(x,t) = a \operatorname{sech}^2(C(x-ct)).$$

Numerical results (I): a test-case



Initial data and numerical solution for $t = 4$.

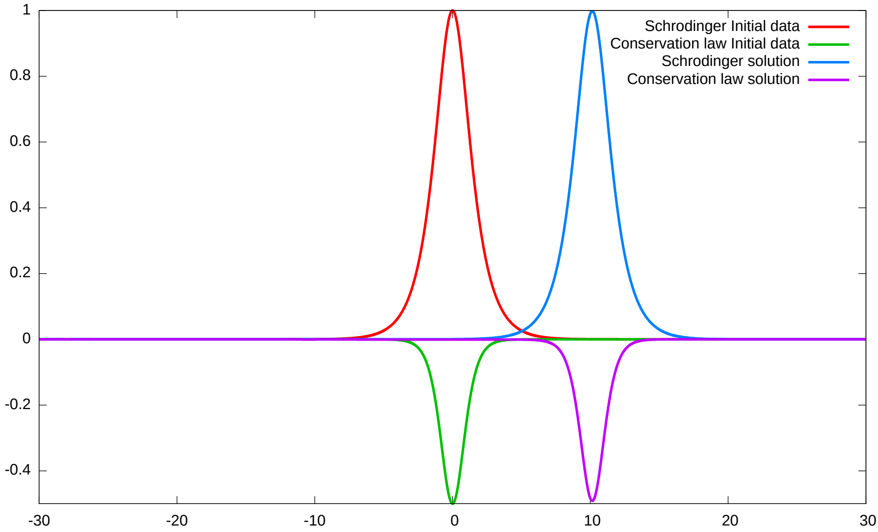
Numerical results (II): Full system, linear flux

$$\begin{cases} i\partial_t u + \partial_{xx} u = |u|^2 u + vu \\ \partial_t v + \gamma \partial_x v = \partial_x(|u|^2) \end{cases}$$

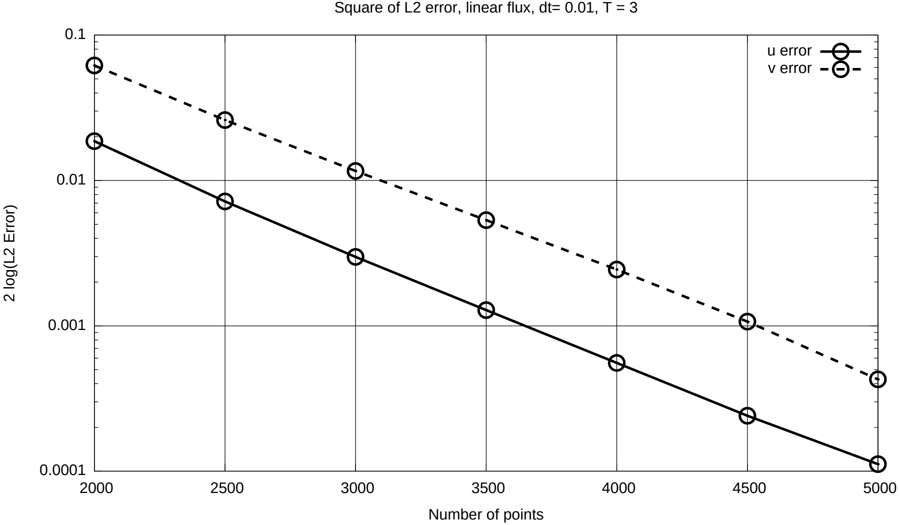
Exact solutions: Traveling waves

$$u(x,t) = e^{it\lambda} e^{icx/2} C \operatorname{sech}^2(C(x-ct)),$$
$$v(x,t) = a \operatorname{sech}^2(C(x-ct)).$$

Numerical results (II): Full system, linear flux



Numerical results (II): Full system, linear flux



L^2 error between the exact solution and the computed solution, linear flux.

Numerical results (III): Full system, nonlinear flux

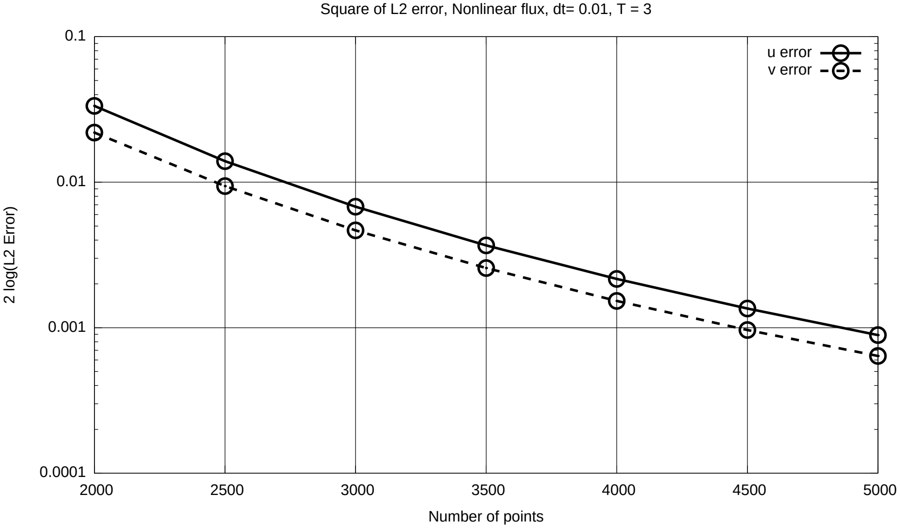
$$\begin{cases} i\partial_t u + \partial_{xx} u = vu \\ \partial_t v + \partial_x v^2 = \partial_x(|u|^2) \end{cases}$$

Exact solutions: Standing waves

$$(u, v) = (e^{ibt} r(x), -r(x)),$$

$$r(x) = b(3/2)\,\mathrm{sech}^2(\sqrt{b}x/2)$$

Numerical results (III): Full system, nonlinear flux



L^2 error between the exact solution and the computed solution, nonlinear flux.

Numerical results (IV): Full system

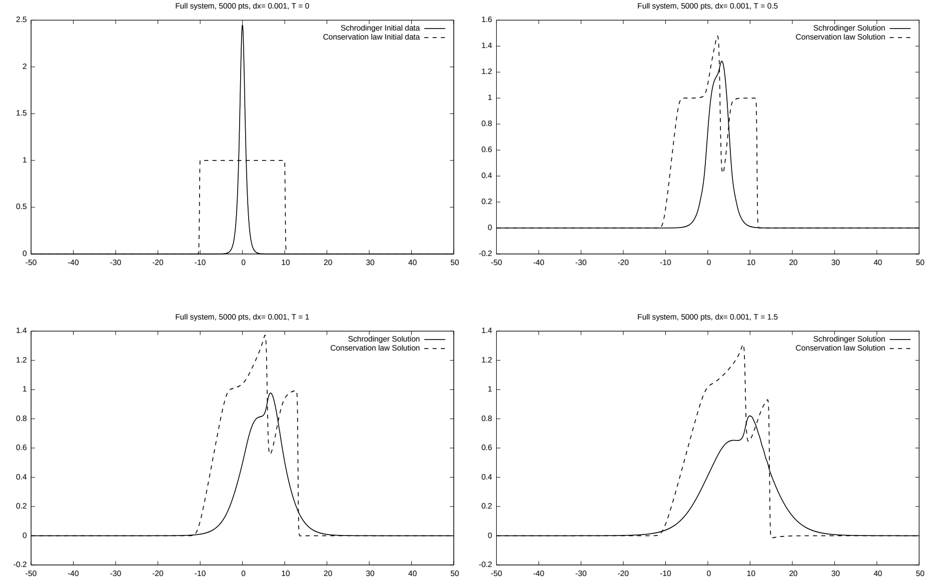
$$\begin{cases} i\partial_t u + \partial_{xx} u = |u|^2 u + g(v) u \\ \partial_t v + \partial_x f(v) = \partial_x (g'(v)|u|^2) \end{cases}$$

Initial data:

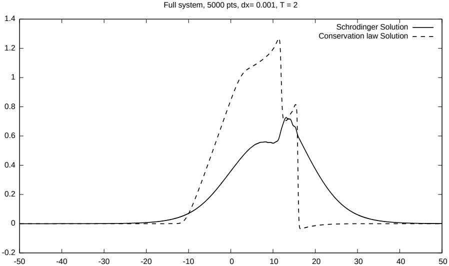
$$u_0(x) = e^{5ix/2}\sqrt{6}\,\mathrm{sech}(\sqrt{3}x),$$
$$v_0(x) = \chi_{[-10,10]}$$

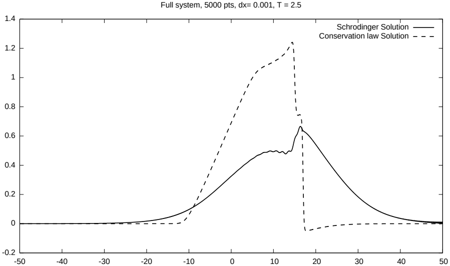
on the spatial domain $[-50, 50]$.

Numerical results (IV): Full system

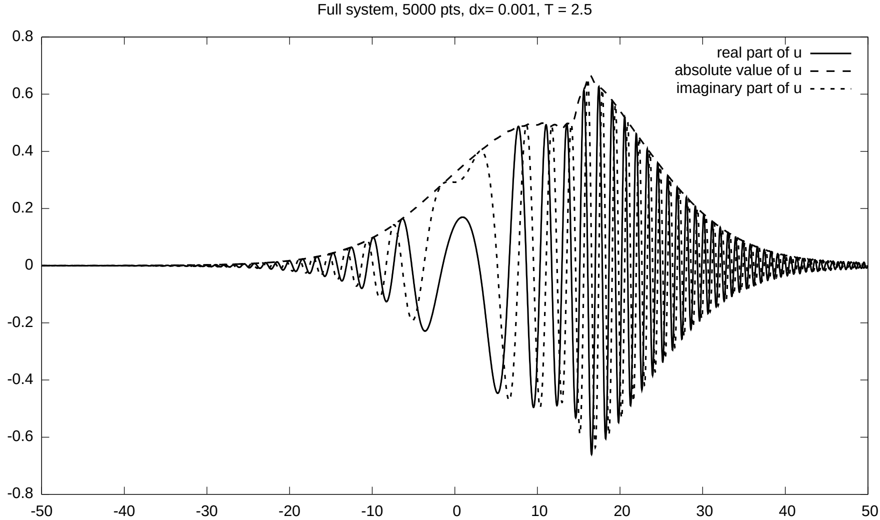


Numerical results (IV): Full system





Numerical results (IV): Full system



Summary

• We have considered a system of short wave long wave interaction equations and a Schrödinger–Conservation law system.

• For both systems we have proved **convergence** results for semidiscrete schemes. The convergence is **strong** for the Schrödinger–Conservation law system.

• We used Compensated Compactness to deal with the nonlinear equations.

• The proof of convergence uses mostly tools from hyperbolic conservation laws but **relies heavily on the interaction** between the two equations.

• The **numerical implementation** of the algorithm was successfully tested against known exact solutions and shows **formation of new waves** due to the interaction terms.

Open questions

- The **fundamental** uniform bound $\|v^h(t)\|_\infty \leq M$ **needs** $\operatorname{supp} g'$ compact (g is the coupling function).
- We have **not** observed explosion of $\|v^h(t)\|_\infty$ if $\operatorname{supp} g'$ is not compact.
- Is there a hidden "maximum principle" for the Schrödinger–Burgers system (i.e., $g(v) \equiv v$)?

Thank you for your attention!

http://ptmat.fc.ul.pt/∼pamorim/