WELL-POSEDNESS RESULTS TO COMPRESSIBLE TWO-PHASE MODELS

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Paris, February 26, 2014
OUTLINE

1. NSAC system
2. NSCH system
3. $2^{nd}$ law of thermodynamics
4. Well-posedness results
Helmholtz energy density $\psi = \psi(\rho, c, \phi)$, $\phi := |\nabla c|^2$

- phase field (mass fraction) $c : J \times \overline{G} \to [0, 1]$ corresponding to concentration of one of two phases

- fluid moves with velocity $u : J \times \overline{G} \to \mathbb{R}^3$, different apparent densities $\rho_1 = c\rho$, $\rho_2 = (1 - c)\rho$,
  $\rho$ - total mass density, $\rho_j$ satisfy mass balance equation

$$\partial_t \rho_j + \nabla \cdot (\rho_j u) + J_j = 0$$

with $J_1 + J_2 = 0$, $J_j$ - transition rates
phase field (mass fraction) \( c : J \times \overline{G} \rightarrow [0, 1] \)

fluid moves with velocity \( u : J \times \overline{G} \rightarrow \mathbb{R}^3 \),
\( \rho_1 = c\rho, \rho_2 = (1 - c)\rho, \rho \) - total mass density,
\( \rho_j \) satisfy mass balance equation

\[
\partial_t \rho_j + \nabla \cdot (\rho_j u) + J_j = 0
\]

with \( J_1 + J_2 = 0 \), \( J_j \) - transition rates

\[
\Rightarrow \partial_t \rho + \nabla \cdot (\rho u) = 0
\]

suppose that \( J := J_1 \) is given by

\[
J := \frac{1}{\epsilon} \frac{\delta \Psi}{\delta c}
\]

with

\[
\Psi := \int_G \rho \psi(\rho, c, \phi) \, dx
\]

\( \epsilon \) - relaxation time, \( \frac{\delta \Psi}{\delta c} \) - generalised chemical potential
NSAC (proposed by Truskinovsky/Blesgen)

- Helmholtz energy density $\psi = \psi(\rho, c, \phi), \phi := |\nabla c|^2$
- suppose that $\mathcal{J} := \mathcal{J}_1$ is given by

$$\mathcal{J} := \frac{1}{\epsilon} \frac{\delta \Psi}{\delta c}$$

with

$$\Psi := \int_G \rho \psi(\rho, c, \phi) \, dx$$

computing $\frac{\delta \Psi}{\delta c}$ yields

$$\frac{\delta \Psi}{\delta c} = \partial_c (\rho \psi) - \nabla \cdot (\partial_c (\rho \psi)) = \partial_c (\rho \psi) - \nabla \cdot (2 \rho \partial_\phi \psi \nabla c)$$

- consider mass balance equation of $\rho_1$

$$0 = \partial_t \rho_1 + \nabla \cdot (\rho_1 u_1) + \mathcal{J} = \partial_t (\rho c) + \nabla \cdot (\rho cu) + \mathcal{J}$$

$$\Leftrightarrow$$

$$\partial_t (\rho c) + \nabla \cdot (\rho cu) - \frac{1}{\epsilon} \left( \nabla \cdot (2 \rho \partial_\phi \psi \nabla c) - \partial_c (\rho \psi) \right) = 0$$
So far we have
\[
\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad J \times G,
\]
\[
\partial_t (\rho c) + \nabla \cdot (c \rho u) + J = 0, \quad J \times G,
\]
with
\[
J = \frac{1}{\epsilon} \left( -\nabla \cdot (2\rho \partial \phi \psi \nabla c) + \partial_c (\rho \psi) \right)
\]
and
\[
\rho \partial \phi \psi (\rho, c, \phi) > 0 \quad \forall \rho > 0, \, c \in [0, 1], \, \phi \geq 0.
\]

- balance of momentum
\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot \mathcal{T} = \rho f_{ext}
\]
with Cauchy stress $\mathcal{T}$. 
NSAC: Constitutive equations

- Assume

\[ \mathcal{T} = S + \mathcal{P} \]

* \( S \) - Newtonian viscous stress, \( \mathcal{P} \) - pressure tensor

- Newtonian viscous stress:

\[
S = 2\eta \mathcal{D}(u) + \lambda \nabla \cdot u \mathcal{I}, \quad \mathcal{D}(u) = \frac{1}{2} (\partial_i u_j + \partial_j u_i)
\]

assumptions on the coefficients

\[
\eta(\rho, \theta, c) > 0, \quad 2\eta(\rho, \theta, c) + \lambda(\rho, \theta, c) > 0
\]

- Pressure tensor \( \mathcal{P} \):

\[
\mathcal{P} = -\rho^2 \partial_\rho \psi \mathcal{I} - \nabla c \otimes \partial_{\nabla c}(\rho \psi) = -\rho^2 \partial_\rho \psi \mathcal{I} - 2\rho \partial_\psi \nabla c \otimes \nabla c
\]

\( \nabla c \otimes \partial_{\nabla c}(\rho \psi) \) - Ericksen’s stress represents capillarity.
NSAC: mathematical problem

Let $J = [0, T]$ and $G \subset \mathbb{R}^n$ be a domain (with $C^2$ boundary $\Gamma$). Consider the compressible Navier-Stokes-Allen-Cahn system

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad J \times G,$$

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (S + P) = \rho f_{ext}, \quad J \times G, \quad (1)$$

$$\partial_t (c \rho) + \nabla \cdot (c \rho u) - \nabla \cdot (2\rho \partial_\phi \psi \nabla c) + \partial_c (\rho \psi) = 0, \quad J \times G,$$

with initial data

$$\rho(0) = \rho_0, \quad u(0) = u_0, \quad c(0) = c_0, \quad G \quad (2)$$

and boundary conditions

non-slip: $u = 0$

pure-slip: $(u|\nu) = 0, \quad Q(\nu)S \cdot \nu = 0, \quad Q := \mathcal{I} - \nu \otimes \nu$ \quad (3)

Dirichlet: $c = 0$

Neumann: $(\nabla c|\nu) = 0$
**NSCH** *(proposed by Lowengrub & Truskinovsky)*

- phase field (mass fraction) \( c : J \times \overline{G} \rightarrow [0, 1] \)
- fluid moves with velocity \( u : J \times \overline{G} \rightarrow \mathbb{R}^3 \)
- Helmholtz energy density \( \psi = \psi(\rho, c, \phi), \phi := |\nabla c|^2 \)
- \( \rho_1 = c\rho, \rho_2 = (1 - c)\rho, \rho - \text{total mass density,} \rho_j \text{ satisfy mass balance equation} \)

\[
\partial_t \rho_j + \nabla \cdot (\rho_j u) + \nabla \cdot \mathcal{J}_j = 0
\]

with

\[
\mathcal{J}_1 + \mathcal{J}_2 = 0 \quad \Rightarrow \quad \partial_t \rho + \nabla \cdot (\rho u) = 0
\]

suppose that \( \mathcal{J} := \mathcal{J}_1 \) is given by (Fick’s law)

\[
\mathcal{J} = m\nabla \mu
\]

with mobility \( m \) and generalised chemical potential \( \mu \),

\[
\rho \mu = \partial_c (\rho \psi) - \nabla \cdot (2\rho \partial_\phi \psi \nabla c).
\]
NSCH: MATHEMATICAL PROBLEM

Let $J = [0, T]$ and $G \subset \mathbb{R}^n$ be a domain (with $C^2$ boundary $\Gamma$). Consider the compressible Navier-Stokes-Allen-Cahn system

\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \quad J \times G, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (S + P) &= \rho f_{\text{ext}}, \quad J \times G, \\
\partial_t (c \rho) + \nabla \cdot (c \rho u) - \nabla \cdot (m \nabla \mu) &= 0, \quad J \times G, \\
\partial_c (\rho \psi) - \nabla \cdot (2 \rho \partial_\phi \psi \nabla c) &= \rho \mu, \quad J \times G,
\end{align*}

with initial data

\begin{align*}
\rho(0) = \rho_0, \quad u(0) = u_0, \quad c(0) = c_0
\end{align*}

and boundary conditions

- non-slip: $u = 0$
- pure-slip: $(u|\nu) = 0, \quad Q(\nu)S \cdot \nu = 0, \quad Q := I - \nu \otimes \nu$
- Neumann: $(\nabla \mu|\nu) = 0, \quad (\nabla c|\nu) = 0$
Features/Advantages/Differences

classical description/sharp interfaces:
   ▶ problem with merging, reconnecting, and hitting interfaces
   ▶ contact angle conditions, jump conditions of stress tensor across interface, quasilinear evolution equations on boundary

diffuse interface models NSAC/NSCH:
   ▶ interfaces described by phase field (molecular mixing near interfaces)
   ▶ capillary effects inherent in additional stress tensor;
   ▶ Korteweg case: density serves order parameter;
      van der Waals pressure law is needed (“unphysical” pressure law)
DIFFICULTIES (in case of strong solutions)

- system of hyperbolic-parabolic equations
- quasilinear partial differential equations (even for constant coefficients)
  - $\rho \partial_t u, \nabla \cdot (\rho \partial_\phi \psi \nabla c)$;
  - $\nabla \cdot (\nabla c \otimes \rho \partial_\phi \psi \nabla c)$ – quasilinearity of highest order, similar to quasilinear elliptic operators: $\nabla \cdot (a(\nabla v)\nabla v)$;
- $n \geq 1$ – dimension of the problem
- strong coupled equations
- avoid vacuum, e.g. in case of NSAC

\[
\rho \partial_t u + \rho \nabla u \cdot u - \nabla \cdot S(u) + \nabla (\rho^2 \partial_\rho \psi) + \nabla \cdot (2 \rho \partial_\phi \psi \nabla c \otimes \nabla c) = \rho f_{ext}
\]
\[
\rho \partial_t c + \rho u \cdot \nabla c - \nabla \cdot (2 \rho \partial_\phi \psi \nabla c) + \partial_c (\rho \psi) = 0
\]
Typical situation/Configurations

Interested in Helmholtz energy of the form

\[ \rho \psi (\rho, c, \phi) = \rho [c \psi_1 + (1 - c) \psi_2] + \rho \left( W(c) + \frac{\delta}{2} \phi \right), \quad \phi = |\nabla c|^2. \]

- convex combination of energies \( \psi_1 \) and \( \psi_2 \) of the pure phases
- \( \delta \) - a measure of thickness for the interface
- typically: \( W \) - double-well potential, e.g.

\[ W(c) = k_1 [c \ln(c) + (1 - c) \ln(1 - c)] + k_2 c (1 - c), \quad k_1, k_2 \in \mathbb{R} \]

- \( \partial_{\nabla c} (\rho \psi) = 2 \rho \partial_{\phi} (\rho \psi) \nabla c = \rho \delta \nabla c \) and

\[ \mathcal{P} = -\rho^2 \partial_{\psi} \mathbf{I} - \delta \rho \nabla c \otimes \nabla c, \]
\[ \rho \mu = \partial_c (\rho \psi) - \nabla \cdot (\rho \delta \nabla c). \]

- depending on the choice of \( \psi_j \) different kinds of fluid mixtures are modelled (two compressible mixtures, one compressible and one incompressible fluid, two incompressible fluids)
2. NSCH modelling: Lowengrub, Truskinovsky (1998)
3. well-posedness results to NSAC:
   - Feireisl et al. (2010): existence of global weak solutions in case of $\delta \rho = 1$, i.e.
     $$P = -\pi I - (\nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 I)$$
     Problem: Energy estimates do not provide any bound for $\nabla c$ in vacuum zones.
   - Shijin Ding, and Yinhua Li and Wanglong Luo (2012), Global solutions in 1D
   - Alt & Witterstein (2011), sharp interface limits
   - K. (2012), strong well-posedness to NSAC
4. well-posedness results to NSCH:
   - Abels & Feireisl, global weak solution for a simplified model
   - many articles to incompressible NSCH
   - K., Zacher (2013), strong well-posedness to the original model
2\textsuperscript{nd} LAW OF THERMODYNAMICS

Considering the non-isothermal counterparts of NSAC and NSCH we have

**THEOREM**

The thermodynamically closed systems of NSAC and NSCH are thermodynamically and mechanically consistent and the following equations hold:

\[
\int_G \partial_t (\rho s) \, dx = \int_G \beta \left| \frac{\nabla \theta}{\theta} \right|^2 \, dx + \int_G \frac{1}{\theta} \mathcal{S} : \mathcal{D} \, dx + \int_G \frac{\xi}{\rho} |\mathcal{J}|^2 \, dx,
\]

\[
\int_G \partial_t (\rho s) \, dx = \int_G \beta \left| \frac{\nabla \theta}{\theta} \right|^2 \, dx + \int_G \frac{1}{\theta} \mathcal{S} : \mathcal{D} \, dx + \int_G \frac{1}{m} |\mathcal{J}|^2 \, dx.
\]

\(\theta\) - temperature, \(\beta\) - heat conducting coefficient, \(s\) - entropy density
WELL-POSEDNESS OF NSAC: SETTING

We are looking for strong solutions in the $L_p$-setting. Consider first the equation for $c$ in $L_p(J;L_p(G))$:

$$
\partial_t (\rho c) + \nabla \cdot (c \rho u) - \nabla \cdot (\rho \delta \nabla c) + \partial_c (\rho \psi) = 0
\quad \Leftrightarrow
\rho \partial_t c + \rho \nabla c \cdot u - \nabla \cdot (\rho \delta \nabla c) + \partial_c (\rho \psi) = 0
$$

The natural regularity class is

$$
c \in H^1_p (J; L_p (G)) \cap L_p (J; H^2_p (G)).
$$

Notice: $\nabla \rho$ occurs in the Allen-Cahn equation, i.e. we need at least

$$
\rho \in L_p (J; H^1_p (G)).
$$
**Setting NSAC**

Consider the Navier-Stokes equation in $L_p(J; L_p(G; \mathbb{R}^n))$:

\[
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot S - \nabla \cdot \mathcal{P} = \rho f_{\text{ext}}
\]

\[
\nabla \cdot \mathcal{P} \sim \nabla^2 c \quad \text{and again} \quad \nabla \rho
\]

\[
\nabla \cdot S \sim \nabla^2 u, \, \nabla c, \, \nabla \rho
\]

This is compatible with the regularity of $c$, since $c \in L_p(J; H^2_p(G))$ and thus

\[
\nabla \cdot \mathcal{P} \sim \nabla^2 c \in L_p(J; L_p(G; \mathbb{R}^n)).
\]

The natural regularity class for $u$ is:

\[
u \in H^1_p(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)).\]

Note: need again $\rho \in L_p(J; H^1_p(G))$. 

Since $\rho$ is governed by the hyperbolic equation

$$\partial_t \rho + \nabla \rho \cdot u = -\rho \nabla \cdot u,$$

we need $u \in L^p(J; H^2_p(G; \mathbb{R}^n)) \cap L^1(J; C^1(\overline{G}))$. Recall

$$u \in H^1_p(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)) =: Z_1(J),$$

and the embedding

$$H^2_p(G; \mathbb{R}^n) \hookrightarrow C^1(\overline{G}; \mathbb{R}^n), \quad p > n.$$

Using this regularity the continuity equation yields

$$\rho \in C^1(J; L_p(G)) \cap C(J; H^1_p(G)).$$
Seek solutions \((u, c, \rho)\) in the regularity class \(Z_1(J) \times Z_2(J) \times Z_3(J)\),

\[
Z_1(J) = H^1_p(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)), \\
Z_2(J) = H^1_p(J; L_p(G)) \cap L_p(J; H^2_p(G)), \\
Z_3(J) = C^1(J; L_p(G)) \cap C(J; H^1_p(G)).
\]
**Well-posedness result: NSAC**

**Theorem (K., ARMA 2012)**

Let $G \subset \mathbb{R}^n$ be a bounded domain with $C^2$ boundary $\Gamma$, $J_0 = [0, T_0]$, and let $p > n + 2$. Assume that

1. $\mu, \lambda, \delta, \psi_1, \psi_2, W$ sufficiently smooth;
2. $\mu(\rho, c) > 0$, $2\mu(\rho, c) + \lambda(\rho, c) > 0$, $\delta(c) > 0$ for all $(\rho, c) \in (0, \infty)^2$;
3. $f_{\text{ext}} \in L_p(J_0; L_p(G; \mathbb{R}^n))$;
4. $w_0 := (u_0, c_0, \rho_0) \in V$,
   
   $$V := \{(u, c, \rho) \in W_p^{2-2/p}(G; \mathbb{R}^n) \times W_p^{2-2/p}(G) \times H_p^1(G) : 0 < \rho(x), 0 \leq c(x) \leq 1, \forall x \in \overline{G}, (u(y)|\nu(y)) \geq 0, \forall y \in \Gamma\};$$
5. compatibility conditions.

Then there exists $0 < T \leq T_0$ such that the NSAC-system has a unique solution $w := (u, c, \rho) \in Z_1(J) \times Z_2(J) \times Z_3(J)$ on $J = [0, T]$.

The map $w_0 \rightarrow w(t)$ generates a local semiflow on $V_c := \{\phi \in V : \phi$ satisfies (V)\}. 
**Well-posedness of NSCH: the setting**

We are looking for strong solutions in the $L_p$-setting. Consider first the Cahn-Hilliard equation in $L_p(J; L_p(G))$:

$$\rho \partial_t c + \rho \nabla c \cdot u - \nabla \cdot (m \nabla \mu) = 0,$$

$$\partial_c(\rho \psi) - \nabla \cdot (\rho \delta \nabla c) = \rho \mu.$$

The natural regularity class is

$$c \in H^1_p(J; L_p(G)) \cap L_p(J; H^4_p(G)).$$

Notice: $\nabla^3 \rho$ occurs in the Allen-Cahn equation, i.e. we need at least

$$\rho \in L_p(J; H^3_p(G)).$$
**Setting in case of NSCH**

ρ is governed by the hyperbolic equation

\[ \partial_t \rho + \nabla \cdot (\rho u) = 0. \]

Usually,

\[ u \in L_1(J; C^1(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)) \]

\[ \Downarrow \]

\[ \rho \in C^1(J; L_p(G)) \cap C(J; H^3_p(G)). \]

Cahn-Hilliard eq.: ρ has to be in \( L_p(J; H^3_p(G)) \) at least. We therefore need

\[ u \in L_p(J; H^4_p(G; \mathbb{R}^n)). \]

One can prove

\[ u \in L_1(J; C^1(G; \mathbb{R}^n)) \cap L_p(J; H^4_p(G; \mathbb{R}^n)) \]

\[ \Downarrow \]

\[ \rho \in C^1(J; H^2_p(G)) \cap C(J; H^3_p(G)). \]
**Setting in case of NSCH**

Bear in mind that we need to have

\[
 u \in L_1(J; C^1(G; \mathbb{R}^n)) \cap L_p(J; H^4_p(G; \mathbb{R}^n)).
\]

To obtain this regularity for \( u \), we consider the Navier-Stokes equation in \( L_p(J; H^2_p(G; \mathbb{R}^n)) \):

\[
 \rho \partial_t u + \rho \nabla u \cdot u - \nabla \cdot S = \nabla \cdot \mathcal{P} + \rho f_{\text{ext}}
\]

\[
 \nabla \cdot \mathcal{P} \sim \partial_{x_i} \nabla c, \ \nabla \rho
\]

Note that

\[
 c \in H^1_p(J; L_p(G)) \cap L_p(J; H^4_p(G)) \hookrightarrow H^{1/2}_p(J; H^2_p(G))
\]

\[
 \downarrow
\]

\[
 \partial_{x_i} \nabla c \in H^{1/2}_p(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)).
\]

Regularity of \( u \) implies \( \nabla \rho \in C^1(J; H^1_p(G; \mathbb{R}^n)) \cap C(J; H^2_p(G; \mathbb{R}^n)) \).

Therefore

\[
 \nabla \cdot \mathcal{P} \in H^{1/2}_p(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)) =: X_1.
\]
SETTING IN CASE OF NSCH

Taking \( X_1 = \mathcal{H}_{p}^{1/2}(J; L_p(G; \mathbb{R}^n)) \cap L_p(J; H^2_p(G; \mathbb{R}^n)) \) as the base space for the Navier-Stokes equation one expects that

\[
u \in \mathcal{H}_{p}^{3/2}(J; L_p(G; \mathbb{R}^n)) \cap H^1_p(J; H^2_p(G; \mathbb{R}^n)) \cap L_p(J; H^4_p(G; \mathbb{R}^n)).
\]

Using this regularity the continuity equation yields

\[
\rho \in H^{2 + 1/4}_p(J; L_p(G)) \cap C^1_p(J; H^2_p(G)) \cap C(J; H^3_p(G)).
\]

Seek solutions \((u, c, \rho)\) in

<table>
<thead>
<tr>
<th>REGULARITY CLASS</th>
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<tr>
<td>( Z(J) = Z_1(J) \times Z_2(J) \times Z_3(J), )</td>
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**Well-posedness result: NSCH**

**Theorem (K., Zacher 2013)**

Let $G \subset \mathbb{R}^n$ be a bounded domain with $C^4$ boundary $\Gamma$, $J_0 = [0, T_0]$, and let $p > p^* := \max\{4, n\}$, $p \neq 5$. Assume that

(I) $\rho \psi = \overline{\psi}(\rho, c) + \rho(W(c) + \frac{\delta}{2}|\nabla c|^2)$;

(II) $\eta, \lambda, \delta, m, \overline{\psi}, W$ sufficiently smooth;

(III) $\eta(\rho, c), 2\eta(\rho, c) + \lambda(\rho, c), \delta(\rho, c), m(\rho, c) > 0$ for all $(\rho, c) \in (0, \infty)^2$;

(IV) $f_{ext} \in X_1 = H^{1/2}_p(J_0; L_p(G; \mathbb{R}^n)) \cap L_p(J_0; H^3_p(G; \mathbb{R}^n))$;

(V) $w_0 := (u_0, c_0, \rho_0) \in V := \{(u, c, \rho) \in W^{4-2/p}_p(G; \mathbb{R}^n) \times W^{4-4/p}_p(G) \times H^3_p(G) : 0 < \rho(x) \ \forall x \in \overline{G}\}$;

(VI) compatibility conditions (There are many of them.).

Then there exists $0 < T \leq T_0$ such that the NSCH-system has a unique solution $w := (u, c, \rho) \in Z(J)$ on $J = [0, T]$. The map $w_0 \rightarrow w(t)$ generates a local semiflow on $V_c := \{\phi \in V : \phi \text{ satisfies (VI)}\}$. 
**Outline of the Proof**

1. Suppose $\rho_0 \in H^3_p(G)$, $p > p^*$, $\rho_0 > 0$ in $\bar{G}$, and $(u|\nu) \geq 0$ on $[0, T] \times \Gamma$. Then the continuity equation, together with $\rho|_{t=0} = \rho_0$ and $u \in Z_1([0, T])$, has a unique positive solution.

   $$\Rightarrow \rho = L[u] \rho_0 \in Z_3([0, T]).$$

2. Insert $\rho = L[u] \rho_0$ into momentum and phase field equations. (nonlocal, fully nonlinear)

3. Find appropriate fixed point formulation and subset $\Sigma \subset Z_1(J) \times Z_2(J)$.

4. Contraction mapping principle

5. Problem with contraction:
   - Contraction cannot be proved in $Z_1(J) \times Z_2(J)$. (loss of regularity!)
   - Well-known in the theory of symmetric quasilinear hyperbolic systems.
   - Kato/Lax resolve this problem by studying contraction in a larger space.
   - Find an appropriate space $\mathcal{W}([0, T])$, such that
     $Z_1(J) \times Z_2(J) \subset \mathcal{W}([0, T])$, contraction in $\mathcal{W}(J)$,
     $\forall \{\Phi_n\} \subset \Sigma$ with $\Phi_n \to \Phi$ in $\mathcal{W}(J) \Rightarrow \Phi \in \Sigma$
**Lack of Regularity**

Let $\rho_i = L[u_i]\rho_0$ with $u_i \in Z_1(J)$, i.e.

$$\partial_t \rho_i + \nabla \rho_i \cdot u_i = -\rho_i \nabla \cdot u_i, \quad (t, x) \in J \times G,$$

$$\rho_i(0) = \rho_0, \quad x \in G,$$

and $\rho_i \in C^1(J; L_p(G)) \cap C(J; H^3_p(G))$.

Then $(\rho_1 - \rho_2, u_1 - u_2) =: (q, v)$ satisfies

$$\partial_t q + \nabla q \cdot u_1 = -q \nabla \cdot u_1 - \rho_2 \nabla \cdot v - \nabla \rho_2 \cdot v =: F,$$

$$q(0) = 0.$$

We need $F \in C(J; H^3_p(G))$ to get $q \in C^1(J; L_p(G)) \cap C(J; H^3_p(G))$.

**Note:**

1. $\rho_i \nabla \cdot u_i, q \nabla \cdot u_1 \in C(J; H^3_p(G))$
2. $\nabla \rho_2 \cdot v \in C(J; H^2_p(G))$

**Conclusion:** $\rho_1 - \rho_2 \notin C^1(J; L_p(G)) \cap C(J; H^3_p(G))$ and thus

$$\|\rho_1 - \rho_2\|_{C^1(J; L_p(G)) \cap C(J; H^3_p(G))} \not\leq k \|u_1 - u_2\|_{Z_1(J)}, \quad 0 < k < 1.$$
**Basic Ideas: Fixed Point Equation**

Linearize quasilinear terms (freeze coefficients):

\[
\rho_0 \partial_t u + A u + B c + C \mu = F_1(u, c, \mu, L[u] \rho_0), \quad J \times G,
\]
\[
\rho_0 \partial_t c - \nabla \cdot (m_0 \nabla \mu) = F_2(u, c, \mu, L[u] \rho_0), \quad J \times G,
\]
\[
-\mu - \nabla \cdot (\delta_0 \nabla c) = F_3(c, L[u] \rho_0), \quad J \times G,
\]
\[
u \cdot \nabla (\delta_0 \nabla c) = \rho_0 \nabla \cdot (\delta_0 \nabla c), \quad J \times \Gamma,
\]
\[
\begin{align*}
\rho_0 \partial_t u + A u + B c + C \mu &= F_1(u, c, \mu, L[u] \rho_0), \\
\rho_0 \partial_t c - \nabla \cdot (m_0 \nabla \mu) &= F_2(u, c, \mu, L[u] \rho_0), \\
-\mu - \nabla \cdot (\delta_0 \nabla c) &= F_3(c, L[u] \rho_0),
\end{align*}
\]

Linearization maintains divergence structure! More abstractly,

\[
L_{11} u + L_{12} (c, \mu) = \mathcal{F}_1(u, c, \mu), \quad \mathcal{F}_i - \text{nonlocal, fully nonlinear.}
\]

where

\[
A u := -\nabla \cdot (2 \mu_0 \mathcal{D}(u) + \lambda_0 \nabla \cdot u \mathcal{I}),
\]
\[
B c := [\rho_0^2 \partial_\rho \delta_0 + \rho_0 \delta_0] \nabla c_0 \cdot \nabla^2 c,
\]
\[
C \mu := -\rho_0 \nabla \cdot (\delta_0 \nabla c_0 \mu).
\]
Basic Ideas: Fixed Point Equation

1. Define

\[ \Lambda : \mathbb{Z}([0, T]) := \mathbb{Z}_1([0, T]) \times \mathbb{Z}_2([0, T]) \times \mathbb{Z}_\mu([0, T]) \to \mathbb{Z}([0, T]) \]

by

\[ \Lambda(u, c, \mu) := (\hat{u}, \hat{c}, \hat{\mu}), \]

\[ \mathcal{L}_{11}\hat{u} + \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = \mathcal{F}_1(u, c, \mu), \]

\[ \mathcal{L}_{22}(\hat{c}, \hat{\mu}) = \mathcal{F}_2(u, c, \mu). \]

2. Define subset \( \Sigma \subset \mathbb{Z}([0, T]) \). For \( T \in (0, T_0) \) and \( r \in (0, 1) \) let

\[ \Sigma := \{(u, c, \mu) \in \mathbb{Z}([0, T]) : (u, \partial_t u, c, \mu)_{|t=0} = (u_0, u_\bullet, c_0, \mu_0), \]

\[ u = 0, \partial_\nu c = \partial_\nu \mu = 0 \text{ on } \Gamma, \quad \|(\bar{u}, \bar{c}, \bar{\mu}) - (u, c, \mu)\|_{\mathbb{Z}([0, T])} \leq r \}. \]

\( \Sigma \) – closed ball in \( \mathbb{Z}([0, T]) \) with centre \( (\bar{u}, \bar{c}, \bar{\mu}) \) and radius \( r \).

3. Reference function \( (\bar{u}, \bar{c}, \bar{\mu}) \in \mathbb{Z}(J_0), J_0 = [0, T_0] \), is given as solution of

\[ \mathcal{L}_{11}\bar{u} + \mathcal{L}_{12}(\bar{c}, \bar{\mu}) = \mathcal{F}_1(u_0, c_0, \mu_0), \]

\[ \mathcal{L}_{22}(\bar{c}, \bar{\mu}) = \mathcal{F}_2(u_0, c_0, \mu_0). \]
**Basic Ideas: Things to Do**

Fixed point mapping: $\Lambda(u, c, \mu) := (\hat{u}, \hat{c}, \hat{\mu})$,

$$\mathcal{L}_{11}\hat{u} + \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = \mathcal{F}_1(u, c),$$

$$\mathcal{L}_{22}(\hat{c}, \hat{\mu}) = \mathcal{F}_2(u, c).$$  \hspace{1cm} (7)

1. $\Lambda$ is well-defined: $\exists! (\hat{u}, \hat{c}, \hat{\mu}) \in \mathcal{Z}([0, T])$ of (7).

2. For sufficiently small $T$ and $r$:

   (I) $\Lambda$ leaves $\Sigma$ invariant.

   (II) $\Lambda$ is a strict contraction in

   $$\mathcal{W}([0, T]) := W_1([0, T]) \times W_2([0, T]) \times W_3([0, T])$$

   with

   $$W_1(J) := H_2^{5/4}(J; L_2(G)) \cap H_2^{1/2}(J; H_2^2(G; \mathbb{R}^n)) \cap L_2(J; H_2^3(G; \mathbb{R}^n));$$

   $$W_2(J) := H_2^{3/4}(J; L_2(G)) \cap L_2(J; H_2^3(G));$$

   $$W_3(J) := H_2^{1/4}(J; L_2(G)) \cap L_2(J; H_2^1(G)).$$

   There holds, e.g.,

   $$Z_2([0, T]) := H_p^1([0, T]; L_p(G)) \cap L_p([0, T]; H_p^4(G)) \hookrightarrow W_2([0, T]).$$

   (III) $\Sigma$ is closed in $W_1([0, T]) \times W_2([0, T])$. 
**Basic ideas: some auxiliary results**

1. The system

\[
\mathcal{L}_{11}\hat{u} + \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = f_1, \\
\mathcal{L}_{22}(\hat{c}, \hat{\mu}) = f_2,
\]

has maximal $L_p$-regularity, i.e.

\[
\exists!(\hat{u}, \hat{c}, \hat{\mu}) \in \mathcal{Z}([0, T]) \iff f_1, f_2 \text{ have certain regularity.}
\]

Use: The problem decouples.

Solve second equation, $(\hat{c}, \hat{\mu}) = \mathcal{L}_{22}^{-1} f_2$. Solve equation for $\hat{u}$

\[
\mathcal{L}_{11}\hat{u} + \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = f_1
\]

\[
\uparrow
\]

\[
\mathcal{L}_{11}\hat{u} = f_3
\]

with $f_3 := f_1 - \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = f_1 - \mathcal{L}_{12}\mathcal{L}_{22}^{-1} f_2$.

\[
\Rightarrow \Lambda \text{ is well-defined.}
\]
BASIC IDEAS: SOME AUXILIARY RESULTS

2. Selfmapping means: Let \((u, c, \mu) \in \Sigma\). Then \((\hat{u}, \hat{c}, \hat{\mu}) = \Lambda(u, c, \mu)\) has to satisfy

\[\| (\hat{u}, \hat{c}, \hat{\mu}) - (\overline{u}, \overline{c}, \overline{\mu}) \|_{Z([0,T])} \leq r.\]

The trick is

\[
(\hat{u}, \hat{c}, \hat{\mu}) = \left( \begin{array}{cc} \mathcal{L}_{11} & \mathcal{L}_{12} \\ 0 & \mathcal{L}_{22} \end{array} \right)^{-1} \left( \begin{array}{c} \mathcal{F}_1(u, c, \mu) \\ \mathcal{F}_2(u, c, \mu) \end{array} \right)
\]

\[
(\overline{u}, \overline{c}, \overline{\mu}) = \left( \begin{array}{cc} \mathcal{L}_{11} & \mathcal{L}_{12} \\ 0 & \mathcal{L}_{22} \end{array} \right)^{-1} \left( \begin{array}{c} \mathcal{F}_1(u_0, c_0, \mu_0) \\ \mathcal{F}_2(u_0, c_0, \mu_0) \end{array} \right)
\]

and

\[
\| (\hat{u}, \hat{c}, \hat{\mu}) - (\overline{u}, \overline{c}, \overline{\mu}) \|_{Z([0,T])} \leq M( \| \mathcal{F}_1(u, c, \mu) - \mathcal{F}_1(u_0, c_0, \mu_0) \|_{X_1([0,T])} \\
+ \| \mathcal{F}_2(u, c, \mu) - \mathcal{F}_1(u_0, c_0, \mu_0) \|_{X_2([0,T])}).
\]

Note: only estimates of differences in space of data,
\[\rho = L[u]\rho \in C([0, T]; H^3_p(G))\) behaves as lower order term
BASIC IDEAS: SOME AUXILIARY RESULTS

3. Contraction in \( \mathcal{W}([0, T]) \) - crucial part.

Recall \( \Lambda(u, c, \mu) = (\hat{u}, \hat{c}, \hat{\mu}) \),

\[
\mathcal{L}_{11}\hat{u} + \mathcal{L}_{12}(\hat{c}, \hat{\mu}) = \mathcal{F}_1(u, c, \mu),
\]
\[
\mathcal{L}_{22}(\hat{c}, \hat{\mu}) = \mathcal{F}_2(u, c, \mu).
\]

Let \( (u_i, c_i, \mu_i) \in \Sigma, \rho_i = L[u_i]\rho_0 \), and \( (\hat{u}_i, \hat{c}_i, \hat{\mu}_i) = \Lambda(u_i, c_i, \mu_i), i = 1, 2. \)

Then \( (\hat{u}_1 - \hat{u}_2, \hat{c}_1 - \hat{c}_2, \hat{\mu}_1 - \hat{\mu}_2) \) satisfies

\[
\mathcal{L}_{11}(\hat{u}_1 - \hat{u}_2) + \mathcal{L}_{12}(\hat{c}_1 - \hat{c}_2, \hat{\mu}_1 - \hat{\mu}_2) = \mathcal{F}_1(u_1, c_1, \mu_1) - \mathcal{F}_1(u_2, c_2, \mu_2),
\]
\[
\mathcal{L}_{22}(\hat{c}_1 - \hat{c}_2, \hat{\mu}_1 - \hat{\mu}_2) = \mathcal{F}_2(u_1, c_2, \mu_2) - \mathcal{F}_2(u_2, c_2, \mu_2).
\]

One can prove

\[
\|(\hat{u}_1, \hat{c}_1, \hat{\mu}_1) - (\hat{u}_2, \hat{c}_2, \hat{\mu}_2)\|_{\mathcal{W}([0,T])} \leq \kappa(T, r)\|(u_1, c_1, \mu_1) - (u_2, c_2, \mu_2)\|_{\mathcal{W}([0,T])},
\]

where \( \kappa(T, r) \to 0 \) as \( T, r \to 0. \)

Use: divergence structure, weak formulation, max. reg. methods, and

\[
\|\rho_1 - \rho_2\|_{C([0,T];H^2(G))} \leq C_1 T^{1/2}\|u_1 - u_2\|_{L^2([0,T];H^3(G;\mathbb{R}^n))}.
\]
Thank you.