Non-isothermal two phase flows of incompressible fluids

Elisabetta Rocca – joint work with M. Eleuteri (Milano) and G. Schimperna (Pavia)

Supported by the FP7-IDEAS-ERC-StG Grant “EntroPhase”
Outline

- The motivation
- The PDEs (equations and inequalities)
- The modelling
- The analytical results in 3D - [Eleuteri, R., Schimperna, WIAS preprint no. 1920 (2014)]
- The expected improvements in 2D
- Some open related problems
The motivation

- A non-isothermal model for the flow of a mixture of two
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density

- Avoid problems related to interface singularities
  \[ \implies \text{use a diffuse interface model} \]
  \[ \implies \text{the classical sharp interface replaced by a thin interfacial region} \]

- A partial mixing of the macroscopically immiscible fluids is allowed
  \[ \implies \phi \text{ is the order parameter, e.g. the concentration difference} \]

- The original idea of diffuse interface model for fluids: Hohenberg and Halperin, ’77
  \[ \implies \text{H-model} \]
  Later, Gurtin et al., ’96: continuum mechanical derivation based on microforces

- Models of two-phase or two-component fluids are receiving growing attention (e.g., Abels, Boyer, Garcke, Grün, Grasselli, Lowengrub, Truskinovski, ...)
The main aim of our contribution [Eleuteri, R., Schimpena, in preparation]

- Including temperature dependence is a widely open issue
  Difficulties: getting models which are at the same time thermodynamically consistent and mathematically tractable

- Our idea: a weak formulation of the system as a combination of total energy balance plus entropy production inequality \( \Rightarrow \) “Entropic formulation”

This method has been recently proposed by [Bulíček-Málek-Feireisl, ’09] for the Navier-Stokes-Fourier system and has been proved to be effective to study e.g.

- nonisothermal models for phase transitions ([Feireisl-Petzeltová-R., ’09]) and
- the evolution of nematic liquid crystals ([Frémont, Feireisl, R., Schimperna, Zarnescu, ’12,’13])
The state variables and physical assumptions

- We want to describe the behavior of a mixture of two incompressible fluids of the same density in terms of the following state variables
  - \( u \): macroscopic velocity (Navier-Stokes),
  - \( p \): pressure (Navier-Stokes),
  - \( \varphi \): order parameter (Cahn-Hilliard),
  - \( \mu \): chemical potential (Cahn-Hilliard),
  - \( \theta \): absolute temperature (Entropic formulation).

- We do not neglect convection and capillarity effects. We assume constant mobility and smooth configuration potential in Cahn-Hilliard. We take temperature dependent coefficients wherever possible. We assume the system being insulated from the exterior.
The PDEs (equations and inequalities)

- A weak form of the momentum balance
  \[ u_t + u \cdot \nabla x u + \nabla x p = \text{div}(\nu(\theta) Du) - \text{div}(\nabla x \varphi \otimes \nabla x \varphi), \quad \text{div} u = 0; \]

- The Cahn-Hilliard system in $H^1(\Omega)'$
  \[ \varphi_t + u \cdot \nabla x \varphi = \Delta \mu, \quad \mu = -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} F'(\varphi) - \theta; \]

- A weak form of the total energy balance
  \[
  \partial_t \left( \frac{1}{2} |u|^2 + e \right) + u \cdot \nabla x \left( \frac{1}{2} |u|^2 + e \right) + \text{div} \left( p u + q - S u \right) \\
  - \text{div} \left( \varepsilon \varphi_t \nabla x \varphi + \mu \nabla x \mu \right) = 0 \quad \text{where} \quad e = \frac{1}{\varepsilon} F(\varphi) + \frac{\varepsilon}{2} |\nabla x \varphi|^2 + \int_1^\theta c_v(s) \, ds;
  \]

- The weak form of the entropy production inequality
  \[
  (\Lambda'(\theta) + \varphi)_t + u \cdot \nabla x (\Lambda(\theta)) + u \cdot \nabla x \varphi - \text{div} \left( \frac{\kappa(\theta) \nabla x \theta}{\theta} \right) \\
  \geq \frac{\nu(\theta)}{\theta} |Du|^2 + \frac{1}{\theta} |\nabla x \mu|^2 + \frac{\kappa(\theta)}{\theta^2} |\nabla x \theta|^2, \quad \text{where} \quad \Lambda'(\theta) = \int_1^\theta \frac{c_v(s)}{s} \, ds.
  \]
We start by specifying two functionals:

- the **free energy** $\Psi$, related to the equilibrium state of the material, and
- the **dissipation pseudo-potential** $\Phi$, describing the processes leading to dissipation of energy (i.e., transformation into heat)

Then we impose the balances of **momentum**, **configuration energy**, and both of **internal energy** and of **entropy**, in terms of these functionals

The **thermodynamical consistency** of the model is then a direct consequence of the solution notion
The **total free energy** is given as a function of the state variables \( E = (\theta, \varphi, \nabla_x \varphi) \)

\[
\Psi(E) = \int_{\Omega} \psi(E) \, dx, \quad \psi(E) = f(\theta) - \theta \varphi + \frac{\varepsilon}{2} |\nabla_x \varphi|^2 + \frac{1}{\varepsilon} F(\varphi)
\]

- \( f(\theta) \) is related to the specific heat \( c_v(\theta) = Q'(\theta) \) by \( Q(\theta) = f(\theta) - \theta f'(\theta) \). In our case we need \( c_v(\theta) \sim c_\delta \theta^\delta \) for some \( \delta \in (1/2, 1) \)

- \( \varepsilon > 0 \) is related to the interfacial thickness

- we need \( F(\varphi) \) to be the classical smooth double well potential \( F(\varphi) \sim \frac{1}{4}(\varphi^2 - 1)^2 \)
Modelling: the dissipation potential

The **dissipation potential** is taken as function of $\delta E = (D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x \theta)$ and $E$

$$
\Phi(\delta E, E) = \int_\Omega \phi(D\mathbf{u}, \nabla_x \theta) \, dx + \left\langle \frac{D\varphi}{Dt}, J^{-1} \frac{D\varphi}{Dt} \right\rangle
$$

$$
= \int_\Omega \left( \frac{\nu(\theta)}{2} |D\mathbf{u}|^2 + I_{\{0\}}(\text{div } \mathbf{u}) + \frac{\kappa(\theta)}{2\theta} |\nabla \theta|^2 \right) \, dx + \left\| \frac{D\varphi}{Dt} \right\|_{H_1^1(\Omega)'}^2
$$

- $D\mathbf{u} = (\nabla_x \mathbf{u} + \nabla_t^t \mathbf{u})/2$ the symmetric gradient
- $\frac{D(\cdot)}{Dt} = (\cdot)_t + \mathbf{u} \cdot \nabla_x (\cdot)$ the material derivative
- $J : H_1^1(\Omega) \to H_1^1(\Omega)'$ the **Riesz isomorphism**

$$
\langle Ju, v \rangle := ((u, v))_{H_1^1(\Omega)} := \int_\Omega \nabla_x u \cdot \nabla_x v \, dx,
$$

$$
H_1^1(\Omega) = \{ \xi \in H^1(\Omega) : \bar{\xi} := |\Omega|^{-1} \int_\Omega \xi \, dx = 0 \}
$$

- $\nu = \nu(\theta) > 0$ the viscosity coefficient, $\kappa = \kappa(\theta) > 0$ the heat conductivity
- **Incompressibility**: $I_0$ the indicator function of $\{0\}$: $I_0 = 0$ if $\text{div } \mathbf{u} = 0$, $+\infty$ otherwise
The dissipation potential was taken as

$$\Phi = \Phi \left( D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x \theta \right) = \int_{\Omega} \phi(D\mathbf{u}, \nabla_x \theta) \, dx + \left< \frac{D\varphi}{Dt}, J^{-1} \frac{D\varphi}{Dt} \right>$$

If a time-dependent set of variables is given such that

- a.e. in $(0, T)$, $\Psi$ and $\Phi$ are finite
- $\mathbf{u}$ is such that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\Gamma$
- $\varphi$ satisfies the mass conservation constraint $\varphi(t, x) = \varphi(0, x) = \varphi_0(x)$ a.e.

then $\mathbf{u}$ is divergence-free and we get

$$\int_{\Omega} \frac{D\varphi}{Dt} = \int_{\Omega} (\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) \, dx = 0$$

Then we can set $\mu_#:=-J^{-1} \frac{D\varphi}{Dt}$, so that $\frac{D\varphi}{Dt} = -J \mu_# = \Delta \mu_#$ and we get

$$\Phi(\delta E, E) = \int_{\Omega} \tilde{\phi}(\delta E, E) \, dx, \quad \text{where} \quad \tilde{\phi}(\delta E, E) = \phi(\delta E, E) + \frac{1}{2} |\nabla_x \mu_#|^2$$
It is obtained (at least for no-flux b.c.'s) as the following gradient-flow problem

\[ \partial_{L^2_#(\Omega), \frac{D\varphi}{Dt}} \Phi + \partial_{L^2_#(\Omega), \varphi_#} \Psi = 0 \]

where \( L^2_#(\Omega) = \{ \xi \in L^2(\Omega) : \bar{\xi} := |\Omega|^{-1} \int_{\Omega} \xi \, dx = 0 \} \), \( \varphi_# = \varphi - \bar{\varphi}_0 \)

Combining the previous relations we then get

\[ J^{-1} \left( \frac{D\varphi}{Dt} \right) = \varepsilon \Delta \varphi - \frac{1}{\varepsilon} \left( F'(\varphi) - F'(\bar{\varphi}) \right) + \theta - \bar{\theta}, \quad \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma, \quad \varphi(t) = \bar{\varphi}_0 \]

Applying the distributional Laplace operator to both hand sides and noting that

\[ -\Delta J^{-1} v = v \text{ for any } v \in L^2_#(\Omega), \]

we then arrive at the Cahn-Hilliard system with Neumann hom. b.c. for \( \mu \) and \( \varphi \)

\[ \frac{D\varphi}{Dt} = \Delta \mu, \quad \mu = -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} F'(\varphi) - \theta, \quad \frac{\partial \varphi}{\partial n} = \frac{\partial \mu}{\partial n} = 0 \text{ on } \Gamma \]  

(CahnHill)

where the auxiliary variable \( \mu \) takes the name of chemical potential.
The **Navier-Stokes system** is obtained as a momentum balance by setting

\[
\frac{Du}{Dt} = u_t + \text{div}(u \otimes u) = \text{div} \sigma,
\]  

(momentum)
The **Navier-Stokes system** is obtained as a momentum balance by setting

\[
\frac{Du}{Dt} = u_t + \text{div}(u \otimes u) = \text{div} \left( \sigma^d + \sigma^{nd} \right),
\]

where the stress \(\sigma\) is split into its

- **dissipative part**

\[
\sigma^d := \frac{\partial \phi}{\partial Du} = \nu(\theta) Du - p I, \quad \text{div } u = 0,
\]

representing kinetic energy which **dissipates** (i.e. is transformed into heat) due to viscosity, and its

- **non-dissipative part** \(\sigma^{nd}\) to be determined later in agreement with Thermodynamics
Nonlocal internal energy balance

The balance of internal energy takes the form

\[
\frac{De}{Dt} + \text{div } q = \nu(\theta)|Du|^2 + \sigma^{nd} : Du + B \frac{D\varphi}{Dt} + \frac{\partial \psi}{\partial \nabla_x \varphi} \cdot \nabla_x \frac{D\varphi}{Dt} + [N]
\]

where \(e = \psi - \theta \psi_\theta\), \(B = B^{nd} + B^d\) and

\[
B^{nd} = \frac{\partial \psi}{\partial \varphi} = \frac{1}{\varepsilon} F'(\varphi) - \theta, \quad B^d = \partial_{L^2(\Omega)} \Phi = J^{-1} \left( \frac{D\varphi}{Dt} \right)
\]

On the right hand side there appears a new (with respect to the standard theory of [FRÉMOND, ’02]) term \([N]\) balancing the nonlocal dependence of the last term in the pseudopotential of dissipation \(\Phi\)

\[
\Phi = \Phi \left( Du, \frac{D\varphi}{Dt} \right) = \int_\Omega \phi \, dx + \left\langle \frac{D\varphi}{Dt}, J^{-1} \frac{D\varphi}{Dt} \right\rangle
\]

It will result from the Second Principle of Thermodynamics that \(\int_\Omega N(x) \, dx = 0\), in agreement with natural expectations.
The Second Law of Thermodynamics

To deduce the expressions for $\sigma^{nd}$ and $N$, we impose validity of the Clausius-Duhem inequality in the form

$$\theta \left( \frac{Ds}{Dt} + \text{div} \left( \frac{q}{\theta} \right) \right) \geq 0$$

where $e = \psi + \theta s$, being $s = -\psi_\theta$ the entropy density and we get

$$\sigma^{nd} = -\varepsilon \nabla_x \varphi \otimes \nabla_x \varphi, \quad N = \frac{1}{2} \Delta (\mu - \bar{\mu})^2$$

and the internal energy balance can be rewritten as

$$(Q(\theta))_t + \mathbf{u} \cdot \nabla_x Q(\theta) + \theta \frac{D\varphi}{Dt} - \text{div}(\kappa(\theta) \nabla_x \theta) = \nu(\theta) |D\mathbf{u}|^2 + |\nabla_x \mu|^2$$

where $Q(\theta) = f(\theta) - \theta f'(\theta)$ and $Q'(\theta) = c_v(\theta)$

The dissipation terms on the right hand side are in perfect agreement with $\Phi$

$$\Phi = \int_\Omega \tilde{\phi} \, dx, \quad \text{where} \quad \tilde{\phi} = \phi + \frac{1}{2} |\nabla_x \mu|^2$$
Following [BULÍČEK, FEIREISL, & MÁLEK], we replace the pointwise internal energy balance by the **total energy balance**

\[
(\partial_t + u \cdot \nabla_x) \left( \frac{|u|^2}{2} + e \right) + \text{div} \left( pu - \kappa(\theta) \nabla_x \theta - (\nu(\theta) Du) u \right)
\]

\[
= \text{div} \left( \varphi_t \nabla_x \varphi + \mu \nabla_x \mu \right)
\]

(energy)

with the internal energy

\[
e = F(\varphi) + \frac{1}{2} |\nabla_x \varphi|^2 + Q(\theta) \quad Q'(\theta) = c_v(\theta)
\]

and the **entropy inequality**

\[
(\Lambda(\theta) + \varphi)_t + u \cdot \nabla_x (\Lambda(\theta) + \varphi) - \text{div} \left( \frac{\kappa(\theta) \nabla_x \theta}{\theta} \right)
\]

\[
\geq \frac{\nu(\theta)}{\theta} |Du|^2 + \frac{1}{\theta} |\nabla_x \mu|^2 + \frac{\kappa(\theta)}{\theta^2} |\nabla_x \theta|^2,
\]

where \( \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s} ds \sim \theta^\delta \)
The PDEs (equations and inequalities)

- a weak form of the momentum balance (in distributional sense)
  \[ u_t + u \cdot \nabla u + \nabla p = \text{div}(\nu(\theta) Du) - \text{div}(\nabla \varphi \otimes \nabla \varphi), \quad \text{div } u = 0; \]

- the Cahn-Hilliard system in \( H^1(\Omega)' \)
  \[ \varphi_t + u \cdot \nabla \varphi = \Delta \mu, \quad \mu = -\Delta \varphi + F'(\varphi) - \theta; \]

- a weak form of the total energy balance (in distributional sense)
  \[ \partial_t \left( \frac{1}{2} |u|^2 + e \right) + u \cdot \nabla \left( \frac{1}{2} |u|^2 + e \right) + \text{div} \left( pu + q - S u \right) \]
  \[ - \text{div} \left( \varphi_t \nabla \varphi + \mu \nabla \mu \right) = 0 \quad \text{where } \quad e = F(\varphi) + \frac{1}{2} |\nabla \varphi|^2 + \int_1^\theta c_v(s) \, ds; \]

- the weak form of the entropy production inequality
  \[ (\Lambda(\theta) + \varphi)_t + u \cdot \nabla \left( \Lambda(\theta) \right) + u \cdot \nabla \varphi - \text{div} \left( \frac{\kappa(\theta) \nabla \theta}{\theta} \right) \]
  \[ \geq \frac{\nu(\theta)}{\theta} |Du|^2 + \frac{1}{\theta} |\nabla \mu|^2 + \frac{\kappa(\theta)}{\theta^2} |\nabla \theta|^2, \quad \text{where } \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s} \, ds. \]
Assumptions on the data and boundary conditions

In order to get a tractable system in 3D, we need to specify assumptions on coefficients in a careful way:

- The viscosity $\nu(\theta)$ is assumed smooth and bounded
- The specific heat $c_v(\theta) \sim \theta^\delta, \ 1/2 < \delta < 1$
- The heat conductivity $\kappa(\theta) \sim 1 + \theta^\beta, \ \beta \geq 2$
- The potential $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$

Concerning B.C.’s, our results are proved for no-flux conditions for $\theta$, $\varphi$, and $\mu$ and complete slip conditions for $u$

$$u \cdot n|_\Gamma = 0 \quad \text{(the fluid cannot exit } \Omega, \text{ it can move tangentially to } \Gamma)$$
$$[Sn] \times n|_\Gamma = 0, \quad \text{where } S = \nu(\theta) Du \quad \text{(exclude friction effects with the boundary)}$$

They can be easily extended to the case of periodic B.C.’s for all unknowns
Existence of global in time solutions

**Theorem**

We can prove existence of at least one global in time weak solution \((u, \varphi, \mu, \theta)\)

\[
\begin{align*}
    u &\in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T; V_n) \\
    \varphi &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega)') \\
    \mu &\in L^2(0, T; H^1(\Omega)) \cap L^{\frac{14}{5}}((0, T) \times \Omega) \\
    \theta &\in L^\infty(0, T; L^{\delta+1}(\Omega)) \cap L^{\beta}(0, T; L^{3\beta}(\Omega)) \cap L^2(0, T; H^1(\Omega)) \\
    \theta &> 0 \ a.e. \ in \ (0, T) \times \Omega, \ \log \theta \in L^2(0, T; H^1(\Omega))
\end{align*}
\]

to system given by (momentum), (CahnHill), (entropy) and (energy), in distributional sense and for finite-energy initial data

\[
\begin{align*}
    u_0 &\in L^2(\Omega), \ \text{div} \ u_0 = 0, \ \varphi_0 \in H^1(\Omega), \ \theta_0 \in L^{\delta+1}(\Omega), \ \theta_0 > 0 \ a.e.
\end{align*}
\]
A priori bounds

- Existence proof based on a classical **a-priori estimates** – **compactness** scheme
- The basic information is contained in the **energy** and **entropy** relations
- Note that the **power-like** growth of the heat conductivity and of the specific heat is required in order to provide sufficient **summability** of the temperature

**Is this sufficient to pass to the limit?**

- The **total energy balance** contains some nasty extra terms $\varphi_t \nabla_x \varphi + \mu \nabla_x \mu$. In particular, $\varphi_t$ lies only in some **negative order** space (cf. (CahnHill))
- Using (CahnHill) and integrating by parts carefully the bad terms transform into

  $$- \Delta \mu^2 + \text{div} \left( (u \cdot \nabla_x \varphi) \nabla_x \varphi \right) + \text{div} (\nabla_x \mu \cdot \nabla_x \nabla_x \varphi) - \text{div} \text{div} (\nabla_x \mu \otimes \nabla_x \varphi)$$

- The above terms can be controlled by getting some **extra-integrability** of $\varphi$ and $\mu$ from (CahnHill). To this aim having a **“smooth” potential $F$** is crucial!
What’s better in 2D?

- Is it possible to say something more in the 2D-case?

- In particular, it would be interesting to see if one might deal with “strong” solutions. Moreover, we would like to drop some restriction on coefficients.

- Let us make one test: in 2D the “extra stress” \( \text{div}(\nabla_x \varphi \otimes \nabla_x \varphi) \) in (momentum)

\[
\begin{align*}
    u_t + u \cdot \nabla_x u + \nabla_x p &= \text{div}(D\!u) - \text{div}(\nabla_x \varphi \otimes \nabla_x \varphi)
\end{align*}
\]

lies in \( L^2 \) as a consequence of the estimates.

- Hence, there is hope to get extra-regularity for constant viscosity \( \nu \) (i.e., independent of temperature).

- Indeed we get

\[
\begin{align*}
    u_t &\in L^2(0, T; L^2(\Omega)) \text{ and } u \in L^{\infty}(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))
\end{align*}
\]
Assumptions in 2D

- Constant viscosity $\nu = 1$
- Constant specific heat $c_v = 1$ (in other words, $f(\theta) = -\theta \log \theta$)
- Power-like conductivity (for simplicity $\kappa(\theta) = \theta^2$)
- Periodic boundary conditions
Main result in 2D

Theorem

We can prove existence of at least one "strong" solution to system given by

\[
\begin{align*}
\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p &= \text{div}(D\mathbf{u}) - \text{div} (\nabla_x \varphi \otimes \nabla_x \varphi) \quad \text{(mom)} \\
\varphi_t + \mathbf{u} \cdot \nabla_x \varphi &= \Delta \mu \quad \text{(CH1)} \\
\mu &= -\Delta \varphi + F'(\varphi) - \theta \quad \text{(CH2)} \\
\theta_t + \mathbf{u} \cdot \nabla_x \theta + \theta (\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) - \Delta \theta^3 &= |D\mathbf{u}|^2 + |\nabla_x \mu|^2 \quad \text{(heat)}
\end{align*}
\]

for finite-energy initial data, namely

\[
\begin{align*}
\mathbf{u}_0 &\in H^1_{\text{per}}(\Omega), \quad \text{div} \mathbf{u}_0 = 0, \\
\varphi_0 &\in H^3_{\text{per}}(\Omega), \\
\theta_0 &\in H^1_{\text{per}}(\Omega), \quad \theta_0 > 0 \ \text{a.e.}, \quad \log \theta_0 \in L^1(\Omega)
\end{align*}
\]
2D: Troubles

- Is the proof just a standard regularity argument? **NO!**
- The main issue is the estimation of $|\nabla_x \mu|^2$ in (heat). From the previous a-priori estimate, this is **only in $L^1$**.
- If one differentiates the Cahn-Hilliard system:
  - (CH1)$_t \times (\nabla)^{-1} \varphi_t$
  
  $$\varphi_{tt} + u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t = \Delta \mu_t \times (\nabla)^{-1} \varphi_t$$
  
  - plus (CH2)$_t \times \varphi_t$
  
  $$\mu_t = -\Delta \varphi_t + F''(\varphi) \varphi_t - \theta_t \times \varphi_t$$
  
  then one faces the term $\theta_t \varphi_t$ and no estimate is available for $\theta_t$
- Only possibility, to **test** (heat) **by** $\varphi_t$
  
  $$\theta_t + u \cdot \nabla \theta + \theta(\varphi_t + u \cdot \nabla \varphi) - \Delta \theta^3 = |D u|^2 + |\nabla_x \mu|^2 \times \varphi_t$$
  
  to let it disappear
The main estimate

- Try $(CH1)_t \times (-\Delta)^{-1} \varphi_t$

\[ \varphi_{tt} + \mathbf{u}_t \cdot \nabla_x \varphi + \mathbf{u} \cdot \nabla_x \varphi_t = \Delta \mu_t \times (-\Delta)^{-1} \varphi_t \]

- plus $(CH2)_t \times \varphi$

\[ \mu_t = -\Delta \varphi_t + F''(\varphi) \varphi_t - \theta_t \times \varphi \]

- plus (heat) $\times (\theta^3 + \varphi_t)$

\[ \theta_t + \mathbf{u} \cdot \nabla_x \theta + \theta (\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) - \Delta \theta^3 = |D\mathbf{u}|^2 + |\nabla_x \mu|^2 \times (\theta^3 + \varphi_t) \]

- getting

\[ \frac{d}{dt} \left( \| \nabla_x \mu \|^2_{L^2} + \| \theta \|^4_{L^4} \right) + \| \varphi_t \|^2_{H^1} + \| \theta^3 \|^2_{H^1} \leq c \int \Omega |\nabla_x \mu|^2 |\varphi_t + \theta^3| \, dx + \text{l.o.t.} \]

where l.o.t. can be easily handled
Having the inequality
\[
\frac{d}{dt} \left( \| \nabla x \mu \|^2_{L^2} + \| \theta \|^4_{L^4} \right) + \| \varphi_t \|^2_{H^1} + \| \theta^3 \|^2_{H^1} \leq c \int_\Omega |\nabla x \mu|^2 |\varphi_t + \theta^3| \, dx + \text{l.o.t.}
\]

one has now to deal with $|\nabla x \mu|^2 |\varphi_t + \theta^3|$

The only way to control it seems the following one:
\[
\int_\Omega \| \varphi_t + \theta^3 \| |\nabla x \mu|^2 \leq \| \varphi_t + \theta^3 \|_{H^1} \| |\nabla x \mu|^2 \|_{(H^1)'}
\]

In 2D we have that $L^p \subset (H^1)'$ for all $p > 1$. But, then, one goes on with
\[
\leq \epsilon \| \varphi_t + \theta^3 \|^2_{H^1} + c \epsilon \| \nabla x \mu \|^4_{L^2 p}
\]

which is bad!
The main idea: a dual Yudovich trick and a regularity estimate

- We know, however, that
  \[ \| v \|_{L^q} \leq c q^{1/2} \| v \|_{H^1} \quad \text{for all } v \in H^1(\Omega), \ q < \infty \]

- Passing to the dual inequality, we infer
  \[ \| \xi \|_{(H^1)^*} \leq c q^{1/2} \| \xi \|_{L^p} \quad \text{for all } \xi \in L^p(\Omega), \ p > 1, \ q = p^* \]

- Interpolating and optimizing w.r.t. \( q \), we arrive at
  \[ \| \xi \|_{(H^1)^*} \leq c \| \xi \|_{L^1} \left( 1 + \log^{1/2} \| \xi \|_{L^2} \right) \quad \text{for all } \xi \in L^2(\Omega) \]

Applying the above to \( \xi = |\nabla_x \mu|^2 \), we get a differential inequality of the form

\[ \frac{d}{dt} \left( \| \nabla_x \mu \|_{L^2}^2 + \| \theta \|_{L^4}^4 \right) + \| \varphi_t \|_{H^1}^2 + \| \theta^3 \|_{H^1}^2 \leq c \| \nabla_x \mu \|_{L^2}^2 \left( \| \nabla_x \mu \|_{L^2}^2 \log \| \nabla_x \mu \|_{L^2}^2 \right) + \ldots \]

Hence, we get a global estimate thanks to a (generalized) Gronwall lemma.
Work in progress and further developments

- **Uniqueness in 2D**

- Convergence to **equilibria** in 2D. Existence of **attractors**

- **Allen-Cahn-type** models

- **Singular potentials** in Cahn-Hilliard (or Allen-Cahn)

- Non-isothermal **nonlocal** models