# *Some convergence results for numerical schemes approximating two phase flow in porous media*

R. Eymard, Université Paris-Est

25-27 février 2014, UPMC

joint work with E. Tillier, Laboratoire de Mathématiques de l'INSA de Rouen

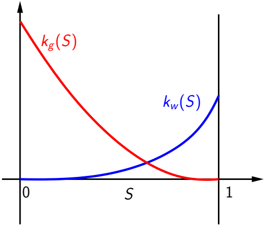
$$\partial_t [S(1-X)] - \partial_x [(1-X)k_w(S)(P_x - \rho_w)] = 0$$

$$\partial_t [SX + \xi(1-S)] \\ -\partial_x [Xk_w(S)(P_x - \rho_w) + \xi k_g(S)(P_x - \rho_g)] = 0$$

with

$$(X \le \overline{X} \text{ and } S = 1) \text{ or } (X = \overline{X} \text{ and } S \le 1)$$

and $\overline{X} < \xi$

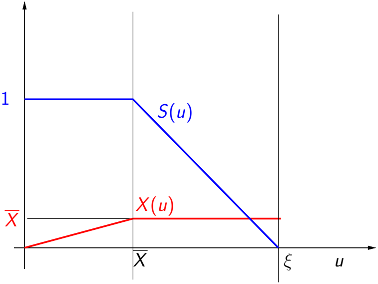



one denotes $u = SX + \xi(1-S)$

which implies

$$X(u) = u \text{ and } S(u) = 1,\ \forall u \in [0, \overline{X}]$$
$$X(u) = \overline{X} \text{ and } S(u) = \frac{\xi - u}{\xi - \overline{X}},\ \forall u \in [\overline{X}, \xi]$$

with $S(u)$ (non strictly) decreasing and $X(u)$ (non strictly) increasing



# Resulting formulation

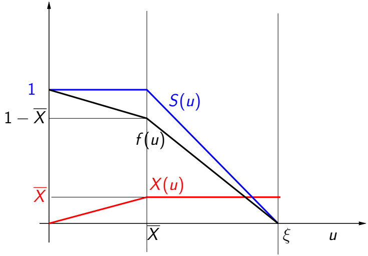
$$\begin{array}{lll} \partial_t f(u) & -\partial_x \left[a(u)v + b(u)\right] & = 0 \\ \partial_t u & -\partial_x \left[c(u)v + d(u)\right] & = 0 \end{array}$$

with

$$\begin{aligned} f(u) &= S(u)(1 - X(u)) \\ a(u) &= (1 - X(u))k_w(S(u)) \\ b(u) &= -(1 - X(u))k_w(S(u))\rho_w \\ c(u) &= X(u)k_w(S(u)) + \xi k_g(S(u)) \\ d(u) &= -X(u)k_w(S(u))\rho_w - \xi k_g(S(u))\rho_g \end{aligned}$$

with $f$ strictly decreasing and

$a + f'c$ strictly positive



## *Particular case :* $\xi = 1$ *or* $\overline{X} = 0$ *then* $f(u) = 1 - u/\xi$

$$\begin{array}{ccc} -\partial_t u & -\partial_x \left[\xi a(u) v + \xi b(u)\right] & = 0 \\ \partial_t u & -\partial_x \left[c(u) v + d(u)\right] & = 0 \end{array}$$

implies : $-\partial_x(\xi a(u)v + \xi b(u) + c(u)v + d(u)) = 0$

$$-(\xi a(u) + c(u))v - \xi b(u) - d(u) = Q$$

leads to

$$\partial_t u + \partial_x \left[ \frac{c(u)Q + \xi b(u)c(u) - \xi a(u)d(u)}{\xi a(u) + c(u)} \right] = 0$$

classical nonlinear scalar hyperbolic problem, with unique solution in the entropy weak sense

numerical applications : $\xi = 0.07$ and $\overline{X} = 0.06....$

## *Can we express* v *w.r.t.* u *in the general case ?*

for regular solution $u$, one gets

$$\begin{array}{lll} f'(u)u_t & -[a(u)v+b(u)]_x & =0 \\ u_t & -[c(u)v+d(u)]_x & =0 \end{array}$$

hence $f'(u)\,[c(u)v+d(u)]_x - [a(u)v+b(u)]_x = 0$

gives $v_x + \beta(u)_x v + \gamma(u)_x = 0$ implies $v = \Phi(u)$

then two scalar nonlinear hyperbolic equations

$u_t - [c(u)\Phi(u)+d(u)]_x = 0$ and

$$w_t - \left[a(f^{-1}(w))\Phi(f^{-1}(w)) + b(f^{-1}(w))\right]_x$$

leads to two entropy solutions such that discontinuous solutions can exist, and in the general case do not satisfy $w = f(u)$ ...

cannot decouple system in the general case

Rankine-Hugoniot relation

from $u_l, v_l$ to $u_r$ in the system
$$\begin{array}{lll} f(u)_t & -[a(u)v + b(u)]_x & = 0 \\ u_t & -[c(u)v + d(u)]_x & = 0 \end{array}$$

then

$V, v_r$ given by
$$\begin{array}{ll} V(f(u_l) - f(u_r)) & = -\big(a(u_l)v_l + b(u_l) - a(u_r)v_r - b(u_r)\big) \\ V(u_l - u_r) & = -\big(c(u_l)v_l + d(u_l) - c(u_r)v_r - d(u_r)\big) \end{array}$$

intermediate shocks for $\kappa$ between $u_l$ and $u_r$ : $V_l(\kappa), v_\kappa$ given by

$$\begin{array}{ll} V_l(\kappa)\ (f(u_l) - f(\kappa)) & = -\big(a(u_l)v_l + b(u_l) - a(\kappa)v_\kappa - b(\kappa)\big) \\ V_l(\kappa)\ (u_l - \kappa) & = -\big(c(u_l)v_l + d(u_l) - c(\kappa)v_\kappa - d(\kappa)\big) \end{array}$$

and $V_r(\kappa), v_r$ given by

$$\begin{array}{ll} V_r(\kappa)\ (f(\kappa) - f(u_r)) & = -\big(a(\kappa)v_\kappa + b(\kappa) - a(u_r)v_r - b(u_r)\big) \\ V_r(\kappa)\ (\kappa - u_r) & = -\big(c(\kappa)v_\kappa + d(\kappa) - c(u_r)v_r - d(u_r)\big) \end{array}$$

Liu criterion : for all $\kappa$ between $u_l$ and $u_r$, then

$\mathrm{sign}(u_r - u_l)(V_l(\kappa) - V) \geq 0$ and $\mathrm{sign}(u_r - u_l)(V_r(\kappa) - V) \leq 0$

# *Consequence : entropy weak solution with Krushkov pairs*

weak sense of

$$\begin{aligned} f(u)_t \quad & - [a(u)v + b(u)]_x & = 0 \\ u_t \quad & - [c(u)v + d(u)]_x & = 0 \end{aligned}$$

and weak sense of Liu criterion

for all $\kappa \in \mathbb{R}$

$$\eta_\kappa^\pm(u) = \mathrm{sign}^\pm(u-\kappa)\big(a(\kappa)(u-\kappa) - c(\kappa)(f(u)-f(\kappa))\big)$$

$$\begin{aligned} \Phi_\kappa^\pm(u) = \mathrm{sign}^\pm(u-\kappa)\big( \quad & -a(\kappa)((c(u)-c(\kappa))v + d(u) - d(\kappa)) \\ & +c(\kappa)((a(u)-a(\kappa))v + b(u) - b(\kappa))\big) \end{aligned}$$

$$\partial_t \eta_\kappa^\pm(u) + \partial_x \Phi_\kappa^\pm(u) \le 0$$

Under Hypotheses

$$\begin{aligned} T > 0,\ \exists M \in \mathbb{R}, \quad & v(x,t) = \overline{v}_0(t) \quad && \text{for a.e. } (x,t) \in (-\infty, M) \times (0,T) \\ & u_0(x) = \overline{u}_0 && \text{for a.e. } x \in (-\infty, M) \end{aligned}$$

$$u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$$

then exists at least one entropy solution on $[0, T]$

proof by convergence of numerical scheme principle : let $u_i^{(n)}$, $u_{i-\frac{1}{2}}^{(n)}$, $v_{i-\frac{1}{2}}^{(n)}$ be given

Problem : define $u_{i+\frac{1}{2}}^{(n)}$, $v_{i+\frac{1}{2}}^{(n)}$ such that scheme

$$\frac{h}{\delta t}(u_i^{(n+1)} - u_i^{(n)}) - \left(c(u_{i+\frac{1}{2}}^{(n)})v_{i+\frac{1}{2}}^{(n)} + d(u_{i+\frac{1}{2}}^{(n)}) - c(u_{i-\frac{1}{2}}^{(n)})v_{i-\frac{1}{2}}^{(n)} - d(u_{i-\frac{1}{2}}^{(n)})\right) = 0$$

$$\frac{h}{\delta t}(f(u_i^{(n+1)}) - f(u_i^{(n)})) - \left(a(u_{i+\frac{1}{2}}^{(n)})v_{i+\frac{1}{2}}^{(n)} + b(u_{i+\frac{1}{2}}^{(n)}) - a(u_{i-\frac{1}{2}}^{(n)})v_{i-\frac{1}{2}}^{(n)} - b(u_{i-\frac{1}{2}}^{(n)})\right) = 0$$

leads to BV stability under CFL condition

Remark : $u = u_{i+\frac{1}{2}}^{(n)}$ and $v = v_{i+\frac{1}{2}}^{(n)}$ should verify

$$\begin{aligned} & f(u_i^{(n)}) + \frac{\delta t}{h}\left(a(u)v + b(u) - a(u_{i-\frac{1}{2}}^{(n)})v_{i-\frac{1}{2}}^{(n)} - b(u_{i-\frac{1}{2}}^{(n)})\right) \\ & -f\left(u_i^{(n)} + \frac{\delta t}{h}\left(c(u)v + d(u) - c(u_{i-\frac{1}{2}}^{(n)})v_{i-\frac{1}{2}}^{(n)} - d(u_{i-\frac{1}{2}}^{(n)})\right)\right) \quad = 0 \end{aligned}$$

# Solution

define $v = g^{(n)}_{i+\frac{1}{2}}(u)$ by

$$f(u_i^{(n)}) + \frac{\delta t}{h}\left(a(u)v + b(u) - a(u^{(n)}_{i-\frac{1}{2}})v^{(n)}_{i-\frac{1}{2}} - b(u^{(n)}_{i-\frac{1}{2}})\right)$$
$$-f\left(u_i^{(n)} + \frac{\delta t}{h}\left(c(u)v + d(u) - c(u^{(n)}_{i-\frac{1}{2}})v^{(n)}_{i-\frac{1}{2}} - d(u^{(n)}_{i-\frac{1}{2}})\right)\right) \quad = 0$$

$u^{(n)}_{i+\frac{1}{2}} = \mathrm{Godunov}(-(cg^{(n)}_{i+\frac{1}{2}} + d), u_i^{(n)}, u_{i+1}^{(n)})$ and $v^{(n)}_{i+\frac{1}{2}} = g^{(n)}_{i+\frac{1}{2}}(u^{(n)}_{i+\frac{1}{2}})$
in the sense
$-(c(u)g^{(n)}_{i+\frac{1}{2}}(u) + d(u))$ maximum on $[u_{i+1}^{(n)}, u_i^{(n)}]$ or minimum on $[u_i^{(n)}, u_{i+1}^{(n)}]$

Then

under CFL condition $u_i^{(n+1)} \in [\min(u^{(n)}_{i-\frac{1}{2}}, u_i^{(n)}, u^{(n)}_{i+\frac{1}{2}}), \max(u^{(n)}_{i-\frac{1}{2}}, u_i^{(n)}, u^{(n)}_{i+\frac{1}{2}})]$
and $u^{(n)}_{i+\frac{1}{2}} \in [\min(u_i^{(n)}, u_{i+1}^{(n)}), \max(u_i^{(n)}, u_{i+1}^{(n)})]$

implies BV-stability $\sum_{i\in\mathbb{Z}} |u_{i+1}^{(n+1)} - u_i^{(n+1)}| \leq \sum_{i\in\mathbb{Z}} |u_{i+1}^{(n)} - u_i^{(n)}|$

# Convergence properties

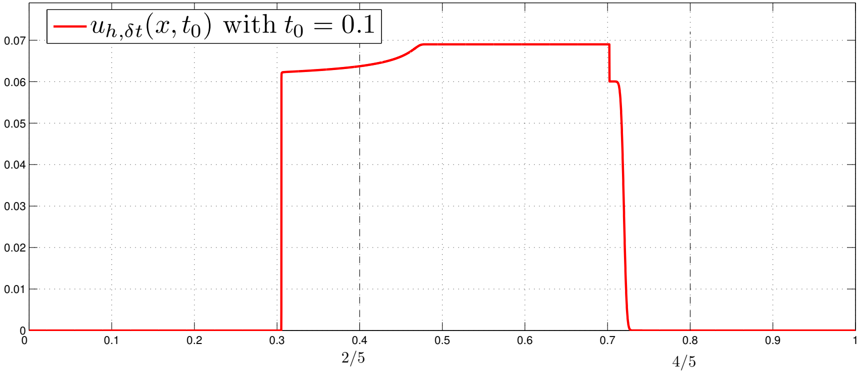
$L^\infty$ bounds on $u$ and $v$

space BV bounds on $u$ and $v$

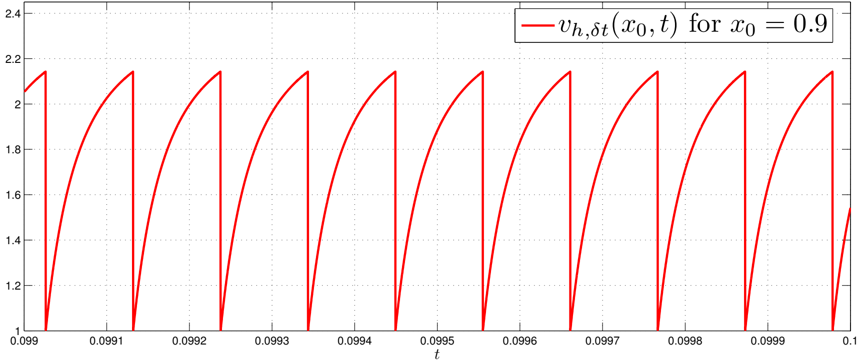
time BV bound on $u$

pass to the limit on the scheme

# *Example : a bubble of gas*



# *Example : a bubble of gas*



## *Open problems*

uniqueness of entropy weak solution ?

problem in bounded domain ? (although no numerical difficulty)