Weierstrass Institute for
Applied Analysis and Stochastics

An incompressible diffuse model with phase transition in the sharp interface limit

G. Aki*, W. Dreyer*, J. Giesselmann**, and C. Kraus*

* Weierstrass Institute for applied analysis and Stochastics, Berlin,
** Institute of Applied Analysis and Numerical Simulation, University of Stuttgart.

Outline

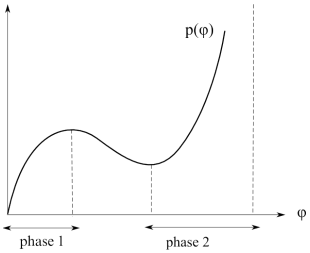
- Description of the model
- Asymptotic setting
- Sharp interface limit:
  - Euler regime
  - Navier-Stokes regime

We consider an incompressible Navier-Stokes-Korteweg model for $(\varphi, \mathbf{v}, \lambda)$

$$\partial_t \varphi + \mathrm{div}(\varphi \mathbf{v}) = c_+ \big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda),$$

$$\rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{1}{M^2}\nabla p(\varphi) + \nabla\lambda = \tfrac{1}{\mathrm{Re}}\big(\mathrm{div}(\eta\,\mathrm{div}\mathbf{v}\mathbf{1} + \hat\eta(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\big) + \tfrac{C}{M^2}\varphi\nabla\Delta\varphi,$$

$$\mathrm{div}\mathbf{v} = c_-\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda),$$



Domain: $(t, \mathbf{x}) \in \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^d$

Boundary conditions:

$$\mathbf{v}|_{\partial\Omega} = 0$$
$$\partial_n(c_+\mu + c_-\lambda)|_{\partial\Omega} = \partial_n\varphi|_{\partial\Omega} = 0$$

Initial conditions:

$$(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_I, \varphi_I)$$

Introduction of the model

We consider an incompressible Navier-Stokes-Korteweg model for $(\varphi, \mathbf{v}, \lambda)$

$$\partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) = c_+\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda),$$
$$\rho(\varphi)(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{1}{M^2}\nabla p(\varphi) + \nabla\lambda = \tfrac{1}{\mathrm{Re}}\big(\mathrm{div}(\eta\,\mathrm{div}\mathbf{v}\mathbf{1} + \hat{\eta}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\big) + \tfrac{C}{M^2}\varphi\nabla\Delta\varphi,$$
$$\mathrm{div}\mathbf{v} = c_-\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda),$$

The compressible Navier-Stokes-Korteweg model for $(\rho, \mathbf{v})$:

$$\partial_t\rho + \mathrm{div}(\rho\mathbf{v}) = 0,$$
$$\rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{1}{M^2}\nabla p(\rho) = \tfrac{1}{\mathrm{Re}}\big(\mathrm{div}(\eta\,\mathrm{div}\mathbf{v}\mathbf{1} + \hat{\eta}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\big) + \tfrac{C}{M^2}\rho\nabla\Delta\rho.$$

Introduction of the model

We consider an incompressible Navier-Stokes-Korteweg model for $(\varphi, \mathbf{v}, \lambda)$

$$
\begin{aligned}
&\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = c_+\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + c_-\lambda),\\
&\rho(\varphi)(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{1}{M^2}\nabla p(\varphi) + \nabla\lambda = \tfrac{1}{\mathrm{Re}}\big(\operatorname{div}(\eta\operatorname{div}\mathbf{v}\mathbf{1} + \hat{\eta}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\big)\\
&\qquad\qquad + \tfrac{C}{M^2}\varphi\nabla\Delta\varphi,\\
&\operatorname{div}\mathbf{v} = c_-\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + c_-\lambda),
\end{aligned}
$$

Model H by Hohenberg and Halperin'77 for $(c, \mathbf{v}, \lambda)$:

$$
\begin{aligned}
&\partial_t c + \operatorname{div}(c\mathbf{v}) = m_{\mathrm{j}}\Delta\mu(c),\\
&\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\lambda = \tfrac{1}{\mathrm{Re}}\big(\operatorname{div}(\hat{\eta}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\big) - \tfrac{C}{M^2}\operatorname{div}(\nabla c\otimes\nabla c),\\
&\operatorname{div}\mathbf{v} = 0.
\end{aligned}
$$

Incompressibility of the mixture

We consider two incompressible true mass densities:

$$\tilde{\rho}_1 = \text{constant} \qquad \tilde{\rho}_2 = \text{constant}$$

Then the mixture has a mass density: $$\rho(\varphi) = \tilde{\rho}_1 \frac{1+\varphi}{2} + \tilde{\rho}_2 \frac{1-\varphi}{2}$$

This is motivated from the partial mass densities for an homogenous mixture of two constituents:

$$\rho = \rho_1 + \rho_2 = \frac{m_1}{V} + \frac{m_2}{V} = \tilde{\rho}_1 \frac{V_1}{V} + \tilde{\rho}_2 \frac{V_2}{V}$$

Here the phase field variable φ is a quantity which is volume fraction related:

$$\frac{V_1}{V} = \frac{1+\varphi}{2}.$$

Balance equations for the basic variables $(\varphi, \mathbf{v}, \lambda)$:

$$\frac{\partial \varphi}{\partial t} + \mathrm{div}(\varphi \mathbf{v} + c_+ \mathbf{j}) = c_+ r,$$
$$\rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v} \times \nabla)\mathbf{v}) - \mathrm{div}(\boldsymbol{\sigma}_{NS}) = \mathrm{div}(\boldsymbol{\sigma}_K)$$
$$\mathrm{div}(\mathbf{v}) = c_-(r - \mathrm{div}(\mathbf{j}))$$

Constitutive relations complete the system:

$$\mathbf{j} = -m_{\mathrm{j}} \nabla(c_+ \mu + c_- \lambda),$$
$$r = -m_{\mathrm{r}}(c_+ \mu + c_- \lambda),$$
$$\mu = (\frac{\partial f}{\partial \varphi} - \nabla \cdot \frac{\partial f}{\partial \nabla \varphi}),$$
$$\boldsymbol{\sigma}^{\mathrm{K}} = -\nabla \varphi \otimes \frac{\partial f}{\partial \nabla \varphi} + (f - \varphi \mu)\mathbf{1},$$
$$\boldsymbol{\sigma}^{\mathrm{NS}} = -\lambda \mathbf{1} + \eta(\varphi)\mathrm{div}(\mathbf{v}) + \hat{\eta}(\varphi)(\nabla \mathbf{v} + \nabla \mathbf{v}^T).$$

The constants:

$$m_{\mathrm{j}}, m_{\mathrm{r}}, c_{\pm} \geq 0$$

And the free energy:

$$f = f(\varphi, \nabla \varphi)$$

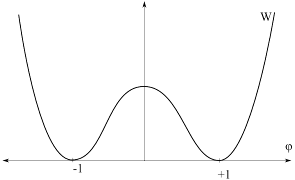
Free energy and the Maxwell points

We consider a scaled free energy density in terms of the phase fraction $\varphi : [0,\infty) \times \Omega \to (-\varepsilon - 1, 1+\varepsilon)$:

$$f(\varphi, \nabla\varphi) = W(\varphi) + \frac{\gamma\varepsilon^2}{2}|\nabla\varphi|^2,$$

with

- $W(\varphi) \geq 0$ is a double-well potential,
- $W : \mathbb{R} \to [0,\infty)$ with $W(-1) = W(1) = 0$

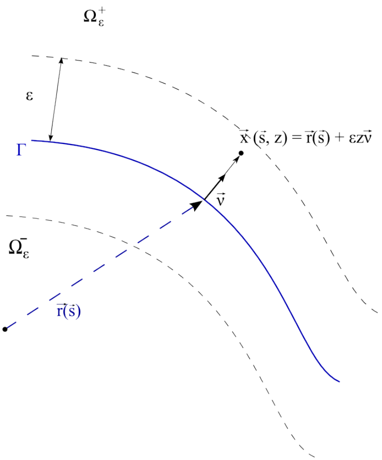


Maxwell construction.

Due to the choice of W above the Maxwell points of W are given by $\varphi^1 = -1$ and $\varphi^2 = 1$.

This scaling of the free energy characterizes an interfacial region with a thickness of order ε.

Decomposition of the interfacial region



- The set of $C^{1,2}$-hypersurfaces:

$$\Gamma(t;\varepsilon) = \{\mathbf{x} \in \Omega \ : \ \varphi_\varepsilon(t,\mathbf{x}) = 0\}.$$

$$\downarrow \quad \text{zero level set limit}$$

$$\Gamma$$

- A local parametrization of Γ:

$$\mathbf{r}(t,\mathbf{s}) : I \times U \to \mathbb{R}^{\mathrm{d}}$$

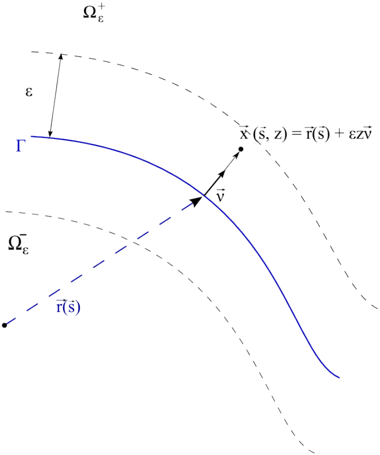
- A new coor. in the neighbourhood of Γ:

$$\mathbf{x}(t,\mathbf{s},z) := (t, \mathbf{r}(t,\mathbf{s}) + \varepsilon z \boldsymbol{\nu}),$$

- The normal component of the interface velocity :

$$w_{\boldsymbol{\nu}} = \partial_t \mathbf{r} \cdot \boldsymbol{\nu}.$$

Decomposition of the interfacial region



- Bulks $\Omega_\varepsilon^\pm \Longleftrightarrow$ outer setting

$$\big(\varphi_\varepsilon(t,\mathbf{x}), \mathbf{v}_\varepsilon(t,\mathbf{x}), \lambda_\varepsilon(t,\mathbf{x})\big)$$

$$\Downarrow$$

expansion in ε

- For a general field f

$$f(t,\mathbf{x}) = F(t,\mathbf{s},z)$$

- Interfacial layer $\Gamma_\varepsilon \Longleftrightarrow$ inner setting

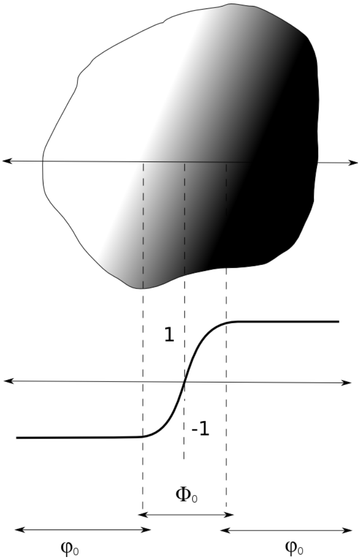
$$\big(\Phi_\varepsilon(t,\mathbf{s},z), \mathbf{V}_\varepsilon(t,\mathbf{s},z), \Lambda_\varepsilon(t,\mathbf{s},z)\big)$$

$$\Downarrow$$

expansion in ε

- Also expand $(\mu_\varepsilon, p_\varepsilon)$ and $(M_\varepsilon, P_\varepsilon)$.

Matching of the inner and the outer setting



- **Matching:**

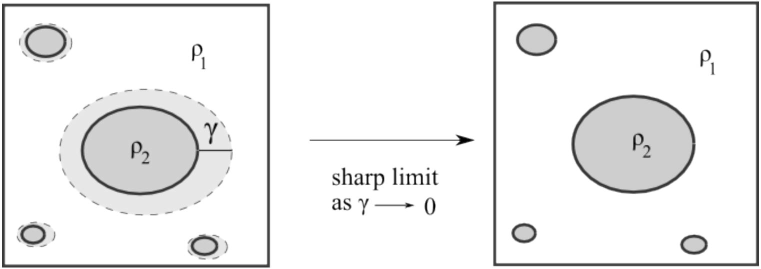
  Large z behavior of inner variables

  $$\approx$$

  Close to Γ behavior of outer variables

- Term-by-term matching:

$$\Phi_0(\mathbf{s}, \pm\infty) \sim \varphi_0|_\Gamma$$
$$\Phi_1(\mathbf{s}, \pm\infty) \sim \varphi_1|_\Gamma + \nabla\varphi_0|_\Gamma \cdot \boldsymbol{\nu}$$



Outer setting $\Longrightarrow$ PDEs in bulks for $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \mu_0, \mu_1, p_0, p_1)$.
Inner setting $\Longrightarrow$ solvable ODEs for $(\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0, M_0, M_1, P_0, P_1)$

Solvability cond. for the inner setting $+$ matching cond. $\Longrightarrow$ interface conditions.

Thus we end up with PDEs for bulks and boundary conditions at the interface.

Diffuse model in Euler (E) regime

We consider low Mach number and low viscosity (high Reynolds number) regime

$$M \sim \varepsilon^{1/2}, \qquad \frac{1}{\mathrm{Re}} \sim \varepsilon.$$

Scaled Navier-Stokes-Korteweg/Cahn-Hilliard system reads

$$\begin{aligned}
&\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \tfrac{1}{\varepsilon} c_+ \big(m_{\mathrm{j}} \Delta - m_{\mathrm{r}}\big)(c_+ \mu(\varphi) + \varepsilon c_- \lambda),\\
&\rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{1}{\varepsilon}\nabla p(\varphi) + \nabla\lambda = \varepsilon \nabla(\eta(\varphi)\operatorname{div}\mathbf{v}) + \varepsilon \operatorname{div}(\hat{\eta}(\varphi)(\nabla \mathbf{v} + \nabla \mathbf{v}^T))\\
&\qquad\qquad + \gamma\varepsilon\varphi\nabla\Delta\varphi,\\
&\operatorname{div}\mathbf{v} = \tfrac{1}{\varepsilon} c_- \big(m_{\mathrm{j}} \Delta - m_{\mathrm{r}}\big)(c_+ \mu(\varphi) + \varepsilon c_- \lambda),
\end{aligned}$$

with the following constitutive relations for the chemical potential $\mu(\varphi)$ and the non-monotone pressure $p(\varphi)$:

$$\mu(\varphi) = W'(\varphi) - \gamma\varepsilon^2\Delta\varphi, \qquad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

Outer and inner systems for (E)

The outer system for $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \mu_0, \mu_1, p_0, p_1)$:

$$
\begin{aligned}
\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)\mu_0 &= 0,\\
\nabla p_0 &= 0,\\
\partial_t\varphi_0 + \mathrm{div}(\varphi_0\mathbf{v}_0) &= c_+\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0),\\
\rho(\varphi_0)(\partial_t\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0) + \nabla p_1 + \nabla\lambda_0 &= 0,\\
\mathrm{div}\mathbf{v}_0 &= c_-\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0).
\end{aligned}
$$

The inner system is an ODE system w.r.t. z:

$$
\mathcal{L}[\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0, M_0, M_1, P_0, P_1] = \mathbf{F}
$$

Note: Equations for M_0, and P_0 decouples from the system above and can be independently solved:

$$
M_0 \equiv 0, \qquad P_0 \equiv 0.
$$

Vanishing leading order chemical potential

M_0 satisfies the following:

$$\partial_{zz} M_0 = 0, \qquad M_0 = W'(\Phi_0) - \gamma \partial_{zz}\Phi_0 \quad \Rightarrow \quad M_0 = 0.$$

Proposition

Let $\mu_0 \in C([0,T);C^2(\bar{\Omega}^{\pm}))$ satisfy the outer system, and the boundary conditions

$$\begin{aligned} \nabla \mu_0 \cdot \boldsymbol{\nu} &= 0 \qquad \text{on} \quad \partial\Omega, \\ \mu_0|_\Gamma &= \lim_{z\to\pm\infty} M_0(\mathbf{s}, z) \qquad \text{on} \quad \Gamma, \end{aligned}$$

where $M_0 \in C([0,T);C^2(\bar{U} \times \mathbb{R}))$ satisfies inner equations with $\partial_z M_0 \to 0$ for $z \to \pm\infty$. Then,

- *μ_0 vanishes in whole Ω,*
- *in particular, we have $\varphi_0 \in \{-1,+1\}$.*

Proof.

- Inner equation + Maxwell construction $\Longrightarrow M_0 \equiv 0$.
- Then we observe that the system

$$\begin{aligned} (m_\mathrm{j}\Delta - m_\mathrm{r})\mu_0 &= 0 \qquad \text{in} \quad \Omega^- \cup \Omega^+, \\ \nabla\mu_0 \cdot \boldsymbol{\nu} &= 0 \qquad \text{on} \quad \partial\Omega, \\ \mu_0|_\Gamma &= 0 \qquad \text{on} \quad \Gamma. \end{aligned}$$

  admits a unique solution, $\mu_0 \equiv 0$, due to the maximum principle.
- Recall that $\mu_0 = W'(\varphi_0)$.
- Then we have $\varphi_0|_{\Omega^\pm} = \pm 1$.

□

Reduction of the outer system

Due to the proposition above, we have a reduced bulk system for the tuple $(\varphi_1, \mathbf{v}_0, \lambda_0)$

$$\begin{cases} \big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu_1 + c_-\lambda_0) = 0 \\ \tilde{\rho}_i(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0)\nabla p_1 + \nabla\lambda_0 = 0 \\ \mathrm{div}\mathbf{v}_0 = 0 \end{cases}$$

where $i = 1, 2$ denotes two phases occurring in $\Omega^\pm$, respectively. Now we can identify Φ_0 uniquely as follows:

Corollary

Let $\Phi_0 \in C^1([0,T) \times U; C^3(\mathbb{R}))$ satisfy inner equations with $\Phi_0 \to \pm 1$ as $z \to \pm\infty$ and $\Phi_0(t, \mathbf{s}, 0) = 0$. Then the function Φ_0 unique and independent of t and $\mathbf{s}$.

Proof.

It directly follows from $M_0 = 0$ such that the equation

$$\gamma\partial_{zz}\Phi_0 - W'(\Phi_0) = 0 \qquad \text{with} \qquad \Phi_0 \to \pm 1 \quad \text{for} \quad z \to \pm\infty,$$

admits a unique solution $\Phi_0(t, \mathbf{s}, z) =: \Phi_0(z)$ which obeys $\Phi_0(0) = 0$. □

Interfacial conditions on velocity field

- Matching inner and outer settings yields a jump condition for the velocity field:

$$\boxed{[\mathbf{v}_0 \cdot \boldsymbol{\nu}]_-^+ = \frac{c_-}{c_+}\big(\langle \mathbf{v}_0 \cdot \boldsymbol{\nu}\rangle_-^+ - w_{\boldsymbol{\nu}}\big)}$$

Here, the jump of a quantity f across the interface and the mean value:

$$[f]_-^+ = f^+ - f^- \qquad \text{and} \qquad \langle f\rangle_-^+ = f^+ + f^-.$$

- In the case of identical fluids, i.e. $\tilde{\rho}_1 = \tilde{\rho}_2$, we recover the continuity of the velocity at the interface, that is $[\mathbf{v}_0 \cdot \boldsymbol{\nu}]_-^+ = 0$
- The continuity of the mass flux through the interface as follows:

$$\boxed{[\rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]_-^+ = 0 \qquad \text{with} \qquad \rho_0^+ = \tilde{\rho}_1,\ \rho_0^- = \tilde{\rho}_2}$$

- The normal velocity of the interface is then given by:

$$\boxed{2w_{\boldsymbol{\nu}} = \langle \mathbf{v}_0 \cdot \boldsymbol{\nu}\rangle_-^+ + \frac{m_j}{2} c_+ [c_+ \nabla\mu_1 \cdot \boldsymbol{\nu} + c_- \nabla\lambda_0 \cdot \boldsymbol{\nu}]_-^+}$$

Young-Laplace law and chemical potential. I

- Inner variables (P_1, M_1) are explicitely Φ_1 related. Furthermore we have the relation

$$c_+\partial_z M_1 + c_-\partial_z \Lambda_0 = 0 \qquad \Longrightarrow \qquad \tilde{L}\Phi_1 = \tilde{f}.$$

- We further perturb Φ_1 such that: $\quad \Psi = \Phi_1 - \tilde{\Phi}_1$ with $\Psi \in W^{3,1}$.

- Therefore we have a an opeartor $L : W^{3,1} \to L^1$ defined as

$$L\Psi := -(c_+ - c_-\Phi_0)\partial_z(W''(\Phi_0)\Psi - \gamma\partial_{zz}\Psi)$$

satisfying the equation

$$L\Psi = f(\Phi_0, V_0, \tilde{\Phi}_1)$$

- where the right hand side is given by

$$\begin{aligned} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1) = &-c_-\partial_z\big(J_0\mathbf{V}_0 \cdot \boldsymbol{\nu}\big) + c_-\partial_z\big((\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\partial_z\mathbf{V}_0 \cdot \boldsymbol{\nu}\big) \\ &+ \kappa\gamma(c_+ - c_-\Phi_0)\partial_{zz}\Phi_0 + (c_+ - c_-\Phi_0)\partial_z(W''(\Phi_0)\tilde{\Phi}_1 - \gamma\partial_{zz}\tilde{\Phi}_1). \end{aligned}$$

Proposition

The equation $L\Psi = f$ admits a solution if and only if

$$\int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)\mathrm{d}z = 0, \qquad \int_{-\infty}^{\infty} \frac{f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)}{c_+ - c_-\Phi_0}\mathrm{d}z = 0$$

holds. This means, in particular,

$$[j_0 \mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]_-^+ = C_1(\Phi_0)\kappa\gamma,$$

$$[\mu_1]_-^+ = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\left(\frac{1}{2}c_- c_+ \, j_0^2\big|_\Gamma + C_2(\Phi_0) j_0\big|_\Gamma\right).$$

Remark 1. The coefficients are given by

$$C_1(\Phi_0) = \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \qquad C_2(\Phi_0) = \int_{-\infty}^{\infty} (\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\left(\frac{1}{\rho(\Phi_0)}\right)_z^2 \mathrm{d}z.$$

Remark 2. We have no longer classical Gibbs-Thompson law valid, but a jump for the chemical potential.

Proof of the proposition

The proof essentially follows from the Fredholm alternative theorem:

$$L\Psi = f \quad \text{is solvable if and only if} \quad \int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)\phi \, \mathrm{d}z = 0$$

for all solutions $\phi(z) \in L^\infty(\mathbb{R})$ of the homogeneous problem for the adjoint operator $L^* : L^\infty(\mathbb{R}) \to W^{-3,1}(\mathbb{R})$ given by

$$L^*\phi = (W''(\Phi_0)\partial_z - \gamma\partial_{zzz})\big((c_+ - c_-\Phi_0)\phi\big).$$

Therefore, we need to find all linearly independent solutions to $L^*\phi = 0$ which are in $L^\infty(\mathbb{R})$.

- First we observe that every constant is a solution, i.e. $\phi_1 =$ constant.
- The other two solutions can be found by reducing the system

$$\phi_2(z) = \frac{c_+/c_-}{c_+ - c_-\Phi_0}, \qquad \phi_3(z) = \int_0^z \phi_2'(\tilde{z}) \int_0^{\tilde{z}} \frac{c_+ - c_-\Phi_0}{(\partial_z\Phi_0)^2} \mathrm{d}\tilde{z}$$

- Observe $\phi_1, \phi_2 \in L^\infty(\mathbb{R})$ and $\phi_3 \notin L^\infty(\mathbb{R})$

The sharp interface limit in the Euler regime

In $\Omega^{\pm}$, respectively, for $i = 1, 2$,

$$\begin{aligned} \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0) &= 0, \\ \tilde{\rho}_i(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0) + \nabla p_1 + \nabla \lambda_0 &= 0, \\ \mathrm{div}\mathbf{v}_0 &= 0, \end{aligned}$$

At the interface

$$\begin{aligned} [j_0 \mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]_-^+ &= \gamma C_1 \kappa, \\ [\mathbf{v}_0 \cdot \boldsymbol{\nu}]_-^+ &= \tfrac{c_-}{c_+} \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}} \rangle_-^+, \\ c_-[\lambda_0]_-^+ &= -c_+[\mu_1]_-^+, \\ [\mu_1]_-^+ &= \tfrac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big(\tfrac{1}{2} c_- c_+ \, j_0^2\big|_\Gamma + j_0|_\Gamma C_2\Big), \\ 2 w_{\boldsymbol{\nu}} &= \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle_-^+ + \tfrac{m_{\mathrm{j}}}{2} c_+ [c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}]_-^+, \end{aligned}$$

Implicitly given the continuity of mass flux: $[\rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]_-^+ = 0.$

Concluding remarks for the Euler regime

- During the phase separation (or transition) it might occure that $\mathbf{v}_0 \cdot \boldsymbol{\nu} = w_{\boldsymbol{\nu}}$, i.e. $j_0 = 0$

$$
\begin{aligned}
[p_1]_-^+ &= \kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \\
[\mathbf{v}_0]_-^+ &= [\mu_1]_-^+ = [\lambda_0]_-^+ = 0, \\
\mu_1|_\Gamma &= \tfrac{1}{2}\kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \\
w_{\boldsymbol{\nu}} &= \mathbf{v}_0|_\Gamma \cdot \boldsymbol{\nu},
\end{aligned}
$$

  where we observe that due to the last relation the chemical potential μ_1 is expected to decouple from the system. However, we have the Lagrange multiplier coupled to the diffusion equation for μ_1.

$$
\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu_1 + c_-\lambda_0) = 0
$$

- In the case of the identical fluids, i.e. $\tilde{\rho}_2 = \tilde{\rho}_1$, the terms with c_- drop out. The chemical potential decouples from the system. Therefore, the whole sharp interface model in the limit recovers the free boundary problem of Euler type.

Diffuse model in Navier-Stokes (N-S) regime with similar densities

Similar densities restriction: $\tilde{\rho}_2 - \tilde{\rho}_1$ is only ε comparable quantity.

Further, we compare the Mach number and Reynolds number as follows:

$$M \sim \varepsilon^{1/2}, \qquad \frac{1}{\mathrm{Re}} \sim 1.$$

which means no viscosity restriction in this case.

Scaled Navier-Stokes-Korteweg/Cahn-Hilliard system reads

$$\partial_t \varphi + \mathrm{div}(\varphi \mathbf{v}) = \tfrac{1}{\varepsilon} c_+ \big(m_{\mathrm{j}} \Delta - m_{\mathrm{r}}\big)(c_+ \mu(\varphi) + \varepsilon c_- \lambda),$$

$$\rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \tfrac{1}{\varepsilon}(\nabla p(\varphi) + \nabla \lambda) = \nabla(\eta(\varphi)\mathrm{div}\mathbf{v}) + \mathrm{div}(\hat{\eta}(\varphi)(\nabla \mathbf{v} + \nabla \mathbf{v}^T)) + \gamma \varepsilon \varphi \nabla \Delta \varphi,$$

$$\mathrm{div}\mathbf{v} = c_- \big(m_{\mathrm{j}} \Delta - m_{\mathrm{r}}\big)(c_+ \mu(\varphi) + \varepsilon c_- \lambda),$$

with the constitutive relations for the chemical potential μ and the non-monotone pressure p:

$$\mu(\varphi) = W'(\varphi) - \gamma \varepsilon^2 \Delta \varphi, \qquad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

Proposition

The leading order chemical potential μ_0 vanishes in whole Ω. In particular, we have

$$\varphi_0 \in \{-1, +1\}.$$

This reduces the outer system:

$$\begin{cases} \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0) = 0, \\ \nabla\lambda_0 = 0, \\ \rho_0(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0) + \nabla p_1 + \nabla\lambda_1 = \mathrm{div}(\hat{\eta}(\varphi_0)(\nabla\mathbf{v}_0 + \nabla{\mathbf{v}_0}^T)), \\ \mathrm{div}\mathbf{v}_0 = 0, \end{cases}$$

for the tuple $(\varphi_1, \mathbf{v}_0, \lambda_0, \lambda_1)$ where $\rho_0 = \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2}$. Furthermore, we can identify $\Phi_0 = \Phi_0(z)$ uniquely as a corollary as before.

Remark. $\nabla\lambda_0 = 0$ implies that $\lambda_0(t, \mathbf{x}) = \lambda_0(t)$.

Interfacial conditions. Navier-Stokes regime with similar densities

- Continuity of the velocity field, leading order Lagrange multiplier and the chemical potential at the interface:

$$\boxed{[\mathbf{v}_0]_-^+ = 0, \qquad [\lambda_0]_-^+ = 0, \qquad [\mu_1]_-^+ = 0}$$

- Interfacial stress balances satisfy the Young-Laplace law:

$$\boxed{[p_1 + \lambda_1]_-^+ \boldsymbol{\nu} - [\hat{\eta}_i(\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^{\mathrm{T}})]_-^+ \boldsymbol{\nu} = \kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z \, \boldsymbol{\nu}}$$

- Gibbs-Thompson law:

$$\boxed{\mu_1|_\Gamma = \tfrac{1}{2}\gamma\kappa \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z}$$

- Interfacial velocity:

$$\boxed{w_{\boldsymbol{\nu}} = \mathbf{v}_0|_\Gamma \cdot \boldsymbol{\nu} + + \tfrac{m_{\mathrm{j}}}{2} c_+ [c_+ \nabla \mu_1 \cdot \boldsymbol{\nu}]_-^+}$$

The limiting model in the (N-S) regime

An incompressible sharp interface model where $\lambda_0(t)$ appears only if $m_{\mathrm{r}} \neq 0$:

$$
\begin{aligned}
c_+\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)\mu_1 + c_- m_{\mathrm{r}}\lambda_0(t) &= 0,\\
\rho_0(\partial_t \mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0) + \nabla p_1 + \nabla\lambda_1 &= \mathrm{div}(\hat\eta(\varphi_0)(\nabla\mathbf{v}_0 + \nabla{\mathbf{v}_0}^T)),\\
\mathrm{div}\mathbf{v}_0 &= 0,
\end{aligned}
$$

equipped with the interface conditions

$$
\begin{aligned}
[p_1+\lambda_1]_-^+\boldsymbol{\nu} - [\hat\eta_i(\nabla\mathbf{v}_0 + \nabla\mathbf{v}_0^{\mathrm{T}})]_-^+\boldsymbol{\nu} &= \kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z\,\boldsymbol{\nu},\\
w_{\boldsymbol{\nu}} &= \mathbf{v}_0|_\Gamma\cdot\boldsymbol{\nu} + + \tfrac{m_{\mathrm{j}}}{2}c_+[c_+\nabla\mu_1\cdot\boldsymbol{\nu}]_-^+,\\
\mu_1|_\Gamma &= \frac{1}{2}\gamma\kappa\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z
\end{aligned}
$$

with

$$
[\mathbf{v}_0]_-^+ = 0, \qquad [\mu_1]_-^+ = 0.
$$

Outlook

Further in formal asymptotic analysis:

- Temperature dependent model
- Semi-compressible model: Introducing the phase fraction φ for an incompressible constituent and keeping the density for the compressible one.

$$\rho_1 = \tilde{\rho}_{\mathrm{L}} \frac{1+\varphi}{2}, \qquad \rho_2 = \tilde{\rho}_{\mathrm{V}} \frac{1-\varphi}{2} \quad \text{with} \quad \rho_{\mathrm{L}} = const \quad \text{and} \quad \rho_{\mathrm{V}} \neq const$$

Open analytical questions:

- Well-posedness of the diffuse model...
- Well-posedness of the limiting sharp models...
- A rigorous sharp interface limit...

Thank You!