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BOUNDARY CONDITIONS FOR NONLINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS.

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ABSTRACT

We propose two formulations of the boundary conditions for nonlinear hyperbolic systems of conservation laws. A first approach is based on the vanishing viscosity method and a second one is related to the Riemann problem. The equivalence between these two conditions is studied. The latter formulation is extended to treat numerically physically relevant boundary conditions. Monodimensional experiments are presented.

INTRODUCTION

We study initial-boundary value problems for nonlinear hyperbolic systems of conservation laws. Recall that with strong Dirichlet boundary conditions the associated problem is not well posed. Generally there is neither existence nor uniqueness. Thus weaker conditions are necessary ; in the linear case by example we know that data are given only on incoming characteristics.

In this paper we define the boundary condition in terms of **admissible values at the boundary**, related to the boundary datum. In our first formulation the set of admissible values is defined thanks to a **boundary entropy inequality** obtained by the vanishing viscosity method and the second set is related to the **resolution of a Riemann problem** at the boundary. The equivalence of these two formulations is established for nonconvex scalar conservation laws and strictly hyperbolic linear systems. The second formulation is naturally applied to Godunov-type numerical schemes : the numerical boundary condition reduces to the computation of a **boundary flux** thanks to some Riemann problem (or partial Riemann problem in physically relevant situations). As an application, outgoing waves from the Sod shock tube are presented.

BOUNDARY ENTROPY INEQUALITY (FIRST FORMULATION)

We consider a nonlinear hyperbolic system of conservation laws in one space dimension :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \qquad ; \quad u(x,t) \in R^n \quad , x > 0 \ , \ t > 0 \tag{1}$$

where $f : R \rightarrow R$ is a smooth flux-function. We suppose that there exists at least a pair (η,q) of entropy-flux in the sense of Lax [9]. The initial boundary value problem obtained by the viscosity method ($\epsilon>0$) :

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \dfrac{\partial}{\partial x} f(u^\epsilon) = \epsilon \dfrac{\partial^2 u^\epsilon}{\partial x^2} & x > 0 \ , \ t > 0 \\ u^\epsilon(x,0) = v_0(x) & x > 0 \\ u^\epsilon(0,t) = u_0(t) & t > 0 \end{cases} \tag{2}$$

admits a unique solution u^ϵ and we study the behaviour of u^ϵ at the boundary as ϵ tends to zero. In fact a discontinuity appears, in general, at the boundary. The following result (essentially formal) yields an inequality at this discontinuity.

Theorem 1. Suppose that u^ϵ is bounded in $W^{1,1}_{loc}(R^+ x R^+, R^n)$ and converges in L^1_{loc} to u as $\epsilon\rightarrow 0$. Then for each admissible pair (η,q) of entropy-flux we have the following **boundary entropy inequality** :

$$q(u(0^+,t)) - q(u_0(t)) - d\eta(u_0(t)).(f(u(0^+,t))-f(u_0(t))) \leq 0 \ , \ t > 0 \tag{3}$$

between the taken value $u(0^+,t)$ and the prescribed value $u_0(t)$ at the boundary.

This result was first derived in [2] in the particular case of scalar conservation laws. The details concerning the derivation of the boundary entropy inequality (3) in the case of systems of conservation laws are presented in [6]. Remark that the latter inequality was independently obtained by other methods [1,12].

Given a state u_0 we define a (first) **set of admissible values at the boundary** :

$$E(u_0) = \{ v \in R^n,\ q(v)-q(u_0)-d\eta(u_0).(f(v)-f(u_0)) \le 0 ,\ \forall\ (\eta,q) \text{ pair of entropy-flux} \}$$

Therefore let us extend the notion of Dirichlet boundary condition and define our (first) **formulation of the boundary condition** :

$$u(0^+,t) \in E(u_0(t)) \ , \ t > 0 \tag{4}$$

The set $E(u_0(t))$ can be entirely explicited for both strictly hyperbolic linear systems and non-convex scalar conservation laws (see [6] for the proofs).

<u>Proposition 1. Strictly hyperbolic linear systems.</u>

Suppose that $f(u) \equiv A.u$, with a constant matrix A characterized by n eigenvalues λ_i (and n associated eigenvectors r_i) satisfying

$$\lambda_1 < \lambda_2 < \ldots < \lambda_p \le 0 < \lambda_{p+1} < \ldots < \lambda_n \tag{5}$$

Then the set $E(u_0)$ is the affine space containing u_0 and generated by the p first eigenvectors of A :

$$E(u_0) = \{ u_0 + \sum_{i=1}^{p} \alpha_i\ r_i \ , \quad \alpha_1,\ldots,\alpha_p \in R \} \ .$$

The interpretation of the boundary condition (4) here is the following : the components of $u(0^+,t)$ on the (n-p) last eigenvectors (i.e. the incoming characteristics) are given by the boundary state $u_0(t)$. With the present approach we recover the classical one in this particular case.

Proposition 2. Scalar conservation laws.

Suppose that the flux f(u) is a C^1 function from R to R. Then the set $E(u_0)$ of the admissible states u is entirely characterized by the **family of inequalities** :

$$\frac{f(u) - f(k)}{u - k} \leq 0 \qquad \forall\, k \in [u,u_0] \cup [u_0,u] \qquad (6)$$

This proposition was previously established in [10], and a geometrical interpretation is presented in [6]. In the particular case of **convex** scalar conservation laws the latter is simpler. Let us specify it for the Burgers equation.

Propositon 3. Burgers equation.

When $u \in R$ and $f(u) \equiv u^2/2$, the set $E(u_0)$ is given by :

(i) if $u_0 \geq 0$, $E(u_0) =]-\infty,-u_0] \cup \{u_0\}$

(ii) if $u_0 \leq 0$, $E(u_0) =]-\infty,0]$.

In the general case of an hyperbolic system of conservation laws, the lack of mathematical entropies does not allow a complete description of this boundary set $E(u_0)$.

APPROACH BY THE RIEMANN PROBLEM (SECOND FORMULATION)

For our second formulation of the boundary condition [5,6] we suppose that each Riemann problem $R(u_L,u_R)$ associated with (1) admits a unique entropy solution denoted by $w(x/t;u_L,u_R)$. Let us define a **second set of admissible values** by :

$$V(u_0) = \{\, w(0^+;u_0,u_R),\ u_R \text{ varying in } R^n \,\}$$

Then we have the following result which generalizes [9] :

Theorem 2 Let v_0, u_0 be constant states of R^n. The problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) = 0 & x > 0 \ , \ t > 0 \\ u(x,0) = v_0 & x > 0 \\ u(0,t) \in \mathbf{V}(u_0) & t > 0 \end{cases} \tag{7}$$

is **well posed** in the class of functions which consist of constant states separated by at most n elementary waves (rarefactions, shocks, contacts).

Proposition 4. Link between the two formulations.

In particular cases of strictly hyperbolic linear systems and (non necessarily convex) scalar conservation laws, **the two sets are identical**:

$$\mathbf{E}(u_0) = \mathbf{V}(u_0) \qquad \forall\, u_0 \in R^n .$$

The advantage of the second formulation is that $\mathbf{V}(u_0)$ can be easily computed. For the p-system, $\mathbf{V}(u_0)$ is exactly the 1-wave containing u_0. And, in [5,6] we have given details on the **V**-sets in the case of barotropic Euler-Saint Venant equations. For more precise relations concerning the **E** and **V** sets in the particular case of 2x2 systems of conservation laws, we refer to [3,6]. Refer also to [11] about a formulation of boundary conditions for weighted conservation laws.

APPLICATION TO THE EULER EQUATIONS OF GAS DYNAMICS

We apply now the ideas developed previously to Godunov-type finite volume numerical schemes [8]. We restrict ourselves to the first order accurate methods. The interval [0,1] is divided into N cells and the numerical approximation of the conservation law (1) at time $t_n = n\Delta t$ in the j° cell is given by :

$$\frac{1}{\Delta t}\left(u_j^{n+1} - u_j^{n} \right) + \frac{1}{\Delta x}\left(f_{j+1/2}^{n} - f_{j-1/2}^{n} \right) = 0 \tag{8}$$

For the internal cells we have classically

$$f_{j+1/2}^{n} = \Phi (u_j^n , u_{j+1}^n) , \quad j = 1,2,\dots,N-1 \tag{9}$$

for some numerical flux function Φ that approaches the flux $f(w(0;u_j^n,u_{j+1}^n))$ of the Riemann problem $R(u_j^n,u_{j+1}^n)$ when $x/t=0$. We suppose that a boundary state u_L (resp u_R) is given for $x \le 0$ (resp $x \ge 1$) and we consider it intuitively as a limiting state for x tending towards $-\infty$ (resp $+\infty$). Thus the **numerical boundary condition** at time t_n results from the interaction of u_L (resp u_R) with the value u_1^n (resp u_N^n) of the field in the first (resp last) cell:

$$f_{1/2}^{n} = \Phi (u_L , u_1^n) \quad ; \quad f_{N+1/2}^{n} = \Phi (u_N^n , u_R) . \tag{10}$$

This kind of numerical boundary condition in terms of a **numerical flux** is natural with the approach of finite volumes. This fact was first recognized by Godunov (e.g. [7]).

The numerical scheme (8)(9)(10) has been applyed to the Sod shock tube [15] for the Euler equations of gas dynamics, i.e. with left and right states $u_L = (\rho_L, v_L, p_L) = (1,0,1)$, $u_R = (\rho_D, v_D, p_D) = (0.125,0,0.1)$, and N=100 cells. We used the Osher upwind scheme [13] and have performed the numerical computation for a time sufficiently long so that the different waves have been gone outside the computational domain [0,1] (see Figure 1). Some results are plotted on Figure 2. The boundary condition (10) appears numerically as transparent for all these nonlinear waves and the physical fields at x = 0 and x = 1 are correct (the difference with the exact solution is first due to the high level of numerical viscosity contained in the first order scheme). More details on this problem with the use of the exact linearized implicit Osher scheme are developed in [4].

We focus now on more realistic boundary conditions for the Euler equations. For most of the internal aerodynamics problems a state $u_0(t)$ is **not** physically given at the boundary. As usual, we distinguish between four cases : the fluid may be sub or super-sonic at the in or out-flow and physical parameters can be associated with each case [16] : **(i)** supersonic inflow : a state u_0 , **(ii)** subsonic inflow : total enthalpy H and physical entropy S , **(iii)** subsonic outflow : static pressure P , **(iv)** supersonic outflow : no numerical datum. We review briefly the main ideas of [4]. For each of the four above cases a manifold (eventually with boundary) **M** is defined by the boundary data; we have respectively

(i) $\mathbf{M} = \{ (\rho_0, v_0, p_0) \}$

(ii) $\mathbf{M} = \left\{ (\rho, v, p) \;/\; \frac{1}{2} v^2 + \frac{\gamma\, p}{\gamma - 1} = H \;,\; p = S\, \rho^{\gamma} \right\}$

(iii) $\mathbf{M} = \{ (\rho, v, p) \;/\; p = P \}$

(iv) $\mathbf{M} = \left\{ (\rho, v, p) \;/\; v - c \geq 0 \;,\; c^2 \equiv \frac{\gamma\, p}{\rho} \right\}$.

Then the formula (10) relative to the computation of the boundary flux is adapted as follows (we consider only the case x = 0). A **partial Riemann problem** P(**M**,z) is posed naturally by the boundary condition between the manifold **M** and the state z located in the (first) cell of the computational domain. We solve this problem in the same manner as Lax did [9] for the classical Riemann problem. A family of codim**M** (equal respectively to 3,2,1,1 in the previous cases) nonlinear waves issued from z intersects **M** at a state I. Interpreting those waves in the (x,t) plane, the solution of P(**M**,z) joins the state I (of **M**) to the state z thanks to a fan of codim**M** waves (Figure 3). Then the boundary flux $f_{1/2}$ is given by

$$f_{1/2} = f(W) \tag{11}$$

where W is the state of this fan located at $x/t = 0^+$. In [4] we have used the Riemann solver of Osher that contains only (eventually multivalued) rarefactions. Thus we have taken into account the (eventual) multiplicity of the states W. Furthermore in the particular case (iii) (given pressure P) and for a sufficiently weak nonlinearity (i.e. p(z) not too far from P) we recover previous results obtained by Osher-Chakravarthy [14]. We have also tested in [4] all thoses boundary conditions (i)-(iv) for one dimensional nozzles using both the explicit and linearized implicit versions of the scheme.

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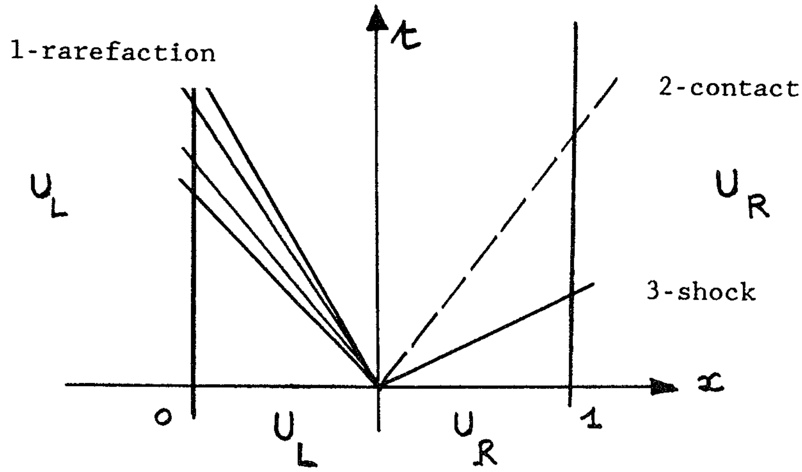

Figure 1. The Sod shock tube for time tending to infinity.

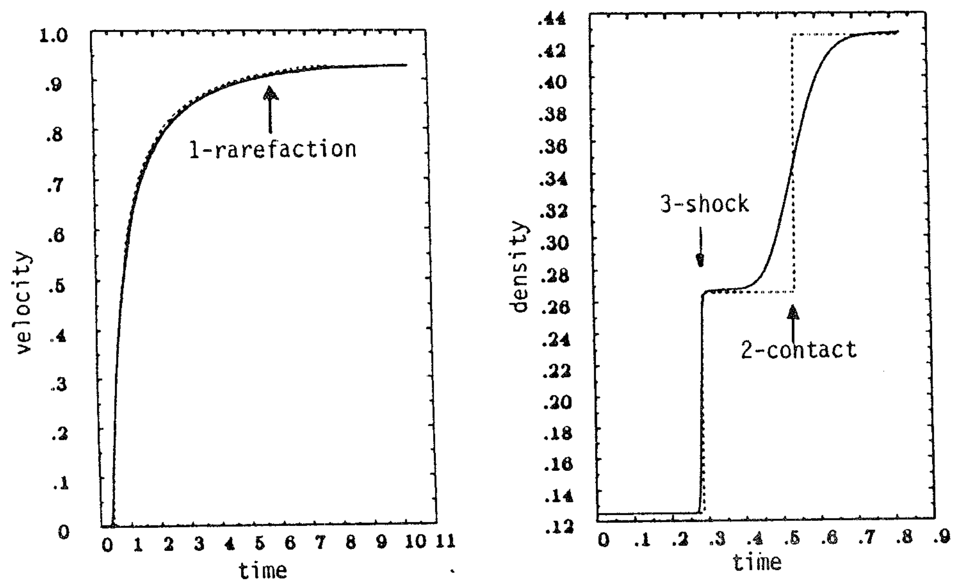

Figure 2. Evolution of the velocity at x=0 (left) and of the density at x=1 (right) for the Sod shock tube with N=100 cells. The dotted line is the exact solution.

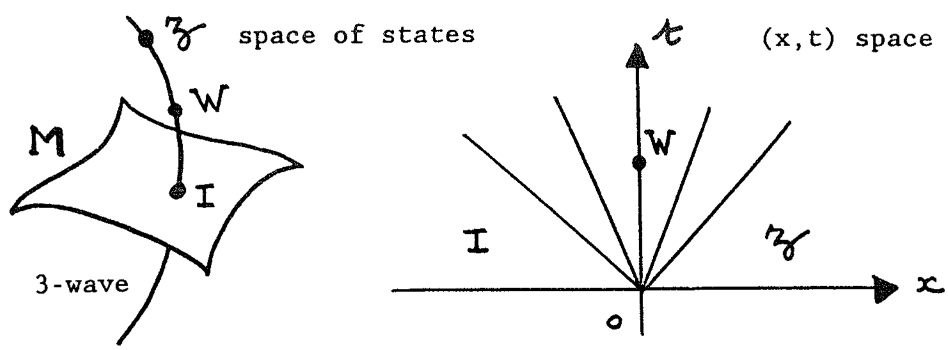

Figure 3. Resolution of the partial Riemann problem p(M,z) in the particular case (iii).