Sharp decay estimates for the Klein Gordon equation on Kerr-AdS

Jacques Smulevici (Université Paris-Sud Orsay)

Joint work with

Gustav Holzegel (Imperial College).

Paris, May 23, 2013.

Introduction

- The most simple (symetric) solutions of the vacuum solutions of the Einstein equation

$$Ric(g) = \Lambda g$$

are Minkowski space, de Sitter space ($\Lambda > 0$) and Anti-de-Sitter space ($\Lambda < 0$, AdS).

- There has been a large amount of work trying to understand the linear and non-linear stability of asymptotically flat/de Sitter spacetimes.

- In the physics litterature, there is a huge research activity based on the study of properties of asymptotically AdS spacetimes.

- Few math results on this subject. In fact, even the stability (or rather non-linear instability) of AdS is not known !

Roughly, our results can be summarized as follows:

- We consider a linear equation $(\Box_g - \alpha)(\psi) = 0$ where $\Box_g - \alpha$ is a Klein Gordon operator associated to a the so-called Kerr-AdS spacetime.

- We prove that solutions ψ of $(\Box_g - \alpha)(\psi) = 0$ satisfies a decay estimate
$$E_{1,loc}[\psi](t) \lesssim \frac{1}{\log(2+t)} E_2[\psi](t=0).$$
where

- $E_{1,loc}(t) =$ "local energy" at time t.

- E_2 second order energy, controls ψ, $\partial\psi$, $\partial^2\psi$ in L^2

Moreover, we prove that the estimate is **sharp**.
The slow decay rate is a consequence of a *stable trapping* phenomenon.

Outline

1. AdS and wave equations in AdS
2. The geometry of Schwarzschild-AdS and Kerr-AdS
3. The sharp log decay result
4. Key points for the proof of decay
5. Key points for the proof of sharpness: quasimodes on Kerr-AdS
6. Epilogue: A non-linear model: Asymptotic stability of Scwharzschild-AdS for the spherically symmetric Einstein-Klein-Gordon system.

Anti-de-Sitter

Fix $\Lambda < 0$. Consider the manifold $\mathbb{R}^4$ with Lorentzian metric

$$g_{AdS} = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\sigma_{S^2},$$

where $d\sigma_{S^2}$ is standard metric on S^2 and $l^2 = -\frac{3}{\Lambda}$.

$$\begin{aligned}\Box_{g_{AdS}}\psi &= \frac{1}{\sqrt{|g_{AdS}|}}\partial_\alpha\left(g^{\alpha\beta}_{AdS}\sqrt{|g_{AdS}|}\partial_\beta\psi\right)\\ &= -\left(1 + \frac{r^2}{l^2}\right)^{-1}\psi_{tt} + \frac{1}{r^2}\partial_r\left(\left(1 + \frac{r^2}{l^2}\right)\psi_r\right) + \frac{1}{r^2}\Delta_{S^2}\psi.\end{aligned}$$

Geometry of AdS

- AdS is static and spherically symmetric,
- but AdS is not globally hyperbolic,
- Standard theory for wave equation on an arbitrary Lorentzian manifold need global hyperbolicity.
- $\rightarrow$ Use appropriate function spaces.

Energy spaces for Klein-Gordon equation on Anti-de-Sitter

Consider the r-weighted energy norms

$$\begin{aligned}
||\psi||_{H^{0,-2}_{AdS}} &= \int_{\mathbb{R}^3} r^{-2}\psi^2 r^2 dr d\sigma_{S^2}, \\
||\psi||_{H^1_{AdS}} &= \int_{\mathbb{R}^3} \left(r^2|\psi_r|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2\right) r^2 dr d\sigma_{S^2}. \\
\|\psi\|^2_{H^2_{AdS}} &= \|\psi\|^2_{H^1_{AdS}} \\
&\quad + \int_{\mathbb{R}^3} \left[r^4 \left(\partial_r\partial_r\psi\right)^2 + r^2|\nabla\!\!\!/\partial_r\psi|^2 + |\nabla\!\!\!/\nabla\!\!\!/\psi|^2\right] r^2 dr \sin\theta d\theta d\phi
\end{aligned}$$

and define the energy norms

$$\begin{aligned}
E_1[\psi] &= ||\partial_t\psi||_{H^{0,-2}_{AdS}} + ||\psi||_{H^1_{AdS}} \\
E_2[\psi] &= ||\partial_{tt}\psi||_{H^{0,-2}_{AdS}} + ||\partial_t\psi||_{H^1_{AdS}} + ||\psi||_{H^2_{AdS}} + \sum_{i=1,2,3} ||\Omega^i\psi||_{H^1_{AdS}}
\end{aligned}$$

For $\psi \in H^k_{AdS}$ imposes decay at infinity.

Dynamics in H^k_{AdS} can be nicely understood in a compactification of the problem.

Ex: take ψ spherically symmetric solution of $(\Box_g - \alpha)\psi = 0$, let $r^\star = arctan \frac{r}{l}$ and $u = r\psi$ then u solves

$$u_{tt} - u_{r^\star r^\star} + V(r^\star)u = 0$$

in a strip $0 \leq r^\star \leq \pi/2$ with Dirichlet data at both boundaries.

- g_{AdS} invariant by vector field $T = \partial_t$ in AdS so get conservation of the following energy

$$\int_{t=const} \Big[(1+r^2)^{-1}\psi_t^2 + (1+r^2)\psi_r^2 + |\nabla\!\!\!/\psi|^2 + \alpha\psi^2 \Big] r^2 dr d\omega.$$

- Note that the conformal wave operator is $\Box_g - \frac{1}{6}R$ which in AdS corresponds to $\alpha = -\frac{2}{l^2}$, i.e. a negative term in the above energy.

- Use Hardy type inequalities to control the α-term

$$\int_{\Sigma_t} \psi^2 r^2 dr d\omega \leq C_H \int_{\Sigma_t} r^4 \psi_r^2 dr d\omega$$

- For any asymptotically AdS spacetime, the equation $\Box_g\psi = \alpha\psi$ is well-posed in the H^k_{AdS} spaces provided that $\alpha > -\frac{9}{4l^2}$. (Breitenlohner-Freedmann, Ishibashi-Wald, Bachelot, Holzegel, Vasy, Warnick).

Wave confinement in AdS

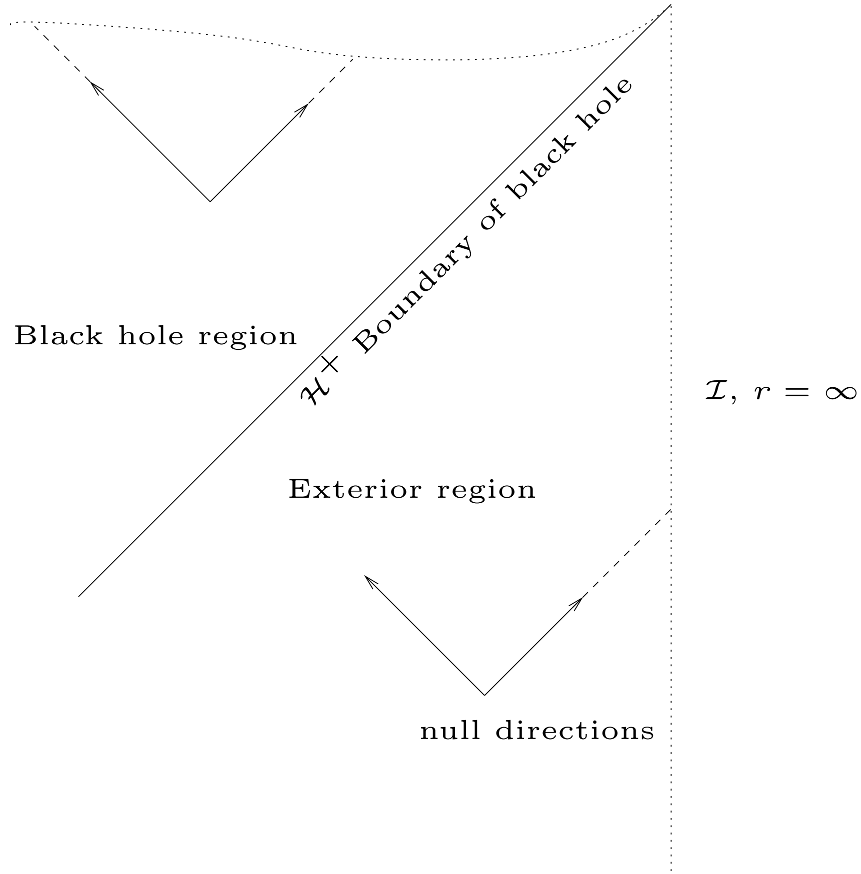
- In AdS, there are periodic finite energy solutions to the wave equation (spectrum of the associated elliptic operator is discrete). So no decay !

- No decay together with the strong nonlinearities in the Einstein equations leads to

  Conjecture 1 (2006: Dafermos-Holzegel, Anderson)**.** *AdS is dynamically unstable.*

  Remark 1: Numerics and heuristics of Bizoń-Rostworowski, see also Dias-Horowitz-Santos.
  Remark 2: Dynamics in AdS may be dependent upon choice of boundary conditions.

Scalar waves in asymptotically AdS black holes



The Schwarzschild-AdS metrics

Let $M, l > 0$ and consider the metric

$$ds^2 = -(1-\mu)dt^2 + (1-\mu)^{-1}dr^2 + r^2 d\sigma^2_{S^2}$$

where $(1-\mu) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$,

- $M = 0, l = \infty$ corresponds to the usual ("flat space") wave equation.
- $M > 0, l = \infty$ corresponds to the Schwarzschild metric,
- $1 - \mu$ has one real root denoted $r_+ > 0$, which depends on M and l.
- The black hole exterior+horizon is $\mathcal{R} = \mathbb{R}_t \times [r_+, \infty) \times S^2$.

The Kerr-AdS black holes

- Let $M > 0, l > 0$ and let a be a real number such that $|a| < l$.
- Schematically, the Kerr-AdS metric takes the form

$$g = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi,$$

  where all coefficients depend on r and θ only and g_{rr} is singular at some $r_+ > 0$.
- As before, $\mathcal{R} = \mathbb{R}_t \times [r_+, \infty) \times S^2$.
- Schwarzschild-AdS correspond to $a = 0$ Kerr-AdS spacetimes.

More precisely,

$$g_{KAdS} = \frac{\Sigma}{\Delta_-} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \left(r^2 + a^2\right)^2 - \Delta_- a^2 \sin^2\theta}{\Xi^2 \Sigma} \sin^2\theta d\phi^2$$
$$-2 \frac{\Delta_\theta \left(r^2 + a^2\right) - \Delta_-}{\Xi \Sigma} a \sin^2\theta \; d\phi dt - \frac{\Delta_- - \Delta_\theta a^2 \sin^2\theta}{\Sigma} dt^2$$

with

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Delta_\pm = \left(r^2 + a^2\right)\left(1 + \frac{r^2}{l^2}\right) \pm 2Mr$$
$$\Delta_\theta = 1 - \frac{a^2}{l^2} \cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}\,.$$

Moreover, r_+ is the largest real root of $\Delta_-(r)$.

Problem: Prove quantitative decay for solutions of $\Box_g \psi = \alpha \psi$ with $(\psi, \psi_t) \in H^k_{AdS} \times H^{k-1}_{AdS}$.

On Schwarzschild/Kerr, huge literature (Wald, Kay-Wald, Whiting, Tataru-Tohaneanu, Tohaneanu, Dafermos-Rodnianski, Blue-Sterbenz, Blue-Soffer, Blue, Luk, Aretakis, Andersson-Blue, Donninger-Schlag-Soffer, Schlue...).

Idem on Schwarzschild-de-Sitter, Kerr-de-Sitter (Dafermos-Rodnianski, Bony-Häfner, Melrose-Sa Barreto-Vasy, Vasy, Dyatlov, ...)

For Schwarzschild-AdS or Kerr-AdS, uniform boundedness results (Holzegel 2009, Holzegel-Warnick 2012) if $|a|$ not too large compared to r_+.

Log decay of Klein-Gordon waves in Kerr-AdS

We prove

Theorem 1 (Holzegel-J.S., 2011-2013)**.** *Let* ψ *be a solution in* H^2_{AdS} *of* $\Box_g\psi + \alpha\psi$ *in* $(\mathcal{R}, g)$*,* g *metric of a Kerr-AdS spacetime such that* $|a|l < r_+^2$*,* $\alpha > -\frac{9}{4l^2}$*. Let* $R > r_+$*. Then, for all* $t \geq 0$*,*

$$E_{1,loc}[\psi](t) := \left(||\psi||_{H^1_{AdS,\{r\geq R\}}} + ||\psi_t||_{H^{0,-2}_{AdS,\{r\geq R\}}} \right)(t) \leq \frac{C}{\log(2+t)} E_2(\psi)(t=0),$$

where $C > 0$ *is some universal constant. Moreover, the estimate is* ***sharp****.*

Remark 1: If $|a|l > r_+^2$, then it is conjectured that not even boundedness of solutions hold! (cf Shlapentokh-Rothman, Cardoso-Dias).

Remark 2: Initially range of parameters smaller (cf recent work of Holzegel-Warnick).

Remark 3: Lower bounds actually holds without restrictions on a.

Sharpness

Let SCH^2_{AdS} be the set of solutions with finite second energy $E_2(\psi)$. Let $t_0^\star \geq 0$ be fixed and define for any non-zero ψ and $t^\star \geq 0$

$$Q\left[\psi\right](t^\star) := \log(2+t^\star)\left[\frac{E_{1,loc}(\psi)(t)}{E_2(\psi)(t_0^\star)}\right].$$

Then there exists a universal constant $C > 0$ such that

$$\limsup_{t^\star \to +\infty} \sup_{\psi \in SCH^2_{AdS}, \psi \neq 0} Q\left[\psi\right](t^\star) > C > 0.$$

Equivalently, sharpness means that the statement

There exists a function $t \to \delta(t)$ *such that* $\delta(t) \to 0$ *as* $t \to +\infty$ *and such that for all solutions* ψ*, we have the estimate*

$$\left(||\psi||_{H^1_{AdS,\{r\geq R\}}} + ||\psi_t||_{H^{0,-2}_{AdS,\{r\geq R\}}} \right)(t) \leq \frac{\delta(t)}{\log(2+t)} E_2(\psi)(t=0),$$

is false.

Elements of proof of decay

Typical elements in analysis of wave equations on black hole spacetimes

- Red-shift
- Superradiance
- Trapping

Red-shift and superradiance

- As usual, Red-shift and superradiance are linked with the failure of $T = \partial_t$ to remain timelike near (or on) the horizon.
- An analogue of the Dafermos-Rodnianski red-shift holds for Kerr-AdS spacetimes (Holzegel 2009).
- Superradiance: ∂_t becomes spacelike near the horizon.
- Natural conserved energy associated to the invariance of g by ∂_t is a priori not coercive.
- However, g also invariant by ∂_ϕ and there is a special combination of the type $K = \partial_t + C(a, M, l)\partial_\phi$ such the conserved energy associated to K is coercive in $r > r_+$ (and degenerate near r_+), provided that $|a|l < r_+^2$. $\left(C = (1 - \frac{a^2}{l^2})\frac{a}{a^2+r_+^2}\right)$.
- The vector field K is called the Hawking-Reall vectorfield.
- We will see another occurence of this in frequency space.

The trapping: the geodesic flow on Kerr-AdS

- is integrable (cf Carter constant).
- If $a = 0$, there exists null geodesics orbiting around $r = 3M$.
- For $a \neq 0$, there still exists periodic null geodesics in a neighbourdhood (of size a) of $r = 3M$.
- But, viewed in $T\mathcal{M}^*$, this behaviour is unstable (normal hyperbolic trapping).
- In asymptotically flat Kerr, this is all the trapping, but in the asymptotically AdS, there is also a trapping at infinity !

Elements of the proof for decay

- Give yourself a frequency cutt-off. Decompose ψ into a high-low frequency $\psi = \psi_{\leq L} + \psi_{>L}$.
- Note that this will be a spacetime frequency decomposition.
- Prove a multiplier estimate on $\psi_{\leq L}$ of the form

$$\int_t ||\psi_{\leq L}||^2_{H^1_{AdS,r\geq R}} \leq e^{CL} E_1(\psi)$$

- For $\psi_{\geq L}$, we would like a Poincaré type inequality

$$||\psi_{>L}||^2_{H^1_{AdS}} \leq \frac{1}{L} E_2(\psi).$$

  However, because of spacetime frequency decomposition, we can only get a spacetime type of Poincaré inequality of the type

$$\int_0^\tau ||\psi_{>L}(t')||^2_{H^1_{AdS}} dt' \leq \frac{1}{L} E_2(\psi)\tau.$$

- Then interpolate.

1-d reduction: The Schwarzschild-AdS

We decompose our solution ψ in spherical harmonics $\psi = \sum_{km} \psi_{km}(t,r) Y_{km}(\theta,\phi)$ and take Fourier in time

$$\widehat{\psi}_{km}(\omega) = \mathcal{F}_t(\psi_{km})$$

to obtain second order ode

$$\omega^2 \widehat{\psi}_{km} = P(r, \partial r)\widehat{\psi}_{km} + (k(k+1)V(r) + R(r))\,\widehat{\psi}_{km}$$

Can be reduced further to standard form by change of radial coordinate ($dr^\star = \frac{dr}{1-\mu}$.) and renormalization $u(r^\star) = u_{km}(r^\star) = r\widehat{\psi}_{km}$. Then,

$$\omega^2 u = -\frac{d^2 u}{(dr^\star)^2} + (k(k+1)V(r^\star) + R(r^\star))\,u.$$

Moreover, in $r^\star$ coordinate, the horizon $r = r_+$ corresponds to $r^\star \to -\infty$ and $r = \infty$ to a finite value of $r^\star$, say $r^\star(r = \infty) = \pi/2$.

Finally, the decay condition on ψ (arising from $E_1(\psi) < \infty$) is translated into Dirichlet boundary conditions for u at $r^\star = \pi/2$.

1-d reduction: Kerr-AdS case

The same procedure for Kerr, using Carter separation of variables leads to an equation of type

$$\omega^2 u = -\frac{d^2 u}{(dr^\star)^2} + \left(\lambda_{km}(a,\omega)V(r^\star) + m^2 W(r^\star) + \omega m U(r^\star) + R(r^\star)\right) u.$$

Here m is the angular frequency associated to ∂_ϕ and $\lambda_{km}(a\omega)$ are angular frequencies corresponging to the eigenvalues of (modified)-oblate-spheroidal operator.

(modified)-oblate-spheroidal-harmonics

The $Q(\omega)_{S^2}$ operator is defined by

$$\begin{aligned} -Q(\omega) f &= \frac{1}{\sin\theta}\partial_\theta\left(\Delta_\theta \sin\theta \partial_\theta f\right) + \frac{\Xi^2}{\Delta_\theta}\frac{1}{\sin^2\theta}\partial^2_{\tilde{\phi}} f \\ &\quad + \Xi\frac{a^2\omega^2}{\Delta_\theta}\cos^2\theta f - 2ia\omega\frac{\Xi}{\Delta_\theta}\frac{a^2}{l^2}\cos^2\theta\ \partial_{\tilde{\phi}} f\ , \end{aligned}$$

where $\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta$ and $\Xi = 1 - \frac{a^2}{l^2}$.

Eigenvalues of $Q(\omega)^2_S$ denoted by $\lambda_{km}(a,\omega)$.

Eigenfunctions $S_{km}(a,\omega)$.

Lemma 1 (estimates for the λ_{km})**.** $\lambda_{km} + a^2\omega^2 \geq |m|(m+1)$.

Superradiance ?

- Recall the Hawking-Reall vectorfield $K = \partial_t + C\partial_\phi$.

In frequency space: need to combine the frequency associated to t and the frequency associated to ϕ, i.e. use Helmholtz equation in the form

$$(\omega - Cm)^2 u = -u'' + V(\omega, m, k, r, \theta)u,$$

to derive mulitplier estimates.

The frequency sets

Let $L > 0$ be a large number. We first do a high-low frequency decomposition:

1. The high frequency set is $\{|\omega - Cm|^2 + \lambda_{km}(\omega) > L\}$

2. The low frequency set is $\{|\omega - Cm|^2 + \lambda_{km}(\omega) \leq L\}$

The low frequency set must also be decomposed to single out the almost stationary frequency set

$$\{|\omega - Cm|^2 \leq L^{1/2}\}$$

We then construct multipliers for all low frequencies.

Quasimodes

To probe the decay of solutions, there is a well known technique in semi-classical analysis, which is the construction of the so-called *quasimodes*.

- A quasimode is an *approximate* solution ψ_ℓ

$$(\Box_g - \alpha)\,\psi_\ell = F_\ell.$$

- A quasimode is periodic in time (like a mode solution) $\psi_\ell = e^{i\omega_\ell t}\varphi_\ell(r,\theta,\phi)$.
- A quasimode is (typically) localized in space.
- Finally, the error F_ℓ goes to zero as ℓ (the frequency scale) goes to infinity.

Quasimodes and sharpness of the main estimate

- Recall Duhamel Formula for inhomogeneous solutions $(\Box_g - \alpha)\psi_{inh} = F$,

$$\psi_{inh}(t) = \psi_{homogeneous}(t) + \int_{t_0}^{t} P(t,s)F(s)ds$$

- $\rightarrow$ the existence of quasimodes translates into lower bounds for the decay estimate.
- If rate of decay of F_ℓ is polynomial in $1/\ell$, then we get that best decay rates for estimates losing 1 derivative cannot be better than a certain polynomial in $1/t$.
- If rate of decay of F_ℓ is of type $e^{-C\ell}$, then we get best decay rate of $(\log t)^{-1}$.

Quasimodes and resonances

- Resonances/quasinormal-modes: complex frequencies of open systems (poles of the meromorphic continuation of truncated resolvent).
- Quasimodes are also strongly related to resonances. Many results in math literature (cf Tang-Zworski, C. de Verdire,..) of type: existence of quasimodes implies existence of resonances with similar frequencies.
- This has been done for Schwarzschild-AdS (Gannot 2012, see also recent work of Warnick). For Schwarzschild-dS (Bony-Häfner) and Kerr-dS (Dyatlov), $|a| << M$, very good description of resonances.
- Cf Numerical work on quasinormal modes for AdS black holes (Festuccia-Liu..)
- Cf recent work of Sbierski for arbitrary Lorentzian manifold (in particular, not stationary): geometric construction of high oscillatory, localized waves near null geodesics.

Existence of quasimodes: the Schwarzschild-AdS case

Recall that after separation of variable, we get equation of type

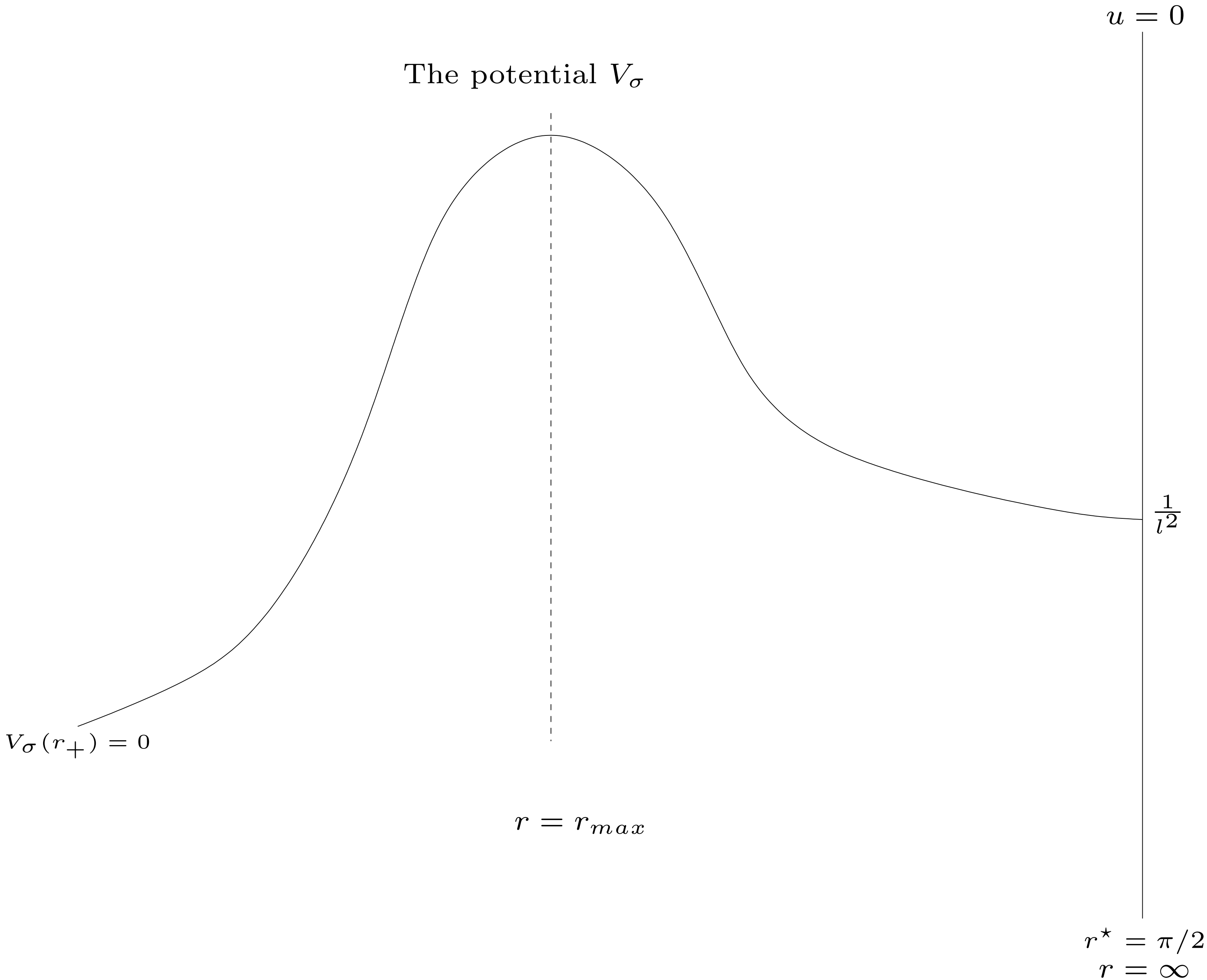
$$\omega^2 u_\ell = -u'' + \left(V_\sigma(\ell(\ell+1)) + V_{junk}\right) u_\ell.$$

Think of $\frac{1}{(\ell(\ell+1))}$ as h^2 a semi-classical parameter. Neglecting lower order terms, we get

$$-u_\ell'' \frac{1}{\ell\left(\ell+1\right)} + V_\sigma u_\ell = \frac{\omega^2}{\ell\left(\ell+1\right)} u_\ell \tag{1}$$

for a potential $V_\sigma\left(r\right)$.

Think of $\kappa_\ell := \frac{\omega^2}{\ell(\ell+1)}$ as a energy level.



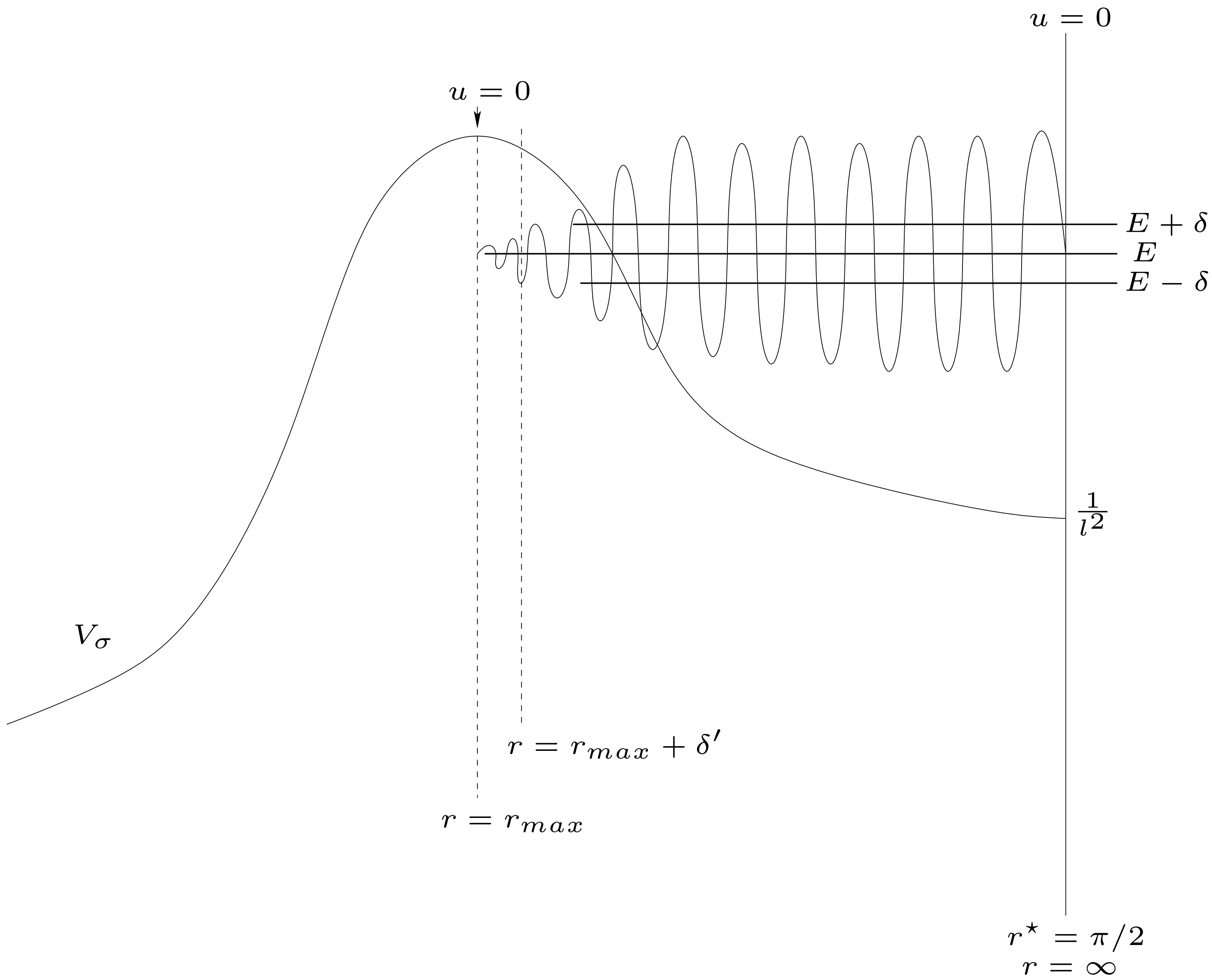
$u = 0$
The potential V_σ
$\frac{1}{l^2}$
$V_\sigma(r_+) = 0$
$r = r_{max}$
$r^\star = \pi/2$
$r = \infty$

- To construct quasimodes, we first construct a sequence of solutions $(u_\ell)_{\ell \in \mathbb{N}}$ to an eigenvalue problem with Dirichlet boundary conditions at $r = 3M$.

- The u_ℓ are solutions to

$$-u_\ell'' \frac{1}{\ell\,(\ell+1)} + V_\sigma u_\ell = \kappa_\ell u_\ell$$

with the κ_ℓ converging to any fixed $E \leq V_{\max}$ as $\ell \to +\infty$.



$u = 0$
$u = 0$
$E + \delta$
E
$E - \delta$
$\frac{1}{l^2}$
V_σ
$r = r_{max} + \delta'$
$r = r_{max}$
$r^\star = \pi/2$
$r = \infty$

The Agmon estimates

In a the region where $V_\sigma \geq \kappa_\ell$, we show that the solutions becomes exponentially small as $\ell \to +\infty$. These are the so-called *Agmon estimates* which in quantum mechanics are used to quantify how small is the tunnel-effect.

Lemma 2. *Let (u, κ) be a solution to the eigenvalue problem. Define for any $\epsilon \in (0,1)$*

$$\phi := (1-\epsilon) d_\kappa.$$

where d_κ is the distance to the classical region. Then, for all ϵ sufficiently small, u satisfies

$$\epsilon^2 \int_{V_\sigma > \kappa} e^{2\frac{\phi}{h}} |u|^2 dr^\star \leq D e^{2a(\epsilon)/h} ||u||^2_{L^2(r^\star_{\max}, \pi/2)},$$

where $D > 0$ is a constant and $a(\epsilon)$ goes to zero as $\epsilon \to 0$, uniformly in $h^{-2} = \ell(\ell+1)$.

End of construction of quasimodes

- We then defined our quasimodes as follows. For each ℓ, define

$$\omega_\ell^2 = \kappa_\ell.\ell(\ell+1)$$

and

$$\psi_\ell = e^{i\omega_\ell t}\chi(r) r u_\ell S_{\ell\,0}(\theta,\phi),$$

where $S_{\ell\,0}(\theta,\phi)$ is a spherical harmonic with angular momentum number ℓ and $\chi(r)$ is cuttoff function with is 1 for $r \geq 3M+\delta$ and 0 for $r \leq 3M$, for some small enough $\delta > 0$.

- Then ψ_ℓ is a solution to the Klein-Gordon equation on Schwarzschild-AdS apart in a small strip of size δ, where the cuttoff function is not constant.

- In this strip, it satisfies

$$\left(\Box_g - \alpha\right)\psi_\ell = F_\ell,$$

with the error being exponentially small in ℓ as $\ell \to +\infty$.

In Kerr-AdS, we want to apply the same technique but the eigenvalue equation becomes non-linear.

Consider Helmholtz equation within axisymmetry. It takes the form

$$-u_\ell'' \frac{1}{\mu_\ell\left(a^2\omega^2\right)} + V_\sigma u_\ell = \frac{\omega^2}{\mu_\ell\left(a^2\omega^2\right)} u_\ell,$$

where $\mu_\ell(a^2\omega^2) := \lambda_{\ell 0}(a^2\omega^2) + a^2\omega^2$.

The operator now depends on ω^2 but ω^2 is constructed from the eigenvalue!

- Solution: for each ℓ, we consider the eigenvalue ω_ℓ as function of a and use the implicit function theorem (IFT) to transport information of the linear eigenvalue problem (when $a = 0$) to the non-linear eigenvalue problem.
- This is jointly done with uniform estimates in a to control the size of the domain in which IFT applies.
- More precisely, for each ℓ, consider the operator

$$Q_\ell\left(a^2, \omega^2\right) u := -u'' + \left(V_\sigma\, \mu_\ell\left(a^2 \omega^2\right) - \omega^2\right) u\,,$$

  When $a = 0$ and $\omega = \omega_\ell$, the previous construction gives us that this operator has a 0 eigenvalue.
- Say this corresponds to the nth eigenvalue of Q_ℓ.
- Define then the $\Lambda_n(a^2, \omega^2)$ to be the nth eigenvalue of $Q_\ell\left(a^2, \omega^2\right)$.
- The IFT is then applied to Λ_n.
- We prove uniform estimates on Λ_n such as uniform bounds away from zero for $\frac{\partial \Lambda_n}{\partial \omega^2}$.

An interesting aspect is that we actually do consider a modified linear problem:

Recall the operators

$$Q_\ell\left(a^2,\omega^2\right)u := -u'' + \left(V_\sigma\,\mu_\ell\left(a^2\omega^2\right) - \omega^2\right)u\,,$$

such that for $a = 0$ it reduces to

$$Q_\ell\left(0,\omega^2\right)u := -u'' + \left(V_\sigma\left(\ell(\ell+1)\right)^{-1} - \omega^2\right)u\,.$$

If V_σ was the Schwarzschild-AdS problem, this analysis of this operator would be the one previously mentionned.

However, we take for V_σ the potential of a Kerr-AdS spacetime of given angular momentum b. We then try to control the $Q_\ell\left(a^2,\omega^2\right)$ for all a from 0 to b.

The introduction of these artificial problems allow us to gain a monotonicity namely, one can show that the energy levels $\frac{\omega_\ell(a)}{\mu_\ell(a^2\omega^2)}$ are decreasing with a^2 !

Since for $a = 0$, we ensure that the energy levels are uniformly below the top of the potential V_σ (uniformly in ℓ), we finally construct solutions to our Dirichlet problem suitable for the Agmon estimates.

Theorem 2 (Holzegel-J.S 2013, Quasimodes for Kerr-AdS). *Let* $(g, \mathcal{R})$ *denote the black hole exterior of a Kerr-AdS spacetime, with mass* $M > 0$*, angular momentum per unit mass* a *and cosmological constant* $\Lambda = -\frac{3}{l^2}$*. Assume that the parameters satisfy* $\alpha < \frac{9}{4}$, $|a| < l$*. Then, for* $\delta > 0$ *sufficiently small, there exists a family of non-zero functions* $\psi_\ell \in H^k_{AdS}$ *for any* $k \geq 0$ *such that*

1. $\psi_\ell\,(t, r, \theta, \varphi) = e^{i\omega_\ell t}\varphi_\ell(r, \theta)$ *(axisymmetric and time-periodic),*
2. $0 < c < \frac{\omega_\ell^2}{\ell(\ell+1)} < C$*, for constants* c *and* C *independent of* ℓ *(uniform bounds on the frequencies),*
3. *for all* $t^\star \geq t^\star_0$*, for all* $k \geq 0$,
   $||\,(\Box_g - \alpha)\,\psi_\ell||_{H^k_{Ads}(\Sigma_{t^\star})} \leq C_k e^{-C_k \ell}||\psi_\ell||_{H^0_{AdS}(\Sigma_{t^\star_0})}$*, for some* $C_k > 0$ *independent of* ℓ *(approximate solutions to the wave equation),*
4. *the support of* $F_\ell := (\Box_g - \alpha)\,\psi_\ell$ *is contained in* $\{r_{max} \leq r \leq r_{max} + \delta\}$ *(spatial localization of the error),*
5. *the support of* $\varphi_\ell(r, \theta)$ *is contained in* $\{r \geq r_{max}\}$ *(spatial localization of the solution).*

A non-linear model problem: spherically symmetric Einstein-Klein-Gordon-system

The Einstein-Klein-Gordon system:

$$\begin{aligned} Ric(g) - \frac{1}{2}Rg + \Lambda g &= 8\pi T[\psi], \\ \Box_g \psi &= \alpha\psi, \end{aligned} \tag{2}$$

where $T[\psi]$ is

$$T[\psi] = d\psi \otimes d\psi - \frac{1}{2} g \left(g(\nabla\psi, \nabla\psi) + \alpha\psi^2 \right).$$

Local existence in H^2_{AdS} (for ψ) and some continuation criterion of solutions are known for this system (Holzegel-J.S. 2011).

Remark 1: spherically symmetric solutions to the $Ric(g) = \Lambda g$ are either AdS or Schwarzschild-AdS, i.e. no spherically-symmetric dynamics in the vacuum, hence the coupled system.

Stability of Schwarzschild-AdS for the spherically-symmetric Einstein-Klein-Gordon system

Theorem 3 (Holzegel, J.S. 2011)**.** *Asymptotic and orbital stability of Schwarzschild-AdS hold.*

Our analysis contains:

- Integrated decay types estimate controlling $\int_t ||\psi||_{H^1_{AdS,\{r\geq R\}}}$.
- Pointwise decay estimate for ψ.
- Bootstrap argument to propagate "good" geometrical properties of Schwarzschild-AdS.