Non-existence of time-periodic vacuum spacetimes

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Outline

1. **2-body problem in general relativity**
   Predictions of post-Newtonian theory

2. **Time-periodic vacuum spacetimes**
   Statement of the main theorem

3. **Final state conjecture**
   Dynamical non-radiating spacetimes
   Relation to black hole uniqueness problem

4. **Proof of the non-existence result**
   Uniqueness results for ill-posed hyperbolic p.d.e.’s
   Null geodesics in spacetimes with positive mass
1. 2-body problem in general relativity
Periodic motion in Newton’s theory

Two-body problem in classical celestial mechanics: Kepler orbits.
Space: $\mathbb{R}^3$, Time: $\mathbb{R}$, Newtonian potential: $\psi(t, x)$

$$F = -m \nabla \psi(t, x)$$

Test body:

$\Delta \psi(t, x) = 4\pi \rho(t, x)$ : mass density

Note: In Newtonian theory the gravitational field can be periodic in time. (Action at a distance)
Post-Newtonian theory

Slow motion / post-Newtonian / weak field approximations:

Einstein-Infeld-Hoffmann, . . . , Damour-Deruelle, Blanchet, . . . , Will-Wiseman, Poisson-Will

Correction to Newtonian potential: $\psi_{\text{post-Newtonian}} = \psi + \frac{1}{c^2}\omega$

In this approximation circular orbits still possible, but ruled out by radiation reaction force in higher orders of the expansion.
**Dissipative dynamics in general relativity**

Periodic motion should **not exist** in general relativity due to the emission of gravitational waves.

Our main result (joint with Spyros Alexakis) is that any time-periodic vacuum spacetime is in fact **time-independent**, at least far away from the sources.
2 Time-periodic vacuum spacetimes
Isolated self-gravitating systems in general relativity

\((\mathcal{M}^{3+1}, g)\) asymptotically flat, and solution to \(\text{Ric}(g) = 0\) outside a spatially compact set.

Future and past null infinities \(\mathcal{I}^+, \mathcal{I}^- \simeq \mathbb{R} \times S^2\) are complete.
**Notion of time-periodicity**

**Definition**

An asymptotically flat spacetime is called *time-periodic* if there exists a discrete isometry $\varphi_a$ with time-like orbits.

Then in fact $\varphi_a$ extends to a map $\varphi_a^+$ on future null infinity where it is an affine translation along the generating geodesics. (Similarly at past null infinity.)
Theorem (Alexakis-S. ’15)

Any asymptotically flat solution \((\mathcal{M}, g)\) to the Einstein vacuum equations arising from a regular initial data set which is time-periodic (near infinity), must be stationary near infinity.

The theorem asserts that there exists a time-like vector field \(T\) on an arbitrarily small neighborhood \(\mathcal{D}\) of infinity such that \(\mathcal{L}_T g = 0\) on \(\mathcal{D}\).
Previous results

Early work:
Papapetrou ’57–’58 (weak field approximation, “non-singular” solutions, strong time-periodicity assumption)

Recent work:
Gibbons-Stewart ’84, Bicak-Scholz-Tod ’10 (contains ideas how to exploit time-periodicity, stationarity inferred under much more restrictive analyticity assumption)

Cosmological setting:
Tipler ’79, Galloway ’84 (spatially closed case)
Dynamical non-radiating spacetimes and the final state conjecture
Gravitational radiation

**Trautmann-Bondi energy:**

The Bondi mass $M(u)$ signifies the amount of energy in the system at time $u$. It is known to be **positive**: $M(u) \geq 0$ (Schoen-Yau, . . ., Chruściel-Jezierski-Leski, Sakovich), and dynamically **monotone decreasing**.

In Christodoulou-Klainerman the Bondi mass loss formula is

$$\frac{\partial M(u)}{\partial u} = -\frac{1}{32\pi} \int_{\mathbb{S}^2} |\Xi|^2 d\mu_{\gamma}$$

and it is shown that $\lim_{u \to -\infty} M(u) = M[\Sigma]$ \(\lim_{u \to \infty} M(u) = 0\).

$|\Xi|(u, \xi)$: **power** of gravitational waves radiated in direction $\xi \in \mathbb{S}^2$, at time $u \in \mathbb{R}$.
Aside: Gravitational wave experiments

\[ v_v(B) = \frac{d_0}{r} \Xi_{AB}(t) \]

Figure: LIGO, Washington
Non-radiating spacetimes

Definition
An asymptotically flat spacetime is called non-radiating if the Bondi mass $M(u)$ is constant along future (and past) null infinity.

Theorem (Alexakis-S. ’15)
Any asymptotically flat solution $(\mathcal{M}^{3+1}, g)$ to the vacuum equations arising from regular initial data which is assumed to be non-radiating, and in addition smooth at null infinity, must be stationary near infinity.

Remark:
Here smooth at null infinity means in particular that the curvature components $\rho$ admit a full asymptotic expansion near null infinity which is well behaved towards spacelike infinity:

$$\rho \sim \sum_{l=0}^{\infty} \kappa_l(u) r^{k-l}, \quad \lim_{u \to -\infty} |\kappa_l(u)| < \infty.$$
**Conjectures**

The **final state conjecture** gives a *characterisation of all possible end states* of the dynamical evolution in general relativity, as a result of a scenario due to Penrose invoking both **weak cosmic censorship** and **black hole uniqueness**.

**Conjecture**

Any smooth asymptotically flat black hole exterior solution to the Einstein vacuum equations which is assumed to be **non-radiating** is *isometric* to the exterior of a *Kerr* solution $(\mathcal{M}, g_{M,a})$.

\[
\begin{align*}
\mathcal{H}^+ & : \text{future event horizon} \\
\mathcal{I}^+ & : \text{future null infinity} \\
\Sigma & : \text{black hole exterior}
\end{align*}
\]
Aside: Soliton resolution conjecture

The energy-critical focusing non-linear wave equation

$$\Box \phi = -\phi^5$$

has soliton solutions

$$\phi_\lambda(t, x) = \frac{1}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right) \quad W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2} \quad \lambda > 0$$

It is expected that for global solutions

$$\|\phi - \left(\phi_L \pm \sum_j \phi_{\lambda_j}\right)\| \rightarrow 0 \quad (t \rightarrow \infty)$$

Proof of the theorems:
extension of a time translation symmetry from infinity
Strategy of Proof

1. Construction of time-like candidate vectorfield $T$, such that by time-periodicity

$$\lim_{r \to \infty} r^k \mathcal{L}_T R = 0 \quad \forall k \in \mathbb{N}$$

2. Use that by virtue of the vacuum Einstein equations,

$$\Box g R = R \ast R$$

thus

"$\Box g \mathcal{L}_T R = R \ast \mathcal{L}_T R$".

Then use our unique continuation theorem which asserts that solutions to wave equations on asymptotically flat spacetimes are uniquely determined if all higher order radiation fields are known, to show that

$$\mathcal{L}_T R = 0.$$
Construction of “candidate” Killing vectorfield $T$

Define $T = \frac{\partial}{\partial u}$: binormal to spheres $S_u^*$. Then extend inwards by Lie transport along geodesics: $[L, T] = 0$, $\nabla_L L = 0$, $g(L, L) = 0$. 

$\Sigma_0$
The components of the curvature fall off at different rates in the distance (peeling):

\[ R_{LL} = O(r^{-1}) \quad R_{LL} = O(r^{-2}) \quad R_{LL} = O(r^{-3}) \]

\[ \lim_{u; r \to \infty} rR_{LL} = A \quad \lim_{u; r \to \infty} r^2 R_{LL} = P \quad \lim_{u; r \to \infty} r^3 R_{LL} = A \]

Using the asymptotic laws obtained in Christodoulou-Klainerman it follows

\[ \Xi = 0 \implies A = -\partial_u \Xi = 0, \quad \partial_u P = -A = 0 \]

however, in general,

\[ \partial_u A = \nabla_\xi P + A. \]
The idea is to differentiate a second time,

\[ \partial_u^2 A = \nabla_\xi \partial_u P = 0 \]

which implies that \( A \) is a linear function in \( u \),

\[ A(u_2, \xi) - A(u_1, \xi) = A_0(\xi)(u_2 - u_1). \]

But by time-periodicity \( A_0(\xi) = 0 \), therefore

\[ \partial_u A = 0. \]

Note, same conclusion if instead of time-periodicity we assume

\[ \lim_{u_1 \to -\infty} |A(u_1, \xi)| < \infty \]

This also yields \( \nabla_\xi P = 0 \), namely that \( P \) is spherically symmetric.
Time-periodicity
and time-independence to all orders

Schematically, this was the first step of an **induction** which proves

\[
\lim_{u; \, r \to \infty} r^k \partial_u R = 0 \quad \forall k \in \mathbb{N}.
\]

In fact, we show at the same time using the **propagation equations** along outgoing null geodesics

\[
\lim_{u; \, r \to \infty} r^k \partial_u g = 0 \quad \lim_{u; \, r \to \infty} r^k \partial_u \Gamma = 0
\]

For example, consider \( \Gamma = \hat{\chi} \). (Recall we already saw \( \Xi = \lim r\hat{\chi} \).)

Schematically, since by construction \([L, \, T] = 0, \, T = \partial_u\):

\[
L\hat{\chi} = -\alpha \quad \alpha = R_{LL}
\]

\[
L\partial_u \hat{\chi} = -\partial_u \alpha
\]
In the time-periodic setting the required regularity can be deduced from a corresponding regularity assumption on the initial data:

\[ g|_{\Sigma} = g_{Kerr}|_{t=0} + g^{\infty} \]

\[ g^{\infty}_{\alpha\beta} \sim \sum_{l} g^{l}_{\alpha\beta}(\vartheta)r^{k-l} \]
Recall strategy of the proof

1. We have now constructed a time-like candidate vectorfield $T$, such that by time-periodicity, or alternatively by assuming a regular expansion,

$$\lim_{r \to \infty} r^k \mathcal{L}_T R = 0 \quad \forall k \in \mathbb{N}$$

2. Use that by virtue of the vacuum Einstein equations

$$\Box g R = R \ast R$$

the Lie derivative of the curvature satisfies a wave equation

"$$\Box g \mathcal{L}_T R = R \ast \mathcal{L}_T R$$".

Then apply our unique continuation theorem to show that

$$\mathcal{L}_T R = 0.$$
Unique continuation from infinity

**Theorem (Alexakis-S.-Shao ’14)**

Let $(\mathcal{M}, g)$ be an asymptotically flat spacetime with **positive mass**, and $L_g$ a linear wave operator

$$L_g = \Box_g + a \cdot \nabla + V$$

with suitably fast decaying coefficients $a$, and $V$. If $\phi$ is a solution to $L_g \phi = 0$ which in addition satisfies

$$\int_{\mathcal{D}} r^k |\phi|^2 + r^k |\partial \phi|^2 < \infty$$

where $\mathcal{D}$ is an arbitrarily small neighborhood of infinity $i^0$, then

$$\phi \equiv 0 \quad : \text{on } \mathcal{D} \subset \mathcal{D}'.$$
Application of the theorem to the Einstein equations

The application of the theorem is not immediate because

\[ \Box_g R = R \star R \]

is not a scalar equation, but a covariant equation for the Riemann curvature tensor. Moreover, \([\Box_g, \mathcal{L}_T] \neq 0\) and differentiating the equation produces additional terms which are not in the scope of the theorem:

\[ \Box_g \mathcal{L}_T R - [\Box_g, \mathcal{L}_T] R = R \star \mathcal{L}_T R + \mathcal{L}_T g \star R^2 \]

These obstacles can be overcome in the general framework of Ionescu-Klainerman '13 for the extension of Killing vectorfields in Ricci-flat manifolds.
**Application of the theorem**

Define *modified* Lie-derivative

\[ W = \mathcal{L}_T R - B \cdot R \quad B = \mathcal{L}_T g + \omega \quad \nabla_L \omega = \mathcal{M}(\mathcal{L}_T g) \]

which then satisfies a covariant equation

\[ \Box W = R \cdot W + \nabla R \cdot \nabla B + R^2 \cdot B + R \cdot \nabla P \]

coupled to o.d.e.'s

\[ \nabla_L B = \mathcal{M}(P, B) \quad \nabla_L P = \mathcal{M}(W, B, P) \]

This is **only** true if \([L, T] = 0\), which we have by construction.

Now choose *Cartesian* coordinates near infinity, such that

\[ \Gamma = O(r^{-2}) \]

and apply our **Carleman estimates**.
Proof of the theorems:
unique continuation from infinity for linear waves
Linear theory on Minkowski space

Consider the linear wave equation on $\mathbb{R}^{3+1}$:

$$\Box \phi = 0$$

The radiation field is defined by

$$\Psi(u, \xi) = \lim_{r \to \infty} (r \phi)(t = u + r, x = r \xi).$$

**Theorem (Friedlander '61)**

For solution $\Box \phi = 0$ there is a 1-1 correspondence

$$(\phi|_{t=0}, \partial_t \phi|_{t=0}) \in \dot{H}^1 \times L^2 \longleftrightarrow \Psi \in L^2.$$

However, no generalisations to perturbations of Minkowski space are known, i.e. for equations

$$\Box_g \phi + a \cdot \nabla \phi + V \phi = 0$$
Counterexamples in linear theory

Question:
Without the finite energy condition, does the vanishing of the radiation field imply the vanishing of the solution?

\[ \psi = 0 \implies \phi \equiv 0 \]

Answer:
No, because \( \phi = \frac{1}{r} \) is a solution, and thus also

\[ \phi_i = \partial x^i \frac{1}{r} \sim \frac{1}{r^2} \]

which is a non-trivial solution with \( \psi = 0 \).

This shows the necessity of the infinite order vanishing assumption:

\[
\lim_{r \to \infty} (r^k \phi)(u + r, r \xi) = 0 \quad \forall k \in \mathbb{N}.
\]
Obstructions to unique continuation from infinity

There is an obstruction related to the behavior of light rays.

Alinhac-Baouendi '83: Unique continuation fails across surfaces which are not pseudo-convex, in general.
**Theorem (Alexakis-S.-Shao ’13)**

Let \((\mathcal{M}, g)\) be a perturbation of Minkowski space, and \(L_g\) a linear wave operator with decaying coefficients. If \(\phi\) is a solution to \(L_g \phi = 0\) which in addition satisfies an infinite order vanishing condition on “at least half” of future and past null infinity, then

\[ \phi \equiv 0 \]

in a neighborhood of infinity.
Pseudo-convexity

The proof crucially relies on the construction of a family of pseudo-convex time-like hypersurfaces.

**Definition**

A time-like hypersurface $H = \{ f = c \}$ is pseudo-convex at a point $p$, if

$$(\nabla^2 f)_p(X, X) < 0$$

for all vectors $X \in T_p\mathcal{M}$ which

(i) are null, $g(X, X) = 0$,

(ii) are tangential to $H$, $g(X, \nabla f) = 0$.

We find a family of pseudo-convex hypersurfaces that foliate a neighborhood of infinity and derive a Carleman inequality to prove the uniqueness result.
Conformal Inversion

in Minkowski space

In fact, we choose

\[ f = \frac{1}{(-u + \epsilon)(v + \epsilon)} \]

\[ u = \frac{1}{2}(t - r), \quad v = \frac{1}{2}(t + r) \]

and consider the conformally inverted metric

\[ \tilde{g} = f^2 g \]

Note this is not the standard Penrose compactification \( \Omega^2 g \)

where

\[ \Omega = \frac{1}{\sqrt{(1 + u^2)(1 + v^2)}} \]

In fact \( \tilde{g} \) is singular.
In Schwarzschild with \( m > 0 \),

\[
    g = -4 \left( 1 - \frac{2m}{r} \right) dudv + r^2 \gamma
\]

where, for arbitrary \( r_0 > 2m \),

\[
    v - u = r^* = \int_{r_0}^{r} \frac{1}{1 - \frac{2m}{r}} \, dr = r + 2m \log |r - 2m| - r_0^*
\]

We set

\[
    f = \frac{1}{(-u)(v)}
\]

and consider the conformally inverted metric

\[
    \tilde{g} = f^2 g.
\]
**Pseudo-convexity**

*in spacetimes with positive mass*

While in Minkowski space

\[-\nabla^2 f(X, X) \sim \frac{\epsilon}{r} \quad \forall X : g(X, X) = g(X, \nabla f) = 0\]

we find that in Schwarzschild

\[-\nabla^2 f(X, X) \sim \frac{2m}{r} \log r - \frac{r_0}{r} > 0\]

for *arbitrarily large* \(r_0 > 2m\).

This is the reason unique continuation from infinity holds in an *arbitrarily small neighborhood of infinity* whenever the spacetime has a *positive mass*. 
Positive mass
and the behaviour of light rays

This is related to ideas of Penrose, Ashtekar-Penrose '90, and Chrusciel-Galloway '04 to characterise the positivity of mass by the behaviour of null geodesics near infinity. (See also Penrose-Sorkin-Woolgar '93)
Thank you for your attention!