The Bounded $L^2$ curvature conjecture in general relativity

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(Joint work with Sergiu Klainerman and Igor Rodnianski)
Cauchy Problem for EE

\((\mathcal{M}, g)\) Lorentzian, \(R\) curvature tensor of \(g\)

Einstein Vacuum equations: \(\text{Ric}_{\alpha\beta} = 0\)

Wave coordinates: \(\Box_g x^\alpha = \frac{1}{\sqrt{|g|}} \partial_\beta (g^{\beta\gamma} \sqrt{|g|} \partial_\gamma) x^\alpha = 0, \alpha = 0, 1, 2, 3\)

\(\Box_g g_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(g, \partial g), \alpha, \beta = 0, 1, 2, 3,\) with \(\mathcal{N}_{\alpha\beta}\) quadratic w.r.t \(\partial g\)

Cauchy data: \((\Sigma_0, g_0, k)\) where \(\Sigma_0 = \{t = 0\},\ g(0, .) = g_0,\ \partial_t g(0, .) = k\)

Question: Under which regularity do we have local existence for EE?
Semilinear Wave Equations

\[
\begin{aligned}
\square \phi &= \mathcal{N}(\phi, \partial \phi), \ (t,x) \in \mathbb{R}^{1+3} \\
\phi(0,.) &= \phi_0 \in H^s(\mathbb{R}^3), \partial_t \phi(0,.) = \phi_1 \in H^{s-1}(\mathbb{R}^3)
\end{aligned}
\]

where \( \mathcal{N} \) is quadratic w.r.t \( \partial \phi \)

Question: For which \( s \) is it locally well posed?

Sobolev embedding in \( \mathbb{R}^3 \): WP for \( s > 5/2 \)

Strichartz for \( \square \phi = 0 \Rightarrow WP \text{ for } s > 2 \) (Ponce-Sideris)

Ill-posed for \( s = 2 \) in general (Cex of Lindblad)

If \( \mathcal{N} = Q_{ij} \) with \( Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_i \psi \partial_j \phi \)

Bilinear estimates for \( Q_{ij} \Rightarrow WP \text{ for } s > 3/2 \) (Klainerman-Machedon)
Quasilinear Wave Equations

\[
\begin{aligned}
\Box_g(\phi) \phi &= \mathcal{N}(\phi, \partial\phi), \quad (t, x) \in \mathbb{R}^{1+3} \\
\phi(0, \cdot) &= \phi_0 \in H^s(\mathbb{R}^3), \quad \partial_t \phi(0, \cdot) = \phi_1 \in H^{s-1}(\mathbb{R}^3)
\end{aligned}
\]

Using Sobolev embedding: WP for \( s > 5/2 \)

Strichartz for \( \Box_g \phi = 0 \) requires \( g \in C^{1,1} \) (Smith)

Strichartz with loss enough: WP for \( s > 2 + 1/4 \) (Bahouri-Chemin)

WP for \( s > 2 \) (Klainerman-Rodnianski for EE, Smith-Tataru for general quasilinear wave equations)

Interesting geometrical hyperbolic equations satisfy the null structure

Goal: prove that EE are WP in \( H^2 \)
Bounded $L^2$ curvature conjecture

**Conjecture.** Let $(\Sigma_0, g_0, k)$ with $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$. Then, EE are WP

Motivations:

- First WP result for a quasilinear wave equation below $H^{2+\epsilon}$
- The assumptions $R \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$ are natural from the point of view of geometry
- Rather than a WP result, it can be viewed as a continuation argument: as long as $R \in L^2$ and $\nabla k \in L^2$ along a spacelike hypersurface, one may extend the solution of EE
- The control of the Eikonal equation $g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ requires $R \in L^2$
Strategy for a proof

A Recast the EE as a quasilinear Yang-Mills theory

B Prove appropriate bilinear estimates for solutions to $\Box g \phi = 0$

C Construct a parametrix for $\Box g \phi = 0$, and obtain the control of the parametrix and of its error term

D Prove a sharp $L^4(M)$ Strichartz estimate for the parametrix

Goal: take inspiration from the proof of Klainerman-Machedon for the WP of Yang-Mills in $H^1(\mathbb{R}^3)$

Achieve Steps B, C and D only assuming $L^2$ bounds on $\mathbb{R}$
WP of Yang-Mills in $H^1(\mathbb{R}^3)$ (Klainerman-Machedon)

\[ \Box A + \nabla_{t,x}(\nabla_{t,x} \cdot A) = [A, \nabla_{t,x} A] + A^3, \quad A = (A_0, A_1, A_2, A_3) \]

Gauge freedom. Choosing the Coulomb gauge $\partial_i A_i = 0$:

\[ \Box A + \nabla_{t,x}(\partial_0 A_0) = [A, \nabla_{t,x} A] + A^3 \]

$P = \text{projector on divergence free vectorfields}$:

\[ \Delta(A_0) = \text{l.o.t} \]
\[ \Box(A_i) = \left( P \left( Q_{jl}(\nabla^{-1} A, A) + \nabla^{-1}(Q_{jl}(A, A)) \right) \right)_i + \text{l.o.t} \]

\[ \| \partial A \|_{L^\infty_t L^2(\mathbb{R}^3)} \lesssim \| Q_{jl}(\nabla^{-1} A, A) \|_{L^2_t L^2(\mathbb{R}^3)} + \| \nabla^{-1}(Q_{jl}(A, A)) \|_{L^2_t L^2(\mathbb{R}^3)} + \text{l.o.t.} \]

Prove two bilinear estimates to conclude
Step A: EE as a quasilinear Yang-Mills theory

Let $e_{\alpha}$ an orthonormal frame on $\mathcal{M}$, i.e. $g(e_{\alpha}, e_{\beta}) = m_{\alpha \beta}$

Let $(A_{\mu})_{\alpha \beta} := (A)_{\alpha \beta}(\partial_{\mu}) = g(D_{\mu} e_{\beta}, e_{\alpha})$

The definition of $R$ yields:

$$R(e_{\alpha}, e_{\beta}, \partial_{\mu}, \partial_{\nu}) = \partial_{\mu}(A_{\nu})_{\alpha \beta} - \partial_{\nu}(A_{\mu})_{\alpha \beta} + (A_{\nu})_{\alpha}^{\lambda}(A_{\mu})_{\lambda \beta} - (A_{\mu})_{\alpha}^{\lambda}(A_{\nu})_{\lambda \beta}$$

$D^{\mu}R_{\alpha \beta \mu \nu} = 0$ yields the tensorial wave equation:

$$(\Box g A)_{\nu} - D_{\nu}(D^{\mu}A_{\mu}) = D^{\mu}([A_{\mu}, A_{\nu}]) + [A^{\mu}, D_{\mu}A_{\nu} - D_{\nu}A_{\mu}] + A_{\nu}^3$$

In view of the Klainerman-Machedon proof, we need in particular a procedure to scalarize the tensorial wave equation and to project on divergence free vectorfields without destroying the null structure
Scalarization and projection procedure

Scalarization: compute $[X, \Box g]$ for any vectorfield $X$ and use it with $X = e_\alpha, \alpha = 0, 1, 2, 3$

Projection: compute $[\mathcal{P}, \Box g]$ where $\mathcal{P} = $ projector on divergence free vectorfields

These commutators generate numerous dangerous terms which need to satisfy the null structure

We check this using the symmetries of $\mathbf{R}$, the Bianchi identities, the link between $\mathbf{A}$ and $\mathbf{R}$, the fact that $A_0$ is better than $A_1, A_2, A_3$, and the Coulomb gauge
The energy estimate for the wave equation

The proof reduces to the control of a scalar function \( \phi \) satisfying

\[
\Box_g (\phi) = \text{null forms} + l.o.t
\]

Let \( \phi \) a scalar function and \( Q_{\alpha\beta} \) its energy momentum tensor:

\[
Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)
\]

\[
\int_{\Sigma_t} Q_{TT} = \int_{\Sigma_0} Q_{TT} + \int_{\mathcal{R}} \Box_g \phi T(\phi) + \int_{\mathcal{R}} Q^{\alpha\beta} D_\alpha T_\beta
\]

Last term in RHS is dangerous \( \Rightarrow \) needs to display the null structure and requires to prove the corresponding trilinear estimate
Step B: the bilinear estimates

We need to estimate the following null forms

\[ \| Q_{ij}(\phi, A) \|_{L^2(\mathcal{M})} \quad \text{and} \quad \| (-\Delta_g)^{-\frac{1}{2}} (Q_{ij}(\partial \phi, A)) \|_{L^2(\mathcal{M})} \]

where \( \phi \) is a scalar function \( \phi \) satisfying

\[ \square_g(\phi) = \text{null forms} + l.o.t \]

To prove these bilinear estimates in a quasilinear setting:

- write \( \phi \) by iterating the basic parametrix of step C (construction and control of the parametrix)
- Rethink the proof of bilinear estimates in the quasilinear setting
- For the second type of bilinear estimate, rely on the structure of \( Q_{ij} \) and a sharp \( L^4(\mathcal{M}) \) Strichartz estimate for the parametrix
Step C: construction and control of the parametrix

\[ S(t, x) = \sum_{\pm} \int_{\mathbb{S}^2} \int_{0}^{+\infty} e^{i\lambda u_{\pm}(t, x, \omega)} f_{\pm}(\lambda \omega) \lambda^2 d\lambda d\omega \]

where \( g^{\alpha\beta} \partial_{\alpha} u_{\pm} \partial_{\beta} u_{\pm} = 0 \) on \( \mathcal{M} \) such that \( u_{\pm}(0, x, \omega) \sim x \cdot \omega \) when \( |x| \to +\infty \) on \( \Sigma_0 \)

Construction: for any \( (\phi_0, \phi_1) \) there exists \( f_{\pm} \) such that

\( S(0, .) = \phi_0, \; \mathbf{D}_T S(0, .) = \phi_1 \) and \( \|\lambda f_{\pm}\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} \)

\[ E(t, x) = \Box_g S(t, x) = i \sum_{\pm} \int_{\mathbb{S}^2} \int_{0}^{+\infty} e^{i\lambda u_{\pm}(t, x, \omega)} \Box_g u_{\pm}(t, x, \omega) f_{\pm}(\lambda \omega) \lambda^3 d\lambda d\omega \]

Control of the error term: \( \|Ef\|_{L^2(\mathcal{M})} \lesssim \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \)
Step C: construction and control of the parametrix

C1 Choice of \( u(0, x, \omega) \) and control of the foliation of \( \Sigma_0 \) by \( u(0, x, \omega) \).

C2 Construction of the parametrix using the control of \( u(0, x, \omega) \) obtained in Step C1

C3 Control the geometry of the foliation of \( M \) by \( u \) solution of the Eikonal equation
\[
 g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0.
\]

C4 Control the error term using the control of \( u(t, x, \omega) \) obtained in Step C3

Goal: Achieve Steps C1-C4 only assuming \( L^2 \) bounds on \( R \)

Steps C1 and C3 rely on the full structure of Einstein equations
Step C: construction and control of the parametrix

- The regularity in $\omega$ of $u$ obtained in Steps C1 and C3 is limited.

- In step C1: a careful choice of $u(0, x, \omega)$ (related to the mean curvature flow) allows us to "squeeze" as much regularity in $x$ and $\omega$ as possible.

- In step C3: $\mathbf{R} \in L^2$ is minimal to obtain a lower bound on the radius of injectivity of level surfaces of the phase $u$.

- Steps C2 and C4 require $L^2$ bounds for Fourier integral operators, and in turn several integration by parts. Classical proofs ($TT^*$ and $T^*T$ arguments) would fail by far.
Thank you!