Unique Continuation from Infinity for Linear Waves

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September 17, 2014
Section 1

Introduction
Problem Statement

Problem

- Consider a linear wave, i.e., solution of
  \[ L_g \phi := \Box_g \phi + a^\alpha D_\alpha \phi + V \phi = 0. \]

- To what extent does "data" for \( \phi \) at infinity (i.e., radiation field) determine \( \phi \) near infinity?
  - Does "vanishing at infinity" imply vanishing near infinity?
  - How does the geometry of the spacetime impact the answer?
  - Waves on various asymptotically flat spacetimes.
**Theorem Statement**

**Theorem**

- **Assume** $L_g$ **as before:**
  - $a^\alpha$, $V$ satisfies asymptotic bounds.
- **Assume** $(M, g)$ **is:**
  - Perturbation of Minkowski spacetime.
  - “Positive-mass spacetime” (including Schwarzschild and Kerr families).
- **Assume** $\phi$ **vanishes (at least to infinite order) on part of null infinity ($J^\pm$).**

*Then, $\phi$ vanishes in the interior near $J^\pm$.***
Some Remarks

- Linear wave equation can be replaced by an inequality:

\[ |\square g \phi| \leq |a| |D \phi| + |V| |\phi|. \]

- Important feature: applicable to nonlinear wave equations.
  - Previous example: general relativity and black hole uniqueness (Alexakis-Ionescu-Klainerman).
  - Hyperbolic analogue of “unique continuation from infinity” problem for time-independent Schrödinger operators \(-\Delta - V\) (Meshkov, etc.).
Problems in Relativity

- Must time-periodic solutions of Einstein’s equations be stationary?
  - Can be reduced to unique continuation for waves at infinity.
  - Past results (Papapetrou, Bičák-Scholtz-Tod) required analyticity.

  *Inheritance of symmetry:* must matter fields coupled to Einstein equations inherit the symmetries of the spacetime?
  - Stationary spacetimes, various matter models (Bičák-Scholtz-Tod)
  - Counterexamples: Klein-Gordon (Bizoń-Wasserman)

- **Goal:** Eliminate analyticity assumption.
Section 2

Background
Unique Continuation

When we do not have existence of solutions, can we still attain uniqueness?

Problem (Unique continuation (UC))

Assume the following:

- \( p(x, D) \) — linear second-order differential operator on domain \( D \subseteq \mathbb{R}^m \).
- \( \phi \) — solution on \( D \) of \( p(x, D)\phi \equiv 0 \).
- \( \Sigma \) — hypersurface in \( D \).

If \( \phi \) and \( d\phi \) vanish on \( \Sigma \), then must \( \phi \) necessarily vanish (locally) on one side of \( \Sigma \)?
Elliptic Equations

UC across $\Sigma$ always holds (Calderón, etc.).

Problem (Strong unique continuation (SUC))

Replace $\Sigma$ by a point $P$:
- If $\phi$ vanishes at $P$, then does $\phi$ also vanish near $P$?

- (Carleman, Aronszajn, Cordes) One now requires infinite-order vanishing of $\phi$ at $P$:

$$\int_{B(P,\delta)} |\phi|^2 r^{-N} < \infty, \quad r(x) = |x - P|.$$
In this case, UC no longer always holds.

(Hörmander) Main criterion for UC for $L_g = \Box_g + a^\alpha D_\alpha + V$ is pseudoconvexity of $\Sigma$.

- If $\Sigma := \{f = 0\}$ is pseudoconvex (w.r.t. $\Box_g$ and direction of increasing $f$), then UC for $L_g$ holds from $\Sigma$ to $\{f > 0\}$.
- (Alinhac) If $\Sigma$ is not pseudoconvex, then there is an $L_g$ for which UC does not hold across $\Sigma$. 
Pseudoconvexity

For wave equations, pseudoconvexity can be defined geometrically:

Definition
\( \Sigma := \{ f = 0 \} \) is pseudoconvex (w.r.t. \( \Box_g \) and increasing \( f \)) iff on \( \Sigma \),

\[
D^2 f(X, X) < 0, \text{ if } g(X, X) = Xf = 0.
\]

- \( -f \) is convex with respect to tangent null directions.
- Any null geodesic that hits \( \Sigma \) tangentially will lie in \( \{ f < 0 \} \) nearby.
Carleman Estimates

- **Carleman estimates**: main tool in proving UC.
- For wave equations, roughly of the form

\[ \| e^{-\lambda F(f)} \cdot \Box_g \phi \|_{L^2}^2 \gtrsim \lambda \| e^{-\lambda F(f)} \cdot D\phi \|_{L^2}^2 + \lambda^3 \| e^{-\lambda F(f)} \cdot \phi \|_{L^2}^2. \]  

(1)

- \( \lambda \gg 1 \) is a constant.
- \( F(f) \) is a reparametrization of \( f \) (e.g., \( \log f \)).
- By standard arguments, (1) implies UC for \( \Box_g \).
Example: Bifurcate Null Cones

- Consider a *bifurcate null cone* in Minkowski space, e.g.,

\[ \Sigma = \mathcal{N}_{r_0} := \{|t| = |r| - r_0\} \subseteq \mathbb{R}^{n+1}. \]

- *(Ionescu-Klainerman)*: Unique continuation from \( \mathcal{N}_{r_0} \) to outer region.
- Applications: black hole uniqueness results (*Alexakis-Ionescu-Klainerman*).
Hyperbolic SUC

What is a hyperbolic analogue for SUC?

- Elliptic ($\mathbb{R}^n$): ($\infty$-order) vanishing at $r^2 = 0 \Rightarrow$ vanishing on $r^2 \ll 1$.
  
  $$r^2 = |x|^2 = (x^1)^2 + \cdots + (x^n)^2.$$ 

- Hyperbolic ($\mathbb{R}^{1+n}$): replace $r^2$ by
  
  $$f = (x^1)^2 + \cdots + (x^n)^2 - (x^0)^2 = r^2 - t^2.$$ 

Vanishing at $f = 0 \Rightarrow$ vanishing for $f \ll 1$?

This is UC from null cone to exterior.
The Minkowski Case

Lemma (Ionescu-Klainerman)

Assume:
- $\phi$ satisfies $\Box \phi + V \phi = 0$.
- $V$ satisfies certain decay assumptions.
- $\phi$ vanishes to infinite order on the null cone $N_0 := \{ f = 0 \}$.

Then, $\phi$ vanishes in the region $0 < f \ll 1$.

Remark: No first-order terms allowed in wave equation.
- Because level sets of $f$ have exactly zero pseudoconvexity.

As before, proof is via a Carleman estimate.
General Cases

(Alexakis-Schlue-S.) New extensions of previous result:

1. Generalizations of vanishing assumptions.
   - If we prescribe exponential, and not just $\infty$-order, vanishing at $N_0$, then the UC theorem applies to a wider class of $V$.
   - In general: correspondence between vanishing condition for $\phi$ and wave operators $\square + V$ for which theorem holds.

2. Geometric robustness: extensions to many non-flat metrics.
   - Main idea: Carleman estimates, proved using entirely geometric methods (covariant derivatives, integration by parts).
Geometric Robustness

Lemma

- Lorentz metric $g$, given in “almost null coordinates”,
  
  $$ U \approx t - r, \quad V \approx t + r. $$

- Level sets of $f := -UV$ are pseudoconvex.

- $\phi$ vanishes at least to $\infty$-order at $\mathcal{N}_0 := \{f = 0\}$.

- Some other technical conditions relating $g$ and pseudoconvexity.

Then, $\phi$ also vanishes on $0 < f \ll 1$.

If pseudoconvexity positive, then first-order terms allowed in wave equation (i.e. $\Box_g + a^\alpha D_\alpha + V$).
Section 3

Unique Continuation from Infinity
The Conformal Inversion

- Consider first Minkowski spacetime, $\mathbb{R}^{1+n}$, with
  \[ g_M = -4dudv + r^2 \gamma. \]

- Recall the conformal inversion,
  \[ \Psi(\xi) := \frac{c\xi}{g_M(\xi, \xi)}. \]

  $\Psi$ is a conformal isometry:
  \[ \Psi^* g_M = u^2 v^2 \cdot g_M = f^2 g_M. \]

  Identifies half of $I^+ \cup I^-$ with $N_0$. 
A Preliminary Result

Lemma
Assume:
- $\phi$ vanishes to infinite/exponential order on half of $I^+ \cup I^-$.  
- $\phi$ satisfies $\Box \phi + V \phi = 0$, and, near infinity,
  
  \[ V \in O((|u||v|)^{-1-\varepsilon}) \quad / \quad V \in O(1). \]

Then, $\phi$ vanishes near infinity.

What about wave equations with first-order terms?
- For this, we must find some pseudoconvexity.
Finding Pseudoconvexity

- Consider “a bit more than half of null infinity”:
  \[ J_\varepsilon := \{ v = \infty, u < \varepsilon \} \cup \{ u = -\infty, v > -\varepsilon \}. \]

- Consider \( f_\varepsilon := (-u + \varepsilon)^{-1}(v + \varepsilon)^{-1} \).
- Positive level sets of \( f_\varepsilon \) are hyperboloids.
  - Level sets focus at boundary of \( J_\varepsilon \).
  - \( \{ f_\varepsilon = 0 \} \) corresponds to \( J_\varepsilon \).
- Level sets \( \{ f_\varepsilon = c \} \) are pseudoconvex.
  - Pseudoconvexity degenerates as \( c \downarrow 0 \).

(Figure by V. Schlue.)
A Warped Inversion

While there is no inversion $\Psi$ adapted to $f_\varepsilon$, the idea of a conformal factor survives.

- Construct a “warped” conformal inversion.
  - Conformal transformation of $g_M$:
    \[
    \bar{g}_M := f_\varepsilon^2 \cdot g_M.
    \]
  - Change of coordinates:
    \[
    U := -(v + \varepsilon)^{-1}, \quad V := (-u + \varepsilon)^{-1}.
    \]
- In “inverted” coordinates,
  \[
  \bar{g}_M = -4dUdV + f_\varepsilon^2 r^2 \cdot \gamma, \quad f_\varepsilon = -UV.
  \]
Geometric Robustness, Revisited

This once again looks like hyperbolic SUC.
- Pseudoconvexity is conformally invariant.
  - Thus, level sets of $f_\varepsilon$ also pseudoconvex in $\bar{g}_M$.
- While $\bar{g}_M$ is not Minkowski, it satisfies our lemma.
- Since level sets of $f_\varepsilon$ are pseudoconvex, we can also treat wave equations with first-order terms.

What if we perturb the Minkowski metric ($g = g_M + \delta$)?
- If $\delta$ (in null coordinates) decays fast enough toward $J_\varepsilon$, then spacetime, after similar inversion, satisfies hyperbolic SUC lemma.
- (These spacetimes have zero mass.)
Main Theorem 1.1

Theorem (Alexakis-Schlue-S., 2013)

Decaying potential case. Consider a metric $g$ over $\mathbb{R}^{n+1}$ of the form

$$g = \mu du^2 - 4Kdudv + \nu dv^2 + \sum_{A,B=1}^{n-1} r^2 \gamma_{AB} dy^A dy^B + \sum_{A=1}^{n-1} (c_A du^A + c_A dv^A dv),$$

with the components satisfying

$$K = 1 + O^\epsilon_1 (r^{-2}), \quad \gamma_{AB} = \dot{\gamma}_{AB} + O^\epsilon_1 (r^{-1}), \quad c_A, c_A = O^\epsilon_1 (r^{-1}), \quad \mu, \nu = O^\epsilon_1 (r^{-3}).$$

(Here, $O^\epsilon_1 (W)$ denotes functions in $O(W)$ up to first derivatives, with constant $\ll \epsilon$.) Consider also a wave operator $L_g := \Box g + a^\alpha D_\alpha + V$, where

$$a^u \in O((\nu + \epsilon)^{-1} r^{-\frac{1}{2}}), \quad a^v \in O((-u + \epsilon)^{-1} r^{-\frac{1}{2}}), \quad a^I \in O(f_\epsilon^{\frac{1}{2}} r^{-\frac{3}{2}}), \quad V \in O(f_\epsilon^{1+\eta}),$$

for some $\eta > 0$. Consider any $C^2$-solution $\phi$ of $L_g \phi = 0$, which in addition vanishes at $I_\epsilon$ faster than any power of $r$ (in an $L^2$-sense). Then, $\phi$ also vanishes near $I_\epsilon$. 

Main Theorem 1.2

Theorem (Alexakis-Schlue-S., 2013)

**Bounded potential case.** Consider \((\mathbb{R}^{n+1}, g)\) as before. Consider also any wave operator \(L_g := \Box_g + a^\alpha D_\alpha + V\), where

\[
\begin{align*}
a^u &\in \mathcal{O}\left((v + \varepsilon)^{-1} f_\varepsilon^{-\frac{1}{3}} \left( r^{\frac{1}{2}} \right) \right), \\
a^v &\in \mathcal{O}\left((-u + \varepsilon)^{-1} f_\varepsilon^{-\frac{1}{3}} \left( r^{\frac{1}{2}} \right) \right), \\
a^I &\in \mathcal{O}\left(\frac{1}{f_\varepsilon^6} \left( r^{-\frac{3}{2}} \right) \right), \\
V &\in \mathcal{O}(1).
\end{align*}
\]

Consider any \(C^2\)-solution \(\phi\) of \(L_g \phi = 0\), which in addition vanishes at \(I_\varepsilon\) faster than any power of \(\exp(r^{4/3})\) (in an \(L^2\)-sense).

Then, \(\phi\) also vanishes near \(I_\varepsilon\).
Remarks on Optimality

- The infinite-order vanishing assumptions for $\phi$ are necessary.
  - At least, when $\phi$ is locally defined near infinity.
- For first theorem, there are counterexamples with $V \in O(f_\epsilon^{1-\eta})$.
- For second theorem, there are counterexamples with $V \in O(f_\epsilon^{-\eta})$.
- Do not expect unique continuation from less than half of null infinity (due to argument of Alinhac).

Remark: in contrast to many earlier results (Helgason, Sá Barreto, etc.), we work only locally near infinity, both for assumption and conclusion.
The Schwarzschild Exterior

- Outer region of Schwarzschild spacetime with mass $m > 0$:
  
  \[ M := \mathbb{R}_t \times (2m, \infty)_r \times S^2, \]
  
  \[ g_S := -(1 - \frac{2m}{r}) \, dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} \, dr^2 + r^2 \gamma. \]

- How does Schwarzschild differ from Minkowski?
  
  - Minkowski: leading order pseudoconvexity comes from anchor point of the hyperboloids.
  - Schwarzschild: *leading order pseudoconvexity from positive mass.*

- This leads to *stronger* UC results than in Minkowski.
Null Coordinates

- **Tortoise coordinate**: fix $r_0 > 2m$, and let
  \[ r_*(r) := \int_{r_0}^{r} \left( 1 - \frac{2m}{s} \right) ds. \]

- Null coordinates then defined by
  \[ u := \frac{1}{2} (t - r_*), \quad v := \frac{1}{2} (t + r_*). \]

- In null coordinates,
  \[ g_s = -4 \left( 1 - \frac{2m}{r} \right) du dv + r^2 \gamma. \]
Pseudoconvexity in Schwarzschild

- Define \( f_{r_0} = -u^{-1}v^{-1} \), whose level sets are hyperboloids which focus at \( \{v = \infty, \, u = 0\} \) and \( \{u = -\infty, \, v = 0\} \).
  - In particular, anchor points depend on choice of \( r_0 \).
- Main observation: level sets of \( f_{r_0} \) are pseudoconvex, regardless of choice of \( r_0 \).
  - Thus, by choosing \( r_0 \) large enough, we get unique continuation from an arbitrarily small part of null infinity (containing \( \iota^0 \)).
We can define an analogous “conformal inversion",

\[ \bar{g}_S := \left(1 - \frac{2m}{r}\right)^{-1} f_0^2 \cdot g_S, \quad U := -v^{-1}, \quad V := -u^{-1}. \]

In the inverted coordinates,

\[ \bar{g}_S = -4dUdV + \left(1 - \frac{2m}{r}\right)^{-1} r^2 \cdot \gamma. \]

Again, this satisfies the hyperbolic SUC lemma.
Perturbations of Schwarzschild

Geometric robustness: process also works for perturbations of $g_S$.
- Includes the entire Kerr family, after coordinate change.

Theorem (Alexakis-Schlue-S., 2013)

The main theorems for near-Minkowski spacetimes have direct analogues for near-Schwarzschild spacetimes, including all Kerr spacetimes. The main difference with the near-Minkowski theorems is the following improvement: (infinite-order) vanishing is required for only an arbitrarily small part of null infinity.
The General Class

Results extend to a general class of dynamical, positive-mass spacetimes.

- Manifold \((M, g)\) given (in almost-null coordinates) by

\[
\mathcal{D} := (-\infty, 0)_u \times (0, \infty)_v \times S^{n-1},
\]

\[
g := \mu du^2 - 4Kdudv + \nu dv^2 + \sum_{A, B=1}^{n-1} r^2 \gamma_{AB} dy^A dy^B + \sum_{A=1}^{n-1} (c_{Au} dy^A du + c_{Av} dy^A dv).
\]

- Similar to near-Minkowski, but we prescribe positive mass.
- Contains perturbations of Schwarzschild as special case.
Asymptotic Assumptions

- **Metric decay:** The components of $g$ satisfy:

  \[ K = 1 - \frac{2m}{r}, \quad \gamma_{AB} = \check{\gamma}_{AB} + O_1 \left( \frac{1}{v - u} \right), \]

  \[ c_{Au}, c_{Av} = O_1 \left( \frac{1}{v - u} \right), \quad \mu, \nu = O_1 \left( \frac{1}{(v - u)^3} \right). \]

- **Positive mass:** $m$ is a function on $m$ satisfying $m \geq m_{\text{min}} > 0$. Moreover, $dm$ satisfies certain decay estimates.

  - In particular, $m$ has limits at null infinity.

- **Radial function:** $r$ is also a (not necessarily spherically symmetric) function satisfying certain asymptotic assumptions.

  - $r$ and $r_* := v - u$ are related like in Schwarzschild: $r_* - r \simeq \log r$. 
Reduction to Hyperbolic SUC

Though more computationally intense, the idea is same as before.

- Level sets of $f := -u^{-1}v^{-1}$ are pseudoconvex.
- Conformal inversion of metric, $ar{g} := K^{-1}f^2 \cdot g$.

Then, $ar{g}$ satisfies the hyperbolic SUC lemma.

- In fact, UC results for perturbations of Minkowski, perturbations of Schwarzschild, and this general class are proved all at once.
Main Theorems 2

Theorem (Alexakis-Schlue-S., 2013)

Consider \((M, g)\) as above. Consider also any wave operator \(L_g := \Box_g + a^\alpha D_\alpha + V\), where

\[
a^u \in \mathcal{O}(v^{-1} r^{-\frac{1}{2}}), \quad a^v \in \mathcal{O}((-u)^{-1} r^{-\frac{1}{2}}), \quad a^I \in \mathcal{O}(f^\frac{1}{3} r^{-\frac{3}{2}}), \quad V \in \mathcal{O}(r^{1+\eta}),
\]

for some \(\eta > 0\). Consider any \(C^2\)-solution \(\phi\) of \(L_g \phi = 0\), which vanishes at \(J = \{v = \infty, u < 0\} \cup \{u = -\infty, v > 0\}\) faster than any power of \(r\) (in an \(L^2\)-sense). Then, \(\phi\) also vanishes near \(J\).

Theorem (Alexakis-Schlue-S., 2013)

Consider \((M, g)\) as above. Consider also any wave operator \(L_g := \Box_g + a^\alpha D_\alpha + V\), where

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\]

Consider any \(C^2\)-solution \(\phi\) of \(L_g \phi = 0\), which vanishes at \(J\) faster than any power of \(\exp(r^{4/3})\) (in an \(L^2\)-sense). Then, \(\phi\) also vanishes near \(J\).
Thank you for your attention!
Section 5

Appendix
Main tool for hyperbolic SUC is Carleman estimate.

(For our main results, this is the inverted setting.)

General form:

$$\| e^{-\lambda F(f)} \Box g \phi \|_{L^2}^2 \gtrsim \lambda \sum_{\alpha} \| e^{-\lambda F(f)} A^\alpha D^\alpha \phi \|_{L^2}^2 + \lambda^3 \| e^{-\lambda F(f)} B \phi \|_{L^2}^2.$$  

- $\lambda \ll 1$.
- $F(f_\epsilon)$ is a reparametrization of $f_\epsilon$.
- $A^\alpha, B$ are positive weights that blow up or decay at $\{f = 0\}$.

Proof of Carleman estimate is purely geometric.
Main Ideas

Carleman estimate can be thought of as an energy estimate for $\Box_g$, but:

1. We want boundary terms to vanish.
2. We want bulk terms to be positive.

Objective (1) achieved by:
- Vanishing assumptions for $\phi$ at $f = 0$.
- Cutoff functions for $f = f_0 > 0$.

Objective (2) achieved using a *positive commutator*.
- Consider wave equation not for $\phi$, but for $\psi = \mathcal{F}\phi$. 
Positive Commutators

To ensure the bulk term is positive:

1. Bulk terms containing derivative of $\phi$ tangent to level sets of $\mathcal{F}$:
   - These are positive only when level sets of $\mathcal{F}$ are pseudoconvex.
   - Thus, $\mathcal{F} = f$ is a candidate.

2. Bulk terms containing $\phi$ and derivative normal to level sets of $\mathcal{F}$:
   - Additional freedom: any reparametrization $F \circ \mathcal{F} = F \circ f$ (where $F' > 0$) produces same level sets.
   - Find reparametrization $F(f)$ so these bulk terms are positive.
   - Many valid choices of $F$—as long as $F$ grows fast enough.
Some Features

Weights $A^\alpha$ and $B$ depend on pseudoconvexity and on choice of $F(f)$.

- $F(f)$ must grow “at least as fast as log $f$” (but cannot be log itself).
- (Ionescu-Klainerman) Choose $F = \log f + \text{correction}$.
  - Decaying potential case: $|V| \lesssim f^{1+}$, requires $\infty$-order vanishing of $\phi$.
- (New) Choose $F = -f^{-2/3}$.
  - Bounded potential case: $|V| \lesssim 1$, requires exponential vanishing of $\phi$. 
Finite-Order Vanishing

Can we somehow do away with the infinite-order vanishing assumption?

- Cannot do so while remaining local near infinity (counterexamples).
- (Alexakis-S.) Yes on Minkowski spacetime, if we have *global* information for $\phi$.

Technical obstruction to finite-order vanishing comes from cutoff function to make boundary terms vanish.

- If we can go from infinity all the way to null cone about origin, then boundary terms vanishing without cutoff function.
- Requires very careful choice of reparametrization of $f$. 
Nonlinear Equations

The finite-order vanishing theorems have a new obstruction:

- Linear potential must also be small.

(Alexakis-S.) However, for some nonlinear equations, we can treat nonlinearity directly within Carleman estimates:

- Focusing, subconformal nonlinearity.
- Defocusing, conformal and superconformal nonlinearity.

In these cases, can eliminate smallness assumption.

(Alexakis-S.) These nonlinear Carleman estimates have other applications:

- Final states.
- Formation of singularities.