Well-posedness results to compressible two-phase models

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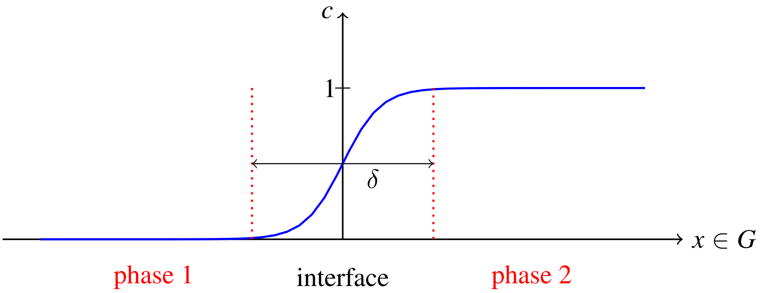
Outline

1. NSAC system
2. NSCH system
3. 2^{nd} law of thermodynamics
4. Well-posedness results

NSAC (PROPOSED BY TRUSKINOVSKY/BLESGEN)

- Helmholtz energy density $\psi = \psi(\rho, c, \phi)$, $\phi := |\nabla c|^2$
- phase field (mass fraction) $c : J \times \overline{G} \to [0, 1]$ corresponding to concentration of one of two phases



- fluid moves with velocity $u : J \times \overline{G} \to \mathbb{R}^3$,
different apparent densities $\rho_1 = c\rho$, $\rho_2 = (1 - c)\rho$,
ρ - total mass density, ρ_j satisfy mass balance equation

$$\partial_t \rho_j + \nabla \cdot (\rho_j u) + \mathcal{J}_j = 0$$

with $\mathcal{J}_1 + \mathcal{J}_2 = 0$, $\mathcal{J}_j$ - transition rates

NSAC (proposed by Truskinovsky/Blesgen)

- phase field (mass fraction) $c : J \times \overline{G} \to [0, 1]$
- fluid moves with velocity $u : J \times \overline{G} \to \mathbb{R}^3$,
  $\rho_1 = c\rho$, $\rho_2 = (1 - c)\rho$, ρ - total mass density,
  ρ_j satisfy mass balance equation

$$\partial_t \rho_j + \nabla \cdot (\rho_j u) + \mathcal{J}_j = 0$$

  with $\mathcal{J}_1 + \mathcal{J}_2 = 0$, $\mathcal{J}_j$ - transition rates

$$\Rightarrow \quad \partial_t \rho + \nabla \cdot (\rho u) = 0$$

- suppose that $\mathcal{J} := \mathcal{J}_1$ is given by

$$\mathcal{J} := \frac{1}{\epsilon} \frac{\delta \Psi}{\delta c}$$

  with

$$\Psi := \int_G \rho \psi(\rho, c, \phi)\, dx$$

  ϵ - relaxation time, $\frac{\delta \Psi}{\delta c}$ - generalised chemical potential

NSAC (proposed by Truskinovsky/Blesgen)

- Helmholtz energy density $\psi = \psi(\rho, c, \phi)$, $\phi := |\nabla c|^2$
- suppose that $\mathcal{J} := \mathcal{J}_1$ is given by

$$\mathcal{J} := \frac{1}{\epsilon}\frac{\delta \Psi}{\delta c}$$

with

$$\Psi := \int_G \rho\psi(\rho, c, \phi)\, dx$$

computing $\frac{\delta \Psi}{\delta c}$ yields

$$\frac{\delta \Psi}{\delta c} = \partial_c(\rho\psi) - \nabla \cdot (\partial_{\nabla c}(\rho\psi)) = \partial_c(\rho\psi) - \nabla \cdot (2\rho\partial_\phi \psi \nabla c)$$

- consider mass balance equation of ρ_1

$$0 = \partial_t \rho_1 + \nabla \cdot (\rho_1 u_1) + \mathcal{J} = \partial_t(\rho c) + \nabla \cdot (\rho c u) + \mathcal{J}$$

$$\Leftrightarrow$$

$$\partial_t(\rho c) + \nabla \cdot (\rho c u) - \frac{1}{\epsilon}\Big(\nabla \cdot (2\rho\partial_\phi \psi \nabla c) - \partial_c(\rho\psi)\Big) = 0$$

NSAC

So far we have

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad J \times G,$$
$$\partial_t (\rho c) + \nabla \cdot (c \rho u) + \mathcal{J} = 0, \quad J \times G,$$

with

$$\mathcal{J} = \frac{1}{\epsilon}\Big(- \nabla \cdot (2\rho \partial_\phi \psi \nabla c) + \partial_c (\rho \psi)\Big)$$

and

$$\rho \partial_\phi \psi(\rho, c, \phi) > 0 \quad \forall \rho > 0,\ c \in [0,1],\ \phi \geq 0.$$

- balance of momentum

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot \mathcal{T} = \rho f_{ext}$$

with Cauchy stress $\mathcal{T}$

NSAC: Constitutive equations

- assume

$$\mathcal{T} = \mathcal{S} + \mathcal{P}$$

  $\mathcal{S}$ - Newtonian viscous stress, $\mathcal{P}$ - pressure tensor

- Newtonian viscous stress:

$$\mathcal{S} = 2\eta\mathcal{D}(u) + \lambda\nabla \cdot u\,\mathcal{I}, \quad \mathcal{D}(u) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)$$

  assumptions on the coefficients

$$\eta(\rho, \theta, c) > 0, \quad 2\eta(\rho, \theta, c) + \lambda(\rho, \theta, c) > 0$$

- pressure tensor $\mathcal{P}$:

$$\mathcal{P} = -\rho^2\partial_\rho\psi\,\mathcal{I} - \nabla c \otimes \partial_{\nabla c}(\rho\psi) = -\rho^2\partial_\rho\psi\,\mathcal{I} - 2\rho\partial_\phi\psi\nabla c \otimes \nabla c$$

  $\nabla c \otimes \partial_{\nabla c}(\rho\psi)$ - Ericksen's stress represents capillarity.

NSAC: MATHEMATICAL PROBLEM

Let $J = [0, T]$ and $G \subset \mathbb{R}^n$ be a domain (with C^2 boundary Γ). Consider the compressible Navier-Stokes-Allen-Cahn system

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0, && J \times G, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (\mathcal{S} + \mathcal{P}) &= \rho f_{ext}, && J \times G, \\ \partial_t(c\rho) + \nabla \cdot (c\rho u) - \nabla \cdot (2\rho \partial_\phi \psi \nabla c) + \partial_c(\rho\psi) &= 0, && J \times G, \end{aligned} \tag{1}$$

with initial data

$$\rho(0) = \rho_0, \quad u(0) = u_0, \quad c(0) = c_0, \quad G \tag{2}$$

and boundary conditions

$$\begin{aligned} \text{non-slip:} &\quad u = 0 \\ \text{pure-slip:} &\quad (u|\nu) = 0, \quad \mathcal{Q}(\nu)\mathcal{S} \cdot \nu = 0, \quad \mathcal{Q} := \mathcal{I} - \nu \otimes \nu \\ \text{Dirichlet:} &\quad c = 0 \\ \text{Neumann:} &\quad (\nabla c|\nu) = 0 \end{aligned} \tag{3}$$

NSCH (PROPOSED BY LOWENGRUB & TRUSKINOVSKY)

- phase field (mass fraction) $c : J \times \overline{G} \to [0, 1]$
- fluid moves with velocity $u : J \times \overline{G} \to \mathbb{R}^3$
- Helmholtz energy density $\psi = \psi(\rho, c, \phi)$, $\phi := |\nabla c|^2$
- $\rho_1 = c\rho$, $\rho_2 = (1 - c)\rho$, ρ - total mass density,
  ρ_j satisfy mass balance equation

$$\partial_t \rho_j + \nabla \cdot (\rho_j u) + \nabla \cdot \mathcal{J}_j = 0$$

with

$$\mathcal{J}_1 + \mathcal{J}_2 = 0 \quad \Rightarrow \quad \partial_t \rho + \nabla \cdot (\rho u) = 0$$

suppose that $\mathcal{J} := \mathcal{J}_1$ is given by (Fick's law)

$$\mathcal{J} = m \nabla \mu$$

with mobility m and generalised chemical potential μ,

$$\rho\mu = \partial_c(\rho\psi) - \nabla \cdot (2\rho\partial_\phi \psi \nabla c).$$

NSCH: MATHEMATICAL PROBLEM

Let $J = [0, T]$ and $G \subset \mathbb{R}^n$ be a domain (with C^2 boundary Γ). Consider the compressible Navier-Stokes-Allen-Cahn system

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0, && J \times G, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (\mathcal{S} + \mathcal{P}) &= \rho f_{ext}, && J \times G, \\ \partial_t (c\rho) + \nabla \cdot (c\rho u) - \nabla \cdot (m \nabla \mu) &= 0, && J \times G, \\ \partial_c (\rho \psi) - \nabla \cdot (2\rho \partial_\phi \psi \nabla c) &= \rho \mu, && J \times G, \end{aligned} \tag{4}$$

with initial data

$$\rho(0) = \rho_0, \quad u(0) = u_0, \quad c(0) = c_0 \tag{5}$$

and boundary conditions

$$\begin{aligned} \text{non-slip:} \quad & u = 0 \\ \text{pure-slip:} \quad & (u|\nu) = 0, \quad \mathcal{Q}(\nu)\mathcal{S} \cdot \nu = 0, \quad \mathcal{Q} := \mathcal{I} - \nu \otimes \nu \\ \text{Neumann:} \quad & (\nabla \mu | \nu) = 0, \quad (\nabla c | \nu) = 0 \end{aligned} \tag{6}$$

Features/Advantages/Differences

classical description/sharp interfaces:

- problem with merging, reconnecting, and hitting interfaces
- contact angle conditions, jump conditions of stress tensor across interface, quasilinear evolution equations on boundary

diffuse interface models NSAC/NSCH:

- interfaces described by phase field (molecular mixing near interfaces)
- capillary effects inherent in additional stress tensor;
- Korteweg case: density serves order parameter;
  van der Waals pressure law is needed ("unphysical" pressure law)

DIFFICULTIES (IN CASE OF STRONG SOLUTIONS)

- system of hyperbolic-parabolic equations
- quasilinear partial differential equations (even for constant coefficients)
  - $\rho\partial_t u, \quad \nabla\cdot(\rho\partial_\phi\psi\nabla c)$;
  - $\nabla\cdot(\nabla c\otimes\rho\partial_\phi\psi\nabla c)$ – quasilinearity of highest order, similar to quasilinear elliptic operators: $\nabla\cdot(a(\nabla v)\nabla v)$;
- $n \geq 1$ – dimension of the problem
- strong coupled equations
- avoid vacuum, e.g. in case of NSAC

$$\rho\partial_t u + \rho\nabla u\cdot u - \nabla\cdot\mathcal{S}(u) + \nabla(\rho^2\partial_\rho\psi) + \nabla\cdot(2\rho\partial_\phi\psi\nabla c\otimes\nabla c) = \rho f_{ext}$$

$$\rho\partial_t c + \rho u\cdot\nabla c - \nabla\cdot(2\rho\partial_\phi\psi\nabla c) + \partial_c(\rho\psi) = 0$$

Typical situation/Configurations

Interested in Helmholtz energy of the form

$$\rho\psi(\rho, c, \phi) = \rho[c\psi_1 + (1-c)\psi_2] + \rho\Big(W(c) + \frac{\delta}{2}\phi\Big), \quad \phi = |\nabla c|^2.$$

- convex combination of energies ψ_1 and ψ_2 of the pure phases
- δ - a measure of thickness for the interface
- typically: W - double-well potential, e.g.

$$W(c) = k_1[c\ln(c) + (1-c)\ln(1-c)] + k_2 c(1-c), \quad k_1, k_2 \in \mathbb{R}$$

- $\partial_{\nabla c}(\rho\psi) = 2\rho\partial_\phi(\rho\psi)\nabla c = \rho\delta\nabla c$ and

$$\mathcal{P} = -\rho^2\partial_\rho\psi\,\mathcal{I} - \delta\rho\nabla c \otimes \nabla c,$$
$$\rho\mu = \partial_c(\rho\psi) - \nabla\cdot(\rho\delta\nabla c).$$

- depending on the choice of ψ_j different kinds of fluid mixtures are modelled (two compressible mixtures, one compressible and one incompressible fluid, two incompressible fluids)

LITERATURE

1. NSAC modelling: Truskinovsky (1993), Blesgen (1999)
2. NSCH modelling: Lowengrub, Truskinovsky (1998)
3. well-posedness results to NSAC:
   - Feireisl et al. (2010): existence of global weak solutions in case of $\delta\rho = 1$, i.e.
     $$\mathcal{P} = -\pi\mathcal{I} - (\nabla c \otimes \nabla c - \tfrac{1}{2}|\nabla c|^2\mathcal{I})$$
     Problem: Energy estimates do not provide any bound for ∇c in vacuum zones.
   - Shijin Ding, and Yinghua Li and Wanglong Luo (2012), Global solutions in 1D
   - Alt & Witterstein (2011), sharp interface limits
   - K. (2012), strong well-posedness to NSAC
4. well-posedness results to NSCH:
   - Abels & Feireisl, global weak solution for a simplified model
   - many articles to incompressible NSCH
   - K., Zacher (2013), strong well-posedness to the original model

2^{nd} LAW OF THERMODYNAMICS

Considering the non-isothermal counterparts of NSAC and NSCH we have

THEOREM

The thermodynamically closed systems of NSAC and NSCH are thermodynamically and mechanically consistent and the following equations hold:

$$\int_G \partial_t(\rho s)\,dx = \int_G \beta \left|\frac{\nabla\theta}{\theta}\right|^2 dx + \int_G \tfrac{1}{\theta}\mathcal{S} : \mathcal{D}\,dx + \int_G \tfrac{\epsilon}{\rho}\,|\mathcal{J}|^2\,dx,$$

$$\int_G \partial_t(\rho s)\,dx = \int_G \beta \left|\frac{\nabla\theta}{\theta}\right|^2 dx + \int_G \tfrac{1}{\theta}\mathcal{S} : \mathcal{D}\,dx + \int_G \tfrac{1}{m}|\mathcal{J}|^2 dx.$$

θ - temperature, β - heat conducting coefficient, s - entropy density

Well-posedness of NSAC: setting

We are looking for strong solutions in the L_p-setting. Consider first the equation for c in $\mathrm{L}_p(J; \mathrm{L}_p(G))$:

$$\partial_t(\rho c) + \nabla \cdot (c\rho u) - \nabla \cdot (\rho\delta\nabla c) + \partial_c(\rho\psi) = 0$$

$$\Leftrightarrow$$

$$\rho\partial_t c + \rho\nabla c \cdot u - \nabla \cdot (\rho\delta\nabla c) + \partial_c(\rho\psi) = 0$$

The natural regularity class is

$$c \in \mathrm{H}^1_p(J; \mathrm{L}_p(G)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G)).$$

Notice: $\nabla\rho$ occurs in the Allen-Cahn equation, i.e. we need at least

$$\rho \in \mathrm{L}_p(J; \mathrm{H}^1_p(G)).$$

Setting NSAC

Consider the Navier-Stokes equation in $\mathrm{L}_p(J;\mathrm{L}_p(G;\mathbb{R}^n))$:

$$\begin{aligned}
&\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) - \nabla\cdot\mathcal{S} - \nabla\cdot\mathcal{P} = \rho f_{ext}\\
&\nabla\cdot\mathcal{P} \sim \nabla^2 c \quad \text{and again} \quad \nabla\rho\\
&\nabla\cdot\mathcal{S} \sim \nabla^2 u,\ \nabla c,\ \nabla\rho
\end{aligned}$$

This is compatible with the regularity of c, since $c\in\mathrm{L}_p(J;\mathrm{H}^2_p(G))$ and thus

$$\nabla\cdot\mathcal{P}\sim\nabla^2 c\in\mathrm{L}_p(J;\mathrm{L}_p(G;\mathbb{R}^n)).$$

The natural regularity class for u is:

$$u\in\mathrm{H}^1_p(J;\mathrm{L}_p(G;\mathbb{R}^n))\cap\mathrm{L}_p(J;\mathrm{H}^2_p(G;\mathbb{R}^n)).$$

Note: need again $\rho\in\mathrm{L}_p(J;\mathrm{H}^1_p(G))$.

Setting NSAC

Since ρ is governed by the hyperbolic equation

$$\partial_t \rho + \nabla\rho \cdot u = -\rho\nabla \cdot u,$$

we need $u \in \mathrm{L}_p(J; \mathrm{H}_p^2(G; \mathbb{R}^n)) \cap \mathrm{L}_1(J; \mathrm{C}^1(\overline{G}))$. Recall

$$u \in \mathrm{H}_p^1(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(G; \mathbb{R}^n)) =: Z_1(J)$$

and the embedding

$$\mathrm{H}_p^2(G; \mathbb{R}^n) \hookrightarrow \mathrm{C}^1(\overline{G}; \mathbb{R}^n), \quad p > n.$$

Using this regularity the continuity equation yields

$$\rho \in \mathrm{C}^1(J; \mathrm{L}_p(G)) \cap \mathrm{C}(J; \mathrm{H}_p^1(G)).$$

Setting NSAC

Seek solutions (u, c, ρ) in the regularity class $Z_1(J) \times Z_2(J) \times Z_3(J)$,

$$Z_1(J) = \mathrm{H}^1_p(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G; \mathbb{R}^n)),$$
$$Z_2(J) = \mathrm{H}^1_p(J; \mathrm{L}_p(G)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G)),$$
$$Z_3(J) = \mathrm{C}^1(J; \mathrm{L}_p(G)) \cap \mathrm{C}(J; \mathrm{H}^1_p(G)).$$

Well-posedness result: NSAC

Theorem (K., ARMA 2012)

Let $G \subset \mathbb{R}^n$ *be a bounded domain with* C^2 *boundary* Γ, $J_0 = [0, T_0]$, *and let* $p > n + 2$. *Assume that*

(I) μ, λ, δ, ψ_1, ψ_2, $\mathcal{W}$ *sufficiently smooth;*

(II) $\mu(\rho, c) > 0$, $2\mu(\rho, c) + \lambda(\rho, c) > 0$, $\delta(c) > 0$ *for all* $(\rho, c) \in (0, \infty)^2$*;*

(III) $f_{ext} \in \mathrm{L}_p(J_0; \mathrm{L}_p(G; \mathbb{R}^n))$*;*

(IV) $w_0 := (u_0, c_0, \rho_0) \in V$,

$$V := \{(u, c, \rho) \in \mathrm{W}_p^{2-2/p}(G; \mathbb{R}^n) \times \mathrm{W}_p^{2-2/p}(G) \times \mathrm{H}_p^1(G) : 0 < \rho(x), \quad 0 \le c(x) \le 1, \ \forall x \in \overline{G}, \quad (u(y)|\nu(y)) \ge 0, \quad \forall y \in \Gamma\};$$

(V) *compatibility conditions.*

Then there exists $0 < T \le T_0$ *such that the NSAC-system has a unique solution* $w := (u, c, \rho) \in Z_1(J) \times Z_2(J) \times Z_3(J)$ *on* $J = [0, T]$.
The map $w_0 \to w(t)$ *generates a local semiflow on* $V_c := \{\phi \in V : \ \phi$ *satisfies (V)*$\}$.

WELL-POSEDNESS OF NSCH: THE SETTING

We are looking for strong solutions in the L_p-setting. Consider first the Cahn-Hilliard equation in $\mathrm{L}_p(J;\mathrm{L}_p(G))$:

$$\begin{aligned} \rho\partial_t c + \rho\nabla c \cdot u - \nabla \cdot (m\nabla\mu) &= 0, \\ \partial_c(\rho\psi) - \nabla \cdot (\rho\delta\nabla c) &= \rho\mu. \end{aligned}$$

The natural regularity class is

$$c \in \mathrm{H}^1_p(J;\mathrm{L}_p(G)) \cap \mathrm{L}_p(J;\mathrm{H}^4_p(G)).$$

Notice: $\nabla^3\rho$ occurs in the Allen-Cahn equation, i.e. we need at least

$$\rho \in \mathrm{L}_p(J;\mathrm{H}^3_p(G)).$$

Setting in case of NSCH

ρ is governed by the hyperbolic equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0.$$

Usually,

$$u \in \mathrm{L}_1(J; \mathrm{C}^1(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G; \mathbb{R}^n))$$
$$\Downarrow$$
$$\rho \in \mathrm{C}^1(J; \mathrm{L}_p(G)) \cap \mathrm{C}(J; \mathrm{H}^1_p(G)).$$

Cahn-Hilliard eq.: ρ has to be in $\mathrm{L}_p(J; \mathrm{H}^3_p(G))$ at least. We therefore need

$$u \in \mathrm{L}_p(J; \mathrm{H}^4_p(G; \mathbb{R}^n)).$$

One can prove

$$u \in \mathrm{L}_1(J; \mathrm{C}^1(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^4_p(G; \mathbb{R}^n))$$
$$\Downarrow$$
$$\rho \in \mathrm{C}^1(J; \mathrm{H}^2_p(G)) \cap \mathrm{C}(J; \mathrm{H}^3_p(G)).$$

Setting in case of NSCH

Bear in mind that we need to have

$$u \in \mathrm{L}_1(J; \mathrm{C}^1(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^4_p(G; \mathbb{R}^n)).$$

To obtain this regularity for u, we consider the Navier-Stokes equation in $\mathrm{L}_p(J; \mathrm{H}^2_p(G; \mathbb{R}^n))$:

$$\rho\partial_t u + \rho\nabla u \cdot u - \nabla \cdot \mathcal{S} = \nabla \cdot \mathcal{P} + \rho f_{ext}$$
$$\nabla \cdot \mathcal{P} \sim \partial_{x_i}\nabla c, \ \nabla\rho$$

Note that

$$c \in \mathrm{H}^1_p(J; \mathrm{L}_p(G)) \cap \mathrm{L}_p(J; \mathrm{H}^4_p(G)) \hookrightarrow \mathrm{H}^{1/2}_p(J; \mathrm{H}^2_p(G))$$
$$\Downarrow$$
$$\partial_{x_i}\nabla c \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G; \mathbb{R}^n)).$$

Regularity of u implies $\nabla\rho \in \mathrm{C}^1(J; \mathrm{H}^1_p(G; \mathbb{R}^n)) \cap \mathrm{C}(J; \mathrm{H}^2_p(G; \mathbb{R}^n))$. Therefore

$$\nabla \cdot \mathcal{P} \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(G; \mathbb{R}^n)) =: X_1.$$

Setting in case of NSCH

Taking $X_1 = \mathrm{H}_p^{1/2}(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(G; \mathbb{R}^n))$ as the base space for the Navier-Stokes equation one expects that

$$u \in \mathrm{H}_p^{3/2}(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{H}_p^1(J; \mathrm{H}_p^2(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(G; \mathbb{R}^n)).$$

Using this regularity the continuity equation yields

$$\rho \in \mathrm{H}_p^{2+1/4}(J; \mathrm{L}_p(G)) \cap \mathrm{C}_p^1(J; \mathrm{H}_p^2(G)) \cap \mathrm{C}(J; \mathrm{H}_p^3(G)).$$

Seek solutions (u, c, ρ) in

Regularity class

$$\begin{aligned}
Z(J) &= Z_1(J) \times Z_2(J) \times Z_3(J), \\
Z_1(J) &= \mathrm{H}_p^{3/2}(J; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{H}_p^1(J; \mathrm{H}_p^2(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(G; \mathbb{R}^n)), \\
Z_2(J) &= \mathrm{H}_p^1(J; \mathrm{L}_p(G)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(G)), \\
Z_3(J) &= \mathrm{H}_p^{2+1/4}(J; \mathrm{L}_p(G)) \cap \mathrm{C}_p^1(J; \mathrm{H}_p^2(G)) \cap \mathrm{C}(J; \mathrm{H}_p^3(G)).
\end{aligned}$$

Well-posedness result: NSCH

Theorem (K., Zacher 2013)

Let $G \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary Γ, $J_0 = [0, T_0]$, and let $p > p^* := \max\{4, n\}$, $p \neq 5$. Assume that

(I) $\rho\psi = \overline{\psi}(\rho, c) + \rho(W(c) + \frac{\delta}{2}|\nabla c|^2)$;

(II) η, λ, δ, m, $\overline{\psi}$, W *sufficiently smooth;*

(III) $\eta(\rho, c)$, $2\eta(\rho, c) + \lambda(\rho, c)$, $\delta(\rho, c)$, $m(\rho, c) > 0$ *for all* $(\rho, c) \in (0, \infty)^2$;

(IV) $f_{ext} \in X_1 = \mathrm{H}_p^{1/2}(J_0; \mathrm{L}_p(G; \mathbb{R}^n)) \cap \mathrm{L}_p(J_0; \mathrm{H}_p^2(G; \mathbb{R}^n))$;

(V)

$$w_0 := (u_0, c_0, \rho_0) \in V := \{(u, c, \rho) \in \mathrm{W}_p^{4-2/p}(G; \mathbb{R}^n) \times \mathrm{W}_p^{4-4/p}(G) \times \mathrm{H}_p^3(G) : 0 < \rho(x)\ \forall x \in \overline{G}\};$$

(VI) *compatibility conditions (There are many of them.).*

Then there exists $0 < T \leq T_0$ such that the NSCH-system has a unique solution $w := (u, c, \rho) \in Z(J)$ on $J = [0, T]$. The map $w_0 \to w(t)$ generates a local semiflow on $V_c := \{\phi \in V : \phi$ satisfies (VI)$\}$.

Outline of the proof

1. Suppose $\rho_0 \in \mathrm{H}^3_p(G)$, $p > p^*$, $\rho_0 > 0$ in $\overline{G}$, and $(u|\nu) \geq 0$ on $[0, T] \times \Gamma$. Then the continuity equation, together with $\rho_{|t=0} = \rho_0$ and $u \in Z_1([0, T])$, has a unique positive solution.

$$\Rightarrow \rho = L[u]\rho_0 \in Z_3([0, T]).$$

2. Insert $\rho = L[u]\rho_0$ into momentum and phase field equations. (nonlocal, fully nonlinear)
3. Find appropriate fixed point formulation and subset $\Sigma \subset Z_1(J) \times Z_2(J)$.
4. Contraction mapping principle
5. Problem with contraction:
   - Contraction cannot be proved in $Z_1(J) \times Z_2(J)$. (loss of regularity!)
   - Well-known in the theory of symmetric quasilinear hyperbolic systems.
   - Kato/Lax resolve this problem by studying contraction in a larger space.
   - Find an appropriate space $\mathcal{W}([0, T])$, such that $Z_1(J) \times Z_2(J) \subset \mathcal{W}([0, T])$, contraction in $\mathcal{W}(J)$, $\forall \{\Phi_n\} \subset \Sigma$ with $\Phi_n \to \Phi$ in $\mathcal{W}(J) \Rightarrow \Phi \in \Sigma$

Lack of regularity

Let $\rho_i = L[u_i]\rho_0$ with $u_i \in Z_1(J)$, i.e.

$$\partial_t \rho_i + \nabla \rho_i \cdot u_i = -\rho_i \nabla \cdot u_i, \quad (t,x) \in J \times G,$$
$$\rho_i(0) = \rho_0, \quad x \in G,$$

and $\rho_i \in C^1(J; L_p(G)) \cap C(J; H_p^3(G))$.

Then $(\rho_1 - \rho_2, u_1 - u_2) =: (q, v)$ satisfies

$$\partial_t q + \nabla q \cdot u_1 = -q \nabla \cdot u_1 - \rho_2 \nabla \cdot v - \nabla \rho_2 \cdot v =: F,$$
$$\varrho(0) = 0.$$

We need $F \in C(J; H_p^3(G))$ to get $q \in C^1(J; L_p(G)) \cap C(J; H_p^3(G))$.

Note:
1. $\rho_i \nabla \cdot u_i, q \nabla \cdot u_1 \in C(J; H_p^3(G))$
2. $\nabla \rho_2 \cdot v \in C(J; H_p^2(G))$

Conclusion: $\rho_1 - \rho_2 \notin C^1(J; L_p(G)) \cap C(J; H_p^3(G))$ and thus

$$\|\rho_1 - \rho_2\|_{C^1(J;L_p(G)) \cap C(J;H_p^3(G))} \not\leq k \|u_1 - u_2\|_{Z_1(J)}, \quad 0 < k < 1.$$

Basic ideas: fixed point equation

Linearize quasilinear terms (freeze coefficients):

$$
\begin{aligned}
\rho_0 \partial_t u + \mathcal{A}u + \mathcal{B}c + \mathcal{C}\mu &= F_1(u, c, \mu, L[u]\rho_0), && J \times G,\\
\rho_0 \partial_t c - \nabla \cdot (m_0 \nabla \mu) &= F_2(u, c, \mu, L[u]\rho_0), && J \times G,\\
-\mu - \nabla \cdot (\delta_0 \nabla c) &= F_3(c, L[u]\rho_0), && J \times G,\\
u = 0, \quad \partial_\nu \mu = 0, \quad \partial_\nu c &= 0, && J \times \Gamma,\\
u(0) = u_0, \quad c(0) &= c_0, && G,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}u &:= -\nabla \cdot (2\mu_0 \mathcal{D}(u) + \lambda_0 \nabla \cdot u \mathcal{I}),\\
\mathcal{B}c &:= [\rho_0^2 \partial_\rho \delta_0 + \rho_0 \delta_0] \nabla c_0 \cdot \nabla^2 c,\\
\mathcal{C}\mu &:= -\rho_0 \nabla c_0 \mu.
\end{aligned}
$$

Linearization maintains divergence structure! More abstractly,

$$
\begin{aligned}
\mathcal{L}_{11} u + \mathcal{L}_{12}(c, \mu) &= \mathcal{F}_1(u, c, \mu),\\
\mathcal{L}_{22}(c, \mu) &= \mathcal{F}_2(u, c, \mu),
\end{aligned}
$$

$\mathcal{F}_i$ – nonlocal, fully nonlinear.

Basic ideas: fixed point equation

1. Define
   $\Lambda : \mathcal{Z}([0,T]) := Z_1([0,T]) \times Z_2([0,T]) \times Z_\mu([0,T]) \to \mathcal{Z}([0,T])$ by
   $\Lambda(u,c,\mu) := (\widehat{u}, \widehat{c}, \widehat{\mu})$,

$$\begin{aligned} \mathcal{L}_{11}\widehat{u} + \mathcal{L}_{12}(\widehat{c},\widehat{\mu}) &= \mathcal{F}_1(u,c,\mu), \\ \mathcal{L}_{22}(\widehat{c},\widehat{\mu}) &= \mathcal{F}_2(u,c,\mu). \end{aligned}$$

2. Define subset $\Sigma \subset \mathcal{Z}([0,T])$. For $T \in (0,T_0)$ and $r \in (0,1)$ let

$$\begin{aligned} \Sigma := \{(u,c,\mu) \in \mathcal{Z}([0,T]) :\ & (u, \partial_t u, c, \mu)_{|t=0} = (u_0, u_\bullet, c_0, \mu_0), \\ & u = 0,\ \partial_\nu c = \partial_\nu \mu = 0 \text{ on } \Gamma, \quad \|(\overline{u},\overline{c},\overline{\mu}) - (u,c,\mu)\|_{\mathcal{Z}([0,T])} \le r\}. \end{aligned}$$

   Σ – closed ball in $\mathcal{Z}([0,T])$ with centre $(\overline{u},\overline{c},\overline{\mu})$ and radius r.
3. Reference function $(\overline{u},\overline{c},\overline{\mu}) \in \mathcal{Z}(J_0)$, $J_0 = [0,T_0]$, is given as solution of

$$\begin{aligned} \mathcal{L}_{11}\overline{u} + \mathcal{L}_{12}(\overline{c},\overline{\mu}) &= \mathcal{F}_1(u_0,c_0,\mu_0), \\ \mathcal{L}_{22}(\overline{c},\overline{\mu}) &= \mathcal{F}_2(u_0,c_0,\mu_0). \end{aligned}$$

Basic ideas: things to do

Fixed point mapping: $\Lambda(u, c, \mu) := (\widehat{u}, \widehat{c}, \widehat{\mu})$,

$$\begin{aligned} \mathcal{L}_{11}\widehat{u} + \mathcal{L}_{12}(\widehat{c}, \widehat{\mu}) &= \mathcal{F}_1(u, c), \\ \mathcal{L}_{22}(\widehat{c}, \widehat{\mu}) &= \mathcal{F}_2(u, c). \end{aligned} \tag{7}$$

1. Λ is well-defined: $\exists!(\widehat{u}, \widehat{c}, \widehat{\mu}) \in \mathcal{Z}([0, T])$ of (7).
2. For sufficiently small T and r:
   - (I) Λ leaves Σ invariant.
   - (II) Λ is a strict contraction in $\mathcal{W}([0, T]) := W_1([0, T]) \times W_2([0, T]) \times W_3([0, T])$ with

$$\begin{aligned} W_1(J) &:= \mathrm{H}_2^{5/4}(J; \mathrm{L}_2(G)) \cap \mathrm{H}_2^{1/2}(J; \mathrm{H}_p^2(G; \mathbb{R}^n)) \cap \mathrm{L}_2(J; \mathrm{H}_2^3(G; \mathbb{R}^n)), \\ W_2(J) &:= \mathrm{H}_2^{3/4}(J; \mathrm{L}_2(G)) \cap \mathrm{L}_2(J; \mathrm{H}_2^3(G)), \\ W_3(J) &:= \mathrm{H}_2^{1/4}(J; \mathrm{L}_2(G)) \cap \mathrm{L}_2(J; \mathrm{H}_2^1(G)). \end{aligned}$$

There holds, e.g.,

$$Z_2([0, T]) := \mathrm{H}_p^1([0, T]; \mathrm{L}_p(G)) \cap \mathrm{L}_p([0, T]; \mathrm{H}_p^4(G)) \hookrightarrow W_2([0, T]).$$

   - (III) Σ is closed in $W_1([0, T]) \times W_2([0, T])$.

Basic ideas: some auxiliary results

1. The system

$$\begin{aligned} \mathcal{L}_{11}\widehat{u} + \mathcal{L}_{12}(\widehat{c}, \widehat{\mu}) &= f_1, \\ \mathcal{L}_{22}(\widehat{c}, \widehat{\mu}) &= f_2, \end{aligned}$$

has maximal L_p-regularity, i.e.

$$\exists!(\widehat{u}, \widehat{c}, \widehat{\mu}) \in \mathcal{Z}([0,T]) \quad \Leftrightarrow \quad f_1, f_2 \text{ have certain regularity.}$$

Use: The problem decouples.
Solve second equation, $(\widehat{c}, \widehat{\mu}) = \mathcal{L}_{22}^{-1} f_2$. Solve equation for $\widehat{u}$

$$\begin{aligned} \mathcal{L}_{11}\widehat{u} + \mathcal{L}_{12}(\widehat{c}, \widehat{\mu}) &= f_1 \\ &\Updownarrow \\ \mathcal{L}_{11}\widehat{u} &= f_3 \end{aligned}$$

with $f_3 := f_1 - \mathcal{L}_{12}(\widehat{c}, \widehat{\mu}) = f_1 - \mathcal{L}_{12}\mathcal{L}_{22}^{-1} f_2$.

$$\Rightarrow \quad \Lambda \quad \text{is well-defined.}$$

Basic ideas: some auxiliary results

2. Selfmapping means: Let $(u,c,\mu)\in\Sigma$. Then $(\widehat{u},\widehat{c},\widehat{\mu})=\Lambda(u,c,\mu)$ has to satisfy

$$\|(\widehat{u},\widehat{c},\widehat{\mu})-(\overline{u},\overline{c},\overline{\mu})\|_{\mathcal{Z}([0,T])}\leq r.$$

The trick is

$$(\widehat{u},\widehat{c},\widehat{\mu})=\begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ 0 & \mathcal{L}_{22}\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{F}_1(u,c,\mu)\\ \mathcal{F}_2(u,c,\mu)\end{pmatrix}$$

$$(\overline{u},\overline{c},\overline{\mu})=\begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ 0 & \mathcal{L}_{22}\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{F}_1(u_0,c_0,\mu_0)\\ \mathcal{F}_2(u_0,c_0,\mu_0)\end{pmatrix}$$

and

$$\begin{aligned}\|(\widehat{u},\widehat{c},\widehat{\mu})-(\overline{u},\overline{c},\overline{\mu})\|_{\mathcal{Z}([0,T])}\leq M\Big(&\|\mathcal{F}_1(u,c,\mu)-\mathcal{F}_1(u_0,c_0,\mu_0)\|_{X_1([0,T])}\\ &+\|\mathcal{F}_2(u,c,\mu)-\mathcal{F}_1(u_0,c_0,\mu_0)\|_{X_2([0,T])}\Big).\end{aligned}$$

Note: only estimates of differences in space of data,
$\rho=L[u]\rho\in\mathrm{C}([0,T];\mathrm{H}^3_p(G))$ behaves as lower order term

Basic ideas: some auxiliary results

3. Contraction in $\mathcal{W}([0,T])$ - crucial part.

Recall $\Lambda(u,c,\mu) = (\widehat{u},\widehat{c},\widehat{\mu})$,

$$\begin{aligned}\mathcal{L}_{11}\widehat{u} + \mathcal{L}_{12}(\widehat{c},\widehat{\mu}) &= \mathcal{F}_1(u,c,\mu),\\ \mathcal{L}_{22}(\widehat{c},\widehat{\mu}) &= \mathcal{F}_2(u,c,\mu).\end{aligned}$$

Let $(u_i,c_i,\mu_i) \in \Sigma$, $\rho_i = L[u_i]\rho_0$, and $(\widehat{u}_i,\widehat{c}_i,\widehat{\mu}_i) = \Lambda(u_i,c_i,\mu_i)$, $i = 1,2$.
Then $(\widehat{u}_1 - \widehat{u}_2, \widehat{c}_1 - \widehat{c}_2, \widehat{\mu}_1 - \widehat{\mu}_2)$ satisfies

$$\begin{aligned}\mathcal{L}_{11}(\widehat{u}_1 - \widehat{u}_2) + \mathcal{L}_{12}(\widehat{c}_1 - \widehat{c}_2, \widehat{\mu}_1 - \widehat{\mu}_2) &= \mathcal{F}_1(u_1,c_1,\mu_1) - \mathcal{F}_1(u_2,c_2,\mu_2),\\ \mathcal{L}_{22}(\widehat{c}_1 - \widehat{c}_2, \widehat{\mu}_1 - \widehat{\mu}_2) &= \mathcal{F}_2(u_1,c_2,\mu_2) - \mathcal{F}_2(u_2,c_2,\mu_2).\end{aligned}$$

One can prove

$$\begin{aligned}&\|(\widehat{u_1},\widehat{c}_1,\widehat{\mu_1}) - (\widehat{u_2},\widehat{c}_2,\widehat{\mu}_2)\|_{\mathcal{W}([0,T])}\\ &\qquad\qquad \le \kappa(T,r)\|(u_1,c_1,\mu_1) - (u_2,c_2,\mu_2)\|_{\mathcal{W}([0,T])},\end{aligned}$$

where $\kappa(T,r) \to 0$ as $T, r \to 0$.
Use: divergence structure, weak formulation, max. reg. methods, and

$$\|\rho_1 - \rho_2\|_{C([0,T];H^2_2(G))} \le C_1 T^{1/2}\|u_1 - u_2\|_{L_2([0,T];H^3_2(G;\mathbb{R}^n))}.$$

Thank you.