Local Energy Decay for Maxwell Fields on Spherical Black Holes

Jacob Sterbenz,
joint with Daniel Tataru

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Spherical "Black Holes"

Metric Axioms

We consider space-times with coordinates $(x^a, X^A) \in \mathbb{R}^2 \times \mathbb{S}^2$ such that the metric $g = g_{ab}dx^a dx^b + r^2\delta_{AB}dx^A dx^B$ satisfies:

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1. (Stationary asymptotic flatness)There exists a spherically symmetric time function t defined for $t \geqslant 0$, such that in the (t, r) coordinates one has $h_{ab} = \text{diag}(-1, 1) + O(r^{-1})$. Furthermore $\partial_t h_{ab} \equiv 0$, and $\partial_r^k h_{ab} = O(r^{-1-k})$ for $k \geqslant 1$.

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2. (Non-degenerate forward global hyperbolicity) There exists a value $r_0 > 0$ such that $r = r_0$ is also space-like, and the sign of $g^{rr} = \langle dr, dr\rangle$ changes only once in $r \geqslant r_0$ while $\partial_r g^{rr}$ never vanishes.

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3. (Strictly hyperbolic trapping) There is a one and only one value $r_{\mathcal{T}}$ in the region where $\langle dr, dr \rangle > 0$ such that the time-like surface $r = r_{\mathcal{T}}$ is trapped set for all null geodesics initially tangent to it, and the flow is normally hyperbolic.

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- For scalar waves one can combine horizon estimates with work of Wunsch-Zworski (’11). (Horizons are modular).

Two Forms and Their Duals

- Now orient $(\mathcal{M}, g_{\alpha\beta})$ so that (dt, dr, dx^2, dx^3) is a positive basis of $T^*(\mathcal{M})$. There is a unique isomorphism $\star : \Lambda^p \to \Lambda^{4-p}$ such that $\langle \omega, \sigma \rangle_g dV_g = \omega \wedge \star\sigma$ where $dV_g = \sqrt{|g|} dt \wedge dr \wedge dV_{\mathbb{S}^2}$, and $dV_{\mathbb{S}^2}$ is the standard volume form on $\mathbb{S}^2$.

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- Let $F_{\alpha\beta}$ be (any) antisymmetric two-tensor on $\mathcal{M}$, and define $F^\star = \star F$. We label its divergences as follows:

$$\nabla^\beta F^\star_{\alpha\beta} = I_\alpha , \qquad \nabla^\beta F_{\alpha\beta} = J_\alpha .$$

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- We call I_α the *magnetic source* and J_α the *electric source* of $F_{\alpha\beta}$. On physical grounds one usually sets $I \equiv 0$, but for mathematical purposes we will not do so here.

The Physical Energy

- The energy momentum tensor of a Maxwell field is given by:
  $Q_{\alpha\beta}[F] = F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} = \frac{1}{2}\left(F_{\alpha\gamma}F_{\beta}{}^{\gamma} + F^{\star}_{\alpha\gamma}F^{\star}_{\beta}{}^{\gamma}\right)$.

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Thus, the energy of F on $t = const$ is equivalent to $\| F(t) \|_{L^2_x}$.
- Since there are no time-like killing fields in a nbd of $g^{rr} = 0$ we need some integrability of F in order to conclude energy estimates. Note that there is no red shift at the level of F or Q! (In fact component-wise some of F is blue shifted).

A Fake Local Energy Decay Estimate

- Based on the previous slide one should expect a local energy decay estimate roughly of the form:
$$\| r^{-\frac{1}{2}-\epsilon} F \|_{L^2(dV_g)[0,T]} \leqslant C_\epsilon \Big(\| F|_{t=0} \|_{L^2(dV_{g|_{t=0}})} + \| r^{\frac{1}{2}+\epsilon} I \|_{L^2(dV_g)[0,T]} + \| r^{\frac{1}{2}+\epsilon} J \|_{L^2(dV_g)[0,T]} \Big) .$$

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- The estimate is false in general (with any regularity loss) due to finite energy bound states.

Local Charges

- Let $\mathcal{S} \subseteq \mathcal{M}$ be a compact space-like two surface homotopic to some sphere $t = const$, $r = const$, and define the quantities (in terms of pull-backs of F):

$$Q_{\mathcal{S}} = \int_{\mathcal{S}} F|_{\mathcal{S}} , \qquad Q^{\star}_{\mathcal{S}} = -\int_{\mathcal{S}} F^{\star}|_{\mathcal{S}} .$$

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- Let $\mathcal{S}'$ be some other space-like two surface such that there exists a tube $\Sigma(\mathcal{S}', \mathcal{S})$ with boundary $\partial\Sigma(\mathcal{S}', \mathcal{S}) = \mathcal{S}' \sqcup \mathcal{S}$. Then by Stokes theorem one has:
$$\begin{aligned} Q_{\mathcal{S}'} - Q_{\mathcal{S}} &= -\int_{\Sigma(\mathcal{S}', \mathcal{S})} (\star I)|_{\Sigma} \,, \\ Q^{\star}_{\mathcal{S}'} - Q^{\star}_{\mathcal{S}} &= -\int_{\Sigma(\mathcal{S}', \mathcal{S})} (\star J)|_{\Sigma} \,. \end{aligned}$$

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- In particular if $F_{\alpha\beta}$ is a free Maxwell field ($I = J = 0$) then $Q_{\mathcal{S}}$ and $Q^{\star}_{\mathcal{S}}$ are constants equal to:
$$Q_{\infty} = \lim_{r \to \infty} \int_{S_{t,r}} F_{\hat{A}\hat{B}}(t, r) dV_{\mathbb{S}^2} \, ,$$
$$Q^{\star}_{\infty} = \lim_{r \to \infty} \int_{S_{t,r}} F_{tr}(t, r) dV_{\mathbb{S}^2} \, .$$
Here $e_{\hat{A}}, e_{\hat{B}}$ are an orthonormal basis of spheres $S_{t,r}$ with respect to the restriction of g.

Explicit Formulas

- The charges are carried by two finite energy solutions to the stationary homogeneous problem:

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Dynamic Projection

- For a general two form F we project it dynamically onto the span of $\overline{F}^{electric}$ and $\overline{F}^{magnetic}$ as follows:
$$\overline{F} = Q^\star(t,r)\overline{F}^{electric} + Q(t,r)\overline{F}^{magnetic} ,$$
where $Q^\star(t,r)$ and $Q(t,r)$ denote the charge integrals taken over the spheres $S_{t,r}$.

Norms

- It turns out to be very convenient to use asymptotically scale invariant norms (e.g. to handle “residual charges”). Thus, we define:

$$\| F \|_{LE[0,T]} = \sup_j \| r^{-\frac{1}{2}} F \|_{L^2(\mathcal{R}_j)[r_0,\infty)\times[0,T]} ,$$

$$\| J \|_{LE^\star[0,T]} = \sum_j \| r^{\frac{1}{2}} J \|_{L^2(\mathcal{R}_j)[r_0,\infty)\times[0,T]} ,$$

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$$\| J \|_{LE^*[0,T]} = \sum_j \| r^{\frac{1}{2}} J \|_{L^2(\mathcal{R}_j)[r_0,\infty)\times[0,T]} ,$$

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Main Theorem

Let F be any two form on on a space-time $(\mathcal{M}, g)$ which satisfies the previous axioms. Define $\overline{F}$ as above. Then one has the uniform bound:

$$\| (w_{ln})^{-1}(F - \overline{F}) \|_{LE[0,T]} \lesssim \| F(0) \|_{L^2(dV_{g|_{t=0}})} + \| w_{ln}(I, J) \|_{LE^*[0,T]} ,$$

where $w(r) = (1 + |\ln|r - r_{\mathcal{T}}||)/(1 + |\ln(r)|)$.

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Price Law type results (Metcalfe-Tataru-Tohaneanu in prep)

Let F solve the Maxwell system with data (I, J) and $F(0)$ on a general asymptotically flat space-time. Suppose F is truncated so that all quantities are supported in the forward "cone" $t > r - R$ (in normalized coords). Then if one assumes the LE estimate for all such fields F one has:

$$|F| \lesssim \kappa \frac{1}{\langle r\rangle\langle t-r\rangle^3} \ , \qquad \kappa = E^m(0) + \| t^{\frac{7}{2}}(I, J) \|_{LE^*} + \| rt^{\frac{7}{2}}\mathcal{L}_{\partial_t}(I, J) \|_{LE^*}$$

Local energy and related decay on various (non-BH) backgrounds

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- Related to a large body of work on resonances (and related resolvent estimates): Burq, Sjöstrand, Vodev, Zworski, ...

Results for BH: scalar fields

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- Higher decay rates related to LE decay obtained by a number of authors: Tataru ('11), Metcalfe-Tataru-Tohaneau ('12), Luk ('12; first application to a DNLW problem!), Blue-Andersson ('09). Higher decay rates also announced by Dafermos-Rodniaski ('09).

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- Results for the full Maxwell system on Kerr with small $|a|$ also recently announced by Andersson-Blue-Nicolas (’13).

First order equations

- We begin by introducing the field quantities:

$$\phi = \frac{1}{2}\epsilon^{AB}F_{AB} , \quad \phi^\star = -\frac{1}{2}\epsilon^{AB}F^\star_{AB} ,$$
$$\not{F}_{aA} = F_{aA} , \quad \not{F}^\star_{aA} = F^\star_{aA} .$$

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Here ϵ is the volume form of δ_{AB}.
- Then a portion of the Maxwell system can be written as:

$$\underline{d}\phi = \star_\delta \not{d}\not{F} - r^2 \star_h I\,, \tag{1}$$

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- From this one sees immediately that the local charge can be subtracted off in terms of the averages:

$$Q(t,r) = Q_\infty + \int_r^\infty\int_{\mathbb{S}^2}(\star_h I)_r(t,s)s^2 dV_{\mathbb{S}^2} ,$$
$$Q^\star(t,r) = Q^\star_\infty + \int_r^\infty\int_{\mathbb{S}^2}(\star_h J)_r(t,s)s^2 dV_{\mathbb{S}^2} .$$

  Thus w/o loss of generality we assume $\overline{F} \equiv 0$.

Reduction to Spin Zero

- Using the first order system and basic Hodge theory one has:
$\| (w_{ln})^{-1} \not{F} \|_{LS[0,T]} \lesssim$
$\| (w_{ln})^{-1} r^{-1} (-\not{\Delta})^{-\frac{1}{2}} (\underline{d}\phi, \underline{d}\phi^\star) \|_{LS[0,T]} + \| (I, J) \|_{LS^*[0,T]}$,

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- Thus all the dynamical information is contained in ϕ and $\phi^\star$ and the game switches to estimating the RHS norms above involving these quantities.

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- To prove estimates we'll use second order equations. These are easiest to compute with respect to a conformal metric.

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dF &= -\Omega^{-2}\tilde{\star} I\,, & dF^\star &= \Omega^{-2}\tilde{\star} J\,,\\
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- Combining we have:

$$\Box^{hodge} F = d^{\tilde{\star}}(\Omega^{-2}\tilde{\star} I) - d(\Omega^{-2} J)\,,$$
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where $\Box^{hodge} = -(dd^{\tilde{\star}} + d^{\tilde{\star}}d)$ is the Hodge Laplacian of $\widetilde{g}$.
- If we choose $\Omega = r^{-1}$ then a little bit of calculation gives:
$$\Box^0\phi = -\nabla^a(r^2 \star_h I)_a - \star_\delta \not{d} J\,,$$
$$\Box^0\phi^\star = -\nabla^a(r^2 \star_h J)_a + \star_\delta \not{d} I\,.$$
Here $\Box^0 = \nabla^a\nabla_a + r^{-2}\not{\Delta}$ is the "spin zero" wave operator.

The Main Estimate

Let ϕ and H be a scalar functions on $[0, T] \times [r_0, \infty) \times \mathbb{S}^2$. Let G_a be a one form in the x^a variables, depending also on $x^A \in \mathbb{S}^2$, with $\star_h \underline{d} G = K$.

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$$\int_{\mathbb{S}^2}\phi dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} G_b dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} H dV_{\mathbb{S}^2} \ \equiv \ 0 \ .$$

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Then one has the local energy decay type estimate estimate:

$$\| (w_{ln})^{-1} r^{-1}\big(\underline{d}(-\not{\Delta})^{-\frac{1}{2}}\phi, r^{-1}\phi\big)\|_{LS[0,T]}$$

$$\lesssim \ \| (-\not{\Delta})^{-\frac{1}{2}} r^{-1}\big(\underline{d}\phi(0) - G(0)\big)\|_{L^2(dV_{g|_{t=0}})} + \| r^{-2}\phi(0)\|_{L^2(dV_{g|_{t=0}})}$$

$$+ \| w_{ln} r^{-1}\big(r^{-1}G, (-\not{\Delta})^{-\frac{1}{2}}K, (-\not{\Delta})^{-\frac{1}{2}}H\big)\|_{LS^*[0,T]} \ .$$

Here $w(r) = (1+\big|\ln|r-r_{\mathcal{T}}|\big|)/(1+|\ln(r)|)$ as usual.

Further reduction of the main theorem

- To simplify matters introduce rescaled spaces:

  $$\| \phi \|_{LS_0} = \sup_{j\geqslant 0} 2^{-\frac{j}{2}} \| \chi_j \phi \|_{L^2(dV)} ,$$

  $$\| \phi \|_{LS_0^*} = \sum_{j\geqslant 0} 2^{\frac{j}{2}} \| \chi_j g \|_{L^2(dV)} ,$$

  and:

  $$E(\phi[t]) = \int_{[r_0,\infty]\times\mathbb{S}^2} (\phi_t^2 + \phi_r^2 + r^{-2}|\not{d}\phi(t)|^2) dr dV_{\mathbb{S}^2} ,$$

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- Then the core part of our main estimate becomes: Let $\Box^0\phi = G$, then one has:

$$\| \partial_r \phi \|_{LS_0[0,T]} + \| (w_{ln})^{-1} (\partial_t \phi, r^{-1} \not{d}\phi) \|_{LS_0[0,T]} \lesssim E^{\frac{1}{2}}(\phi[0]) + \| w_{ln} G \|_{LS_0^*[0,T]} ,$$

  where we assume $\langle \partial_r, \partial_t \rangle_h = 0$ at $r = r_{\mathcal{T}}$.

Waves close to the horizon

- There are three types of waves one needs to contend with close to $g^{rr} = 0$. Call value $r = r_M$.

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  1. *Waves which propagate parallel to the horizon.* These are the easiest to control due to the red shift (microlocally its just the wave equation with positive friction).
  2. *Pure incoming waves from $r = r_2$ with $r_2 > r_M$.* These are the next easiest to control, and are estimated well in terms of the $\partial_t \approx \nabla r$ energy flux across the horizon $r = r_M$.

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  3. *"Turning waves" on which r is concave down.* These are by far the most complicated, and are related to trapping. To handle them one needs essentially microlocal ideas.
- One very useful features of (regular) horizons is that when properly understood these three mechanism allow one to "truncate" the horizon from the problem.

Pure multiplier estimates

Let r_1, r_2, r_3 be three parameters such that $r_0 \leqslant r_1 < r_M < r_2 < r_3 < r_{\mathcal{T}}$. Then there exists a fixed value of r_3 such that for every $k > 1$ one has the following estimates, where the implicit constants are uniform in r_1, r_2 (but may depend on r_3, k):

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- For parallel waves we use the estimate of Dafermos-Rodnianski:

$$\| \chi_{[r_1,r_3]}(\underline{L}\phi, \not{d}\phi) \|_{LS_0[0,T]} \lesssim \| \chi_{[r_3,r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not{d}\phi \|_{LS_0[0,T]} + E^{\frac{1}{2}}_{\geqslant r_1}(\phi[0]) + \| \chi_{[r_1,r_{\mathcal{T}}]}\Box^0\phi \|_{LS_0^*[0,T]} ,$$

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- For incoming waves we use:

$$\| \chi_{[r_1,r_2]} w_k L\phi \|_{LS_0[0,T]} \lesssim \epsilon \| \chi_{[r_1,\infty)}(w_{In})^{-1}\partial_t\phi \|_{LS_0[0,T]} + \| \chi_{[r_1,r_2]} w_k (r - r_M)\underline{L}\phi \|_{LS_0[0,T]} + (r_M - r_1)^{\frac{1}{2}} \| \chi_{[r_3,r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not{d}\phi \|_{LS_0[0,T]} + E^{\frac{1}{2}}_{\geqslant r_1}(\phi[0]) + \epsilon^{-1} \| \chi_{[r_1,\infty)} w_{In}\Box^0\phi \|_{LS_0^*[0,T]} .$$

The "Microlocal" Estimate

To estimate the remaining (turning) waves we use:

$$\begin{aligned}
&\| \chi_{[r_2,r_3]} w_k\big(L\phi, (r-r_M)\underline{L}\phi\big) \|_{LS_0[0,T]} + \| \chi_{[r_2,r_3]} \not{d}\phi \|_{LS_0[0,T]} \\
&\quad + \| \chi_{[r_3,\infty)} \partial_r \phi \|_{LS_0[0,T]} + \| \chi_{[r_3,\infty)} (w_{ln})^{-1} (\partial_t \phi, r^{-1}\not{d}\phi) \|_{LS_0[0,T]} \\
&\qquad \lesssim \ |\ln(r_2 - r_M)|^k \| \chi_{[r_M,r_2]} w_k\big(L\phi, (r-r_M)\underline{L}\phi\big) \|_{LS_0[0,T]} \\
&\qquad\qquad + C_{r_2}\Big(\| \chi_{[r_M,\infty)} w_{ln} \Box^0 \phi \|_{LS_0^*[0,T]} + E^{\frac{1}{2}}_{\geqslant r_M}(\phi[0]) \Big) \ .
\end{aligned}$$

Normalized Coordinates

- In order to prove the last estimate we introduce some normalized coordinates in $r > r_M$. Note that to prove the last estimate above we really only need them in $r > r_M + \epsilon$, so we are free to let them degenerate at $r = r_M$.

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- Then there exists two functions $s = t + b(r)$ and $r_* = r_*(r)$ defined in the region $r > r_M$, such that $r_*(r_\mathcal{T}) = 0$, and such that (s, r_*) are smooth coordinates in $r > r_M$ with:

$$r_* \ \to \ -\infty \ \text{ as } \ r \ \to \ r_M \ , \qquad\qquad r_* \ \to \ \infty \ \text{ as } \ r \ \to \ \infty \ .$$

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$$r_* \to -\infty \quad \text{as} \quad r \to r_M , \qquad\qquad r_* \to \infty \quad \text{as} \quad r \to \infty .$$
- These have the following asymptotics as $r \to r_M$:
$$\partial_r s = -|\det(g_{ab})|^{-\frac{1}{2}}(g^{rr})^{-1} + \widetilde{s}(r) ,$$
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where $\widetilde{s}$ is uniformly bounded with all of its derivatives on $r > r_M$. They are also “asymptotically flat” as $r \to \infty$.

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  where $\widetilde{s}$ is uniformly bounded with all of its derivatives on $r > r_M$. They are also “asymptotically flat” as $r \to \infty$.
- With respect to (s, r_*) the $(1 + 1)$ dimensional Lorentzian metric h can be written as:
  $$h \ = \ \Omega^2(-ds^2 + dr_*^2) \ , \qquad \Omega^2 \ = \ -g_{00} \ = \ |\det(g_{ab})|g^{rr} \ .$$

The conformal equation and estimates

- We can now prove estimates in terms of a rescaled version of $\Box^0$:
  $\Box^0_{RW} = -\partial_s^2 + \partial_{r^*}^2 + V(r_*)\not\Delta = -\langle\partial_t, \partial_t\rangle_g \Box^0$,
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- The normal hyperbolicity assumption is simply that V have a single non-degenerate critical point. Thus, one can prove (rough) estimates in the standard way.
- For applications we need sharper bounds. This is contained in work of Marzoula-Metcalfe-Tataru-Tohaneanu: Let ϕ be a space-time function which is supported in the region $|r_*| \leqslant C$ for some fixed $C > 0$. Then one has the bound:
  $\| \partial_{r_*}\phi \|_{L^2[0,T]} + \| (w_{ln})^{-1}(\partial_s\phi, \not{d}\phi) \|_{L^2[0,T]} \lesssim$
  $E^{\frac{1}{2}}(\phi[0]) + \| w_{ln}\Box^0_{RW}\phi \|_{L^2[0,T]}$.

"Elliptic Renormalization" of residual charges

- We still need to go back and deal with the equation:
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- However, there is a simple "gauge structure" to this equation which is:
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- We can now try to chose χ to as to minimize $G - d\chi$. This turns out to be an elliptic problem reminiscent of the Coulomb gauge.
- The estimates are provided via rescaling and standard elliptic bounds. The main thing here is that we need $w(r)$ to be an A_2 weight. For this we would just need $w \approx |r - r_{\mathcal{T}}|^{-\frac{1}{2}+\epsilon}$ so there is plenty of room.