Weierstrass Institute for
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Non-isothermal two phase flows of incompressible fluids

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Outline

- The motivation
- The PDEs (equations and inequalities)
- The modelling
- The analytical results in 3D - [Eleuteri, R., Schimperna, WIAS preprint no. 1920 (2014)]
- The expected improvements in 2D
- Some open related problems

- A **non-isothermal** model for the flow of a **mixture of two**
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density
- Avoid problems related to interface singularities

  $\Longrightarrow$ use a **diffuse interface model**

  $\Longrightarrow$ the classical sharp interface replaced by a **thin interfacial region**
- A partial mixing of the macroscopically immiscible fluids is allowed

  $\Longrightarrow$ φ **is the order parameter**, e.g. the concentration difference
- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77

  $\Longrightarrow$ **H-model**

  Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

- **Including temperature dependence is a widely open issue**

  Difficulties: getting models which are at the same time *thermodynamically consistent* and *mathematically tractable*

- Our idea: a weak formulation of the system as a combination of *total energy balance* plus *entropy production inequality* $\Longrightarrow$ **"Entropic formulation"**

- This method has been recently proposed by [BULÍČEK-MÁLEK-FEIREISL, '09] for the Navier-Stokes-Fourier system and has been proved to be effective to study e.g.
  - nonisothermal models for **phase transitions** ([FEIREISL-PETZELTOVÁ-R., '09]) and
  - the evolution of **nematic liquid crystals** ([FRÉMOND, FEIREISL, R., SCHIMPERNA, ZARNESCU, '12,'13])

- We want to describe the behavior of a mixture of two incompressible fluids of the same density in terms of the following state variables
  - $\mathbf{u}$: macroscopic **velocity** (Navier-Stokes),
  - p: **pressure** (Navier-Stokes),
  - φ: **order parameter** (Cahn-Hilliard),
  - μ: **chemical potential** (Cahn-Hilliard),
  - θ: **absolute temperature** (Entropic formulation).
- We do not neglect convection and capillarity effects. We assume constant mobility and smooth configuration potential in Cahn-Hilliard. We take temperature dependent coefficients wherever possible. We assume the system being insulated from the exterior.

- a **weak form of the momentum balance**

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x\mathbf{u} + \nabla_x p = \operatorname{div}(\nu(\theta)D\mathbf{u}) - \operatorname{div}(\nabla_x\varphi\otimes\nabla_x\varphi), \quad \operatorname{div}\mathbf{u} = 0;$$

- the **Cahn-Hilliard system** in $H^1(\Omega)'$

$$\varphi_t + \mathbf{u}\cdot\nabla_x\varphi = \Delta\mu, \quad \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta;$$

- a **weak form of the total energy balance**

$$\partial_t\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \operatorname{div}\left(p\mathbf{u} + \mathbf{q} - \mathbb{S}\mathbf{u}\right)$$

$$-\operatorname{div}\left(\varepsilon\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) = 0 \quad \text{where} \quad e = \frac{1}{\varepsilon}F(\varphi) + \frac{\varepsilon}{2}|\nabla_x\varphi|^2 + \int_1^\theta c_v(s)\,\mathrm{d}s;$$

- the weak form of the **entropy production inequality**

$$(\Lambda(\theta) + \varphi)_t + \mathbf{u}\cdot\nabla_x(\Lambda(\theta)) + \mathbf{u}\cdot\nabla_x\varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right)$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s.$$

- We start by specifying two functionals:
  - the **free energy** Ψ, related to the equilibrium state of the material, and
  - the **dissipation pseudo-potential** Φ, describing the processes leading to dissipation of energy (i.e., transformation into heat)

- Then we impose the balances of **momentum**, **configuration energy**, and both of **internal energy** and of **entropy**, in terms of these functionals

- The **thermodynamical consistency** of the model is then a direct consequence of the solution notion

The **total free energy** is given as a function of the state variables $E = (\theta, \varphi, \nabla_x \varphi)$

$$\Psi(E) = \int_\Omega \psi(E)\,\mathrm{d}x, \quad \psi(E) = f(\theta) - \theta\varphi + \frac{\varepsilon}{2}|\nabla_x \varphi|^2 + \frac{1}{\varepsilon}F(\varphi)$$

- $f(\theta)$ is related to the specific heat $c_v(\theta) = Q'(\theta)$ by $Q(\theta) = f(\theta) - \theta f'(\theta)$. In our case we need $c_v(\theta) \sim c_\delta \theta^\delta$ for some $\delta \in (1/2, 1)$
- $\varepsilon > 0$ is related to the interfacial thickness
- we need $F(\varphi)$ to be the classical smooth double well potential $F(\varphi) \sim \frac{1}{4}(\varphi^2 - 1)^2$

The **dissipation potential** is taken as function of $\delta E = (D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x\theta)$ and E

$$\Phi(\delta E, E) = \int_\Omega \phi(D\mathbf{u}, \nabla_x\theta)\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt} \right\rangle$$
$$= \int_\Omega \left(\frac{\nu(\theta)}{2}|D\mathbf{u}|^2 + I_{\{0\}}(\mathrm{div}\,\mathbf{u}) + \frac{\kappa(\theta)}{2\theta}|\nabla_x\theta|^2 \right)\mathrm{d}x + \left\| \frac{D\varphi}{Dt} \right\|^2_{H^1_\#(\Omega)'}$$

- $D\mathbf{u} = (\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u})/2$ the symmetric gradient
- $\frac{D(\cdot)}{Dt} = (\cdot)_t + \mathbf{u}\cdot\nabla_x(\cdot)$ the material derivative
- $J : H^1_\#(\Omega) \to H^1_\#(\Omega)'$ the **Riesz isomorphism**
  $\langle Ju, v\rangle := ((u,v))_{H^1_\#(\Omega)} := \int_\Omega \nabla_x u\cdot\nabla_x v\,\mathrm{d}x,$
  $H^1_\#(\Omega) = \{\xi \in H^1(\Omega) \ : \ \overline{\xi} := |\Omega|^{-1}\int_\Omega \xi\,\mathrm{d}x = 0\}$
- $\nu = \nu(\theta) > 0$ the viscosity coefficient, $\kappa = \kappa(\theta) > 0$ the heat conductivity
- **Incompressibility**: I_0 the indicator function of $\{0\}$: $I_0 = 0$ if $\mathrm{div}\,\mathbf{u} = 0$, $+\infty$ otherwise)

- The **dissipation potential** was taken as

$$\Phi = \Phi\left(D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x \theta\right) = \int_\Omega \phi(D\mathbf{u}, \nabla_x \theta)\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt}\right\rangle$$

- If a time-dependent set of variables is given such that
  - a.e. in $(0,T)$, Ψ and Φ are finite
  - $\mathbf{u}$ is such that $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ
  - φ satisfies the **mass conservation constraint** $\varphi(t,x) = \varphi(0,x) = \varphi_0(x)$ a.e.

  then $\mathbf{u}$ is divergence-free and we get

$$\int_\Omega \frac{D\varphi}{Dt} = \int_\Omega (\varphi_t + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x = 0$$

- Then we can set $\mu_\# := -J^{-1}\frac{D\varphi}{Dt}$, so that $\frac{D\varphi}{Dt} = -J\mu_\# = \Delta\mu_\#$ and we get

$$\Phi(\delta E, E) = \int_\Omega \widetilde{\phi}(\delta E, E)\,\mathrm{d}x, \quad \text{where } \widetilde{\phi}(\delta E, E) = \phi(\delta E, E) + \frac{1}{2}|\nabla_x \mu_\#|^2$$

- It is obtained (at least for no-flux b.c.'s) as the following **gradient-flow problem**

$$\partial_{L^2_\#(\Omega),\frac{D\varphi}{Dt}}\Phi + \partial_{L^2_\#(\Omega),\varphi_\#}\Psi = 0$$

where $L^2_\#(\Omega) = \{\xi \in L^2(\Omega)\ :\ \overline{\xi} := |\Omega|^{-1}\int_\Omega \xi\,\mathrm{d}x = 0\}$, $\varphi_\# = \varphi - \overline{\varphi_0}$

- Combining the previous relations we then get

$$J^{-1}\left(\frac{D\varphi}{Dt}\right) = \varepsilon\Delta\varphi - \frac{1}{\varepsilon}\left(F'(\varphi) - \overline{F'(\varphi)}\right) + \theta - \overline{\theta}, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma, \ \overline{\varphi}(t) = \overline{\varphi_0}$$

- Applying the distributional Laplace operator to both hand sides and noting that $-\Delta J^{-1}v = v$ for any $v \in L^2_\#(\Omega)$, we then arrive at **the Cahn-Hilliard system with Neumann hom. b.c. for μ and φ**

$$\frac{D\varphi}{Dt} = \Delta\mu, \quad \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = \frac{\partial\mu}{\partial\mathbf{n}} = 0 \text{ on } \Gamma \qquad \text{(CahnHill)}$$

where the auxiliary variable μ takes the name of *chemical potential*

The **Navier-Stokes system** is obtained as a momentum balance by setting

$$\frac{D\mathbf{u}}{Dt} = \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) = \operatorname{div} \sigma, \qquad \text{(momentum)}$$

The **Navier-Stokes system** is obtained as a momentum balance by setting

$$\frac{D\mathbf{u}}{Dt} = \mathbf{u}_t + \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) = \mathrm{div}\,(\sigma^d + \sigma^{nd}), \qquad \text{(momentum)}$$

where the stress σ is split into its

- **dissipative part**

$$\sigma^d := \frac{\partial \phi}{\partial D\mathbf{u}} = \nu(\theta) D\mathbf{u} - p\mathbb{I}, \quad \mathrm{div}\,\mathbf{u} = 0,$$

  representing kinetic energy which **dissipates** (i.e. is transformed into heat) due to viscosity, and its

- **non-dissipative part** σ^{nd} to be determined later in agreement with Thermodynamics

The balance of internal energy takes the form

$$\frac{De}{Dt} + \operatorname{div}\mathbf{q} = \nu(\theta)|D\mathbf{u}|^2 + \sigma^{nd} : D\mathbf{u} + B\frac{D\varphi}{Dt} + \frac{\partial\psi}{\partial\nabla_x\varphi}\cdot\nabla_x\frac{D\varphi}{Dt} + \boxed{N}$$

where $e = \psi - \theta\psi_\theta$, $B = B^{nd} + B^d$ and

$$B^{nd} = \frac{\partial\psi}{\partial\varphi} = \frac{1}{\varepsilon}F'(\varphi) - \theta, \quad B^d = \partial_{L^2_\#(\Omega),\frac{D\varphi}{Dt}}\Phi = J^{-1}\left(\frac{D\varphi}{Dt}\right)$$

On the right hand side there appears a new (with respect to the standard theory of [Frémond, '02]) term $\boxed{N}$ balancing the **nonlocal dependence of the last term in the pseudopotential of dissipation** Φ

$$\Phi = \Phi\left(D\mathbf{u}, \frac{D\varphi}{Dt}\right) = \int_\Omega \phi\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt}\right\rangle$$

It will result from the Second Principle of Thermodynamics that $\int_\Omega N(x)\,\mathrm{d}x = 0$, in agreement with natural expectations

To deduce the expressions for σ^{nd} and N, we impose validity of the **Clausius-Duhem inequality** in the form

$$\theta\Big(\frac{Ds}{Dt}+\operatorname{div}\Big(\frac{\mathbf{q}}{\theta}\Big)\Big)\geq 0$$

where $e=\psi+\theta s$, being $s=-\psi_\theta$ the entropy density and we get

$$\sigma^{nd}=-\varepsilon\nabla_x\varphi\otimes\nabla_x\varphi,\quad N=\frac{1}{2}\Delta(\mu-\overline{\mu})^2$$

and the internal energy balance can be rewritten as

$$(Q(\theta))_t+\mathbf{u}\cdot\nabla_x Q(\theta)+\theta\frac{D\varphi}{Dt}-\operatorname{div}(\kappa(\theta)\nabla_x\theta)=\nu(\theta)|D\mathbf{u}|^2+|\nabla_x\mu|^2$$

where $Q(\theta)=f(\theta)-\theta f'(\theta)$ and $Q'(\theta)=:c_v(\theta)$

The **dissipation** terms on the right hand side are in perfect agreement with Φ

$$\Phi=\int_\Omega\widetilde{\phi}\,\mathrm{d}x,\quad \text{where } \widetilde{\phi}=\phi+\frac{1}{2}|\nabla_x\mu|^2$$

Following [BULÍČEK, FEIREISL, & MÁLEK], we replace the pointwise internal energy balance by the **total energy balance**

$$(\partial_t + \mathbf{u}\cdot\nabla_x)\left(\frac{|\mathbf{u}|^2}{2} + e\right) + \operatorname{div}\left(p\mathbf{u} - \kappa(\theta)\nabla_x\theta - (\nu(\theta)D\mathbf{u})\mathbf{u}\right)$$

$$= \operatorname{div}\left(\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) \qquad \text{(energy)}$$

with the internal energy

$$e = F(\varphi) + \frac{1}{2}|\nabla_x\varphi|^2 + Q(\theta) \quad Q'(\theta) = c_v(\theta)$$

and the **entropy inequality**

$$(\Lambda(\theta)+\varphi)_t + \mathbf{u}\cdot\nabla_x(\Lambda(\theta)+\varphi) - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right) \qquad \text{(entropy)}$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s \sim \theta^\delta$$

- a **weak form of the momentum balance** (in distributional sense)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x \mathbf{u} + \nabla_x p = \operatorname{div}(\nu(\theta)D\mathbf{u}) - \operatorname{div}(\nabla_x\varphi\otimes\nabla_x\varphi), \quad \operatorname{div}\mathbf{u} = 0;$$

- the **Cahn-Hilliard system** in $H^1(\Omega)'$

$$\varphi_t + \mathbf{u}\cdot\nabla_x\varphi = \Delta\mu, \quad \mu = -\Delta\varphi + F'(\varphi) - \theta;$$

- a **weak form of the total energy balance** (in distributional sense)

$$\partial_t\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \operatorname{div}\left(p\mathbf{u} + \mathbf{q} - \mathbb{S}\mathbf{u}\right)$$

$$-\operatorname{div}\left(\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) = 0 \quad \text{where} \quad e = F(\varphi) + \frac{1}{2}|\nabla_x\varphi|^2 + \int_1^\theta c_v(s)\,\mathrm{d}s;$$

- the weak form of the **entropy production inequality**

$$(\Lambda(\theta) + \varphi)_t + \mathbf{u}\cdot\nabla_x(\Lambda(\theta)) + \mathbf{u}\cdot\nabla_x\varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right)$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s.$$

- In order to get a tractable system in 3D, we need to specify assumptions on coefficients in a careful way:
  - The **viscosity** $\nu(\theta)$ is assumed **smooth and bounded**
  - The **specific heat** $c_v(\theta) \sim \theta^\delta$, $1/2 < \delta < 1$
  - The **heat conductivity** $\kappa(\theta) \sim 1 + \theta^\beta$, $\beta \geq 2$
  - The potential $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$

- Concerning B.C.'s, our results are proved for **no-flux** conditions for θ, φ, and μ and **complete slip** conditions for $\mathbf{u}$

$$\mathbf{u} \cdot \mathbf{n}_{|\Gamma} = 0 \quad \text{(the fluid cannot exit } \Omega\text{, it can move tangentially to } \Gamma\text{)}$$

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}_{|\Gamma} = 0, \quad \text{where } \mathbb{S} = \nu(\theta) D\mathbf{u} \quad \text{(exclude friction effects with the boundary)}$$

They can be easily extended to the case of periodic B.C.'s for all unknowns

Theorem

We can prove existence of **at least** *one* **global in time weak solution** $(\mathbf{u}, \varphi, \mu, \theta)$

$$\mathbf{u} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0,T; \mathbf{V_n})$$
$$\varphi \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^3(\Omega)) \cap H^1(0,T; H^1(\Omega)')$$
$$\mu \in L^2(0,T; H^1(\Omega)) \cap L^{\frac{14}{5}}((0,T) \times \Omega)$$
$$\theta \in L^\infty(0,T; L^{\delta+1}(\Omega)) \cap L^\beta(0,T; L^{3\beta}(\Omega)) \cap L^2(0,T; H^1(\Omega))$$
$$\theta > 0 \ \textit{a.e. in}\ (0,T) \times \Omega, \quad \log\theta \in L^2(0,T; H^1(\Omega))$$

to system given by (momentum), (CahnHill), (entropy) *and* (energy)*, in distributional sense and for finite-energy initial data*

$$\mathbf{u}_0 \in L^2(\Omega),\ \ \operatorname{div}\mathbf{u}_0 = 0, \quad \varphi_0 \in H^1(\Omega), \quad \theta_0 \in L^{\delta+1}(\Omega), \quad \theta_0 > 0 \ \textit{a.e.}$$

- Existence proof based on a classical **a-priori estimates** – **compactness** scheme
- The basic information is contained in the **energy** and **entropy** relations
- Note that the power-like growth of the heat conductivity and of the specific heat is required in order to provide sufficient summability of the temperature

Is this sufficient to pass to the limit?

- The **total energy balance** contains some nasty extra terms $\varphi_t \nabla_x \varphi + \mu \nabla_x \mu$. In particular, φ_t lies only in some **negative order** space (cf. (CahnHill))
- Using (CahnHill) and integrating by parts carefully the bad terms tranform into

$$-\Delta\mu^2 + \operatorname{div}\big((\mathbf{u}\cdot\nabla_x\varphi)\nabla_x\varphi\big)$$

$$+\operatorname{div}(\nabla_x\mu\cdot\nabla_x\nabla_x\varphi) - \operatorname{div}\operatorname{div}(\nabla_x\mu\otimes\nabla_x\varphi)$$

- The above terms can be controlled by getting some **extra-integrability** of φ and μ from (CahnHill). To this aim having a **"smooth" potential** F is crucial!

- Is it possible to say something more in the **2D-case**?
- In particular, it would be interesting to see if one might deal with **"strong"** solutions. Moreover, we would like to drop some restriction on coefficients
- Let us make one test: in 2D the "extra stress" $\mathrm{div}(\nabla_x\varphi \otimes \nabla_x\varphi)$ **in** (momentum)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x\mathbf{u} + \nabla_x p = \mathrm{div}(D\mathbf{u}) - \mathrm{div}\left(\nabla_x\varphi \otimes \nabla_x\varphi\right)$$

lies in L^2 as a consequence of the estimates

- Hence, there is hope to get extra-regularity **for constant viscosity** ν (i.e., independent of temperature)
- Indeed we get
$\mathbf{u}_t \in L^2(0,T;L^2(\Omega))$ and $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$

Assumptions in 2D

- Constant viscosity $\nu = 1$
- Constant specific heat $c_v = 1$ (in other words, $f(\theta) = -\theta \log \theta$)
- Power-like conductivity (for simplicity $\kappa(\theta) = \theta^2$)
- Periodic boundary conditions

Theorem

We can prove existence of at least one **"strong" solution** *to system given by*

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x\mathbf{u} + \nabla_x p = \operatorname{div}(D\mathbf{u}) - \operatorname{div}\left(\nabla_x\varphi\otimes\nabla_x\varphi\right) \quad \text{(mom)}$$

$$\varphi_t + \mathbf{u}\cdot\nabla_x\varphi = \Delta\mu \quad \text{(CH1)}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta \quad \text{(CH2)}$$

$$\theta_t + \mathbf{u}\cdot\nabla_x\theta + \theta(\varphi_t + \mathbf{u}\cdot\nabla_x\varphi) - \Delta\theta^3 = |D\mathbf{u}|^2 + |\nabla_x\mu|^2 \quad \text{(heat)}$$

for finite-energy initial data, namely

$$\mathbf{u}_0 \in H^1_{\mathrm{per}}(\Omega),\ \ \operatorname{div}\mathbf{u}_0 = 0,$$
$$\varphi_0 \in H^3_{\mathrm{per}}(\Omega),$$
$$\theta_0 \in H^1_{\mathrm{per}}(\Omega),\quad \theta_0 > 0\ \textit{a.e.},\quad \log\theta_0 \in L^1(\Omega)$$

- Is the proof just a standard regularity argument? **NO!**
- The main issue is the estimation of $|\nabla_x \mu|^2$ in (heat). From the previous a-priori estimate, this is only in L^1
- If one differentiates the Cahn-Hilliard system:
  - (CH1)$_t$ $\times(-\Delta)^{-1}\varphi_t$

$$\varphi_{tt} + \mathbf{u}_t \cdot \nabla_x \varphi + \mathbf{u} \cdot \nabla_x \varphi_t = \Delta \mu_t \quad \boxed{\times(-\Delta)^{-1}\varphi_t}$$

  - plus (CH2)$_t$ $\times\varphi_t$

$$\mu_t = -\Delta \varphi_t + F''(\varphi)\varphi_t - \theta_t \quad \boxed{\times\varphi_t}$$

then one faces the term $\theta_t \varphi_t$ and no estimate is available for θ_t

- Only possibility, to **test** (heat) **by** φ_t

$$\theta_t + \mathbf{u} \cdot \nabla_x \theta + \theta(\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) - \Delta \theta^3 = |D\mathbf{u}|^2 + |\nabla_x \mu|^2 \quad \boxed{\times\varphi_t}$$

to let it disappear

- Try $(\text{CH1})_t \times (-\Delta)^{-1}\varphi_t$

$$\varphi_{tt} + \mathbf{u}_t \cdot \nabla_x \varphi + \mathbf{u} \cdot \nabla_x \varphi_t = \Delta \mu_t \quad \times (-\Delta)^{-1}\varphi_t$$

- plus $(\text{CH2})_t \times \varphi_t$

$$\mu_t = -\Delta \varphi_t + F''(\varphi)\varphi_t - \theta_t \quad \times \varphi_t$$

- plus (heat)$\times (\theta^3 + \varphi_t)$

$$\theta_t + \mathbf{u} \cdot \nabla_x \theta + \theta(\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) - \Delta \theta^3 = |D\mathbf{u}|^2 + |\nabla_x \mu|^2 \quad \times (\theta^3 + \varphi_t)$$

- getting

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\nabla_x \mu\|_{L^2}^2 + \|\theta\|_{L^4}^4 \right) + \|\varphi_t\|_{H^1}^2 + \|\theta^3\|_{H^1}^2 \leq c \int_\Omega |\nabla_x \mu|^2 |\varphi_t + \theta^3| \, \mathrm{d}x + \text{l.o.t.}$$

where l.o.t. can be easily handled

Having the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla_x\mu\|_{L^2}^2+\|\theta\|_{L^4}^4\right)+\|\varphi_t\|_{H^1}^2+\|\theta^3\|_{H^1}^2\le c\int_\Omega|\nabla_x\mu|^2|\varphi_t+\theta^3|\,\mathrm{d}x+\text{l.o.t.}$$

- one has now to deal with $|\nabla_x\mu|^2|\varphi_t+\theta^3|$
- The only way to control it seems the following one:

$$\int_\Omega|\varphi_t+\theta^3||\nabla_x\mu|^2\le\|\varphi_t+\theta^3\|_{H^1}\left\||\nabla_x\mu|^2\right\|_{(H^1)'}$$

- In 2D we have that $L^p\subset(H^1)'$ for all $p>1$. But, then, one goes on with

$$\le\epsilon\|\varphi_t+\theta^3\|_{H^1}^2+c_\epsilon\|\nabla_x\mu\|_{L^{2p}}^4$$

which is bad!

- We know, however, that

$$\|v\|_{L^q} \leq cq^{1/2}\|v\|_{H^1} \quad \text{for all } v \in H^1(\Omega),\ q < \infty$$

- Passing to the dual inequality, we infer

$$\|\xi\|_{(H^1)^*} \leq cq^{1/2}\|\xi\|_{L^p} \quad \text{for all } \xi \in L^p(\Omega),\ p > 1,\ q = p^*$$

- Interpolating and optimizing w.r.t. q, we arrive at

$$\|\xi\|_{(H^1)^*} \leq c\|\xi\|_{L^1}\left(1 + \log^{1/2}\|\xi\|_{L^2}\right) \quad \text{for all } \xi \in L^2(\Omega)$$

Applying the above to $\xi = |\nabla_x \mu|^2$, we get a differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla_x\mu\|_{L^2}^2 + \|\theta\|_{L^4}^4\right) + \|\varphi_t\|_{H^1}^2 + \|\theta^3\|_{H^1}^2 \leq c\|\nabla_x\mu\|_{L^2}^2\left(\|\nabla_x\mu\|_{L^2}^2 \log\|\nabla_x\mu\|_{L^2}^2\right) + \ldots$$

Hence, we get a **global estimate** thanks to a (generalized) Gronwall lemma

Work in progress and further developments

- Uniqueness in 2D 

- Convergence to equilibria in 2D. Existence of attractors 

- Allen-Cahn-type models 

- Singular potentials in Cahn-Hilliard (or Allen-Cahn) 

- Non-isothermal nonlocal models