Space-time unfitted FEM for problems with moving discontinuities

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Paris, 03.03.2016

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Outline

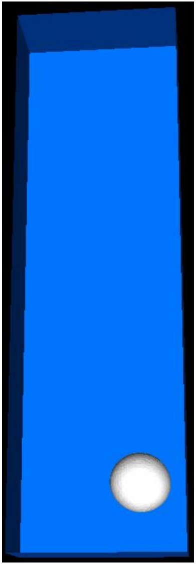
- Motivation: simulation of two-phase **incompressible** flows.
- Space-time FEM for mass transport equation.
- Similar space-time techniques for
  - surfactant transport
  - time dependent Stokes interface problem

Numerical simulation of two-phase incompressible flows

Fluid dynamics in the two phases

system: n-butanol/water

Model: Navier-Stokes equations
+ coupling conditions



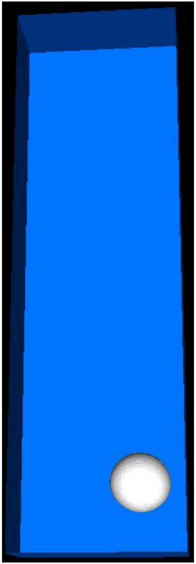
Rising droplet with mass transport

Navier-Stokes equations for fluid dynamics

Convection-diffusion equation

for transport between the fluids

+ Henry coupling condition at interface



Rising droplet with surfactant transport

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Convection-diffusion equation on interface

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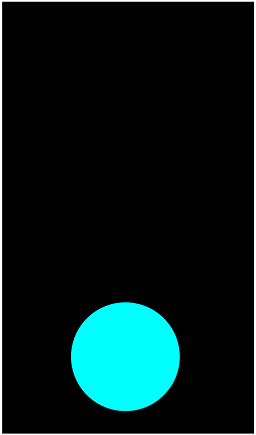
solution



Unfitted FEM for mass transport equation: space-time DG-CutFEM-Nitsche

Strong formulation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \mathbf{w}\cdot\nabla u - \operatorname{div}(\alpha\nabla u) &= f \quad \text{in } \ \Omega_1(t)\cup\Omega_2(t), \ \ t\in[0,T], \\
[\alpha\nabla u\cdot\mathbf{n}]_\Gamma &= 0, \\
[\beta u]_\Gamma &= 0, \\
u(\cdot,0) &= u_0 \quad \text{in } \ \Omega_1(0)\cup\Omega_2(0), \\
u(\cdot,t) &= 0 \quad \text{on } \ \partial\Omega, \quad t\in[0,T].
\end{aligned}
$$

With $\alpha = \alpha_i > 0$, $\beta = \beta_i > 0$.

Assumptions: $u_0 = 0$, $\operatorname{div}\mathbf{w} = 0$, $V_\Gamma = \mathbf{w}\cdot\mathbf{n}$ (Γ transported by $\mathbf{w}$).

Space-time weak formulation

Space-time: $Q_T := \Omega \times (0, T)$,

$$\Gamma_* := \{\, (x,t) \mid x \in \Gamma(t),\ t \in (0,T)\,\}.$$

$$Q_i := \{\, (x,t) \mid x \in \Omega_i(t),\ t \in (0,T)\,\}, \quad i = 1,2.$$

Well-posed weak formulation

Determine $u \in W_\beta$ with $u(\cdot, 0) = 0$ such that

$$\frac{\partial u}{\partial t}(v) - \int_{Q_T} u\mathbf{w} \cdot \nabla v \, dx\, dt + \sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla v \, dx\, dt = \int_{Q_T} fv \, dx\, dt$$

for all $v \in H_0^{1,0}(Q_T)$

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Note:

- Space-time formulation, with appropriate spaces W_β, $H_0^{1,0}(Q_T)$.
- Trial functions are discontinuous across Γ_*.
- Condition $[\beta u]_\Gamma = 0$ essential condition in space W_β.

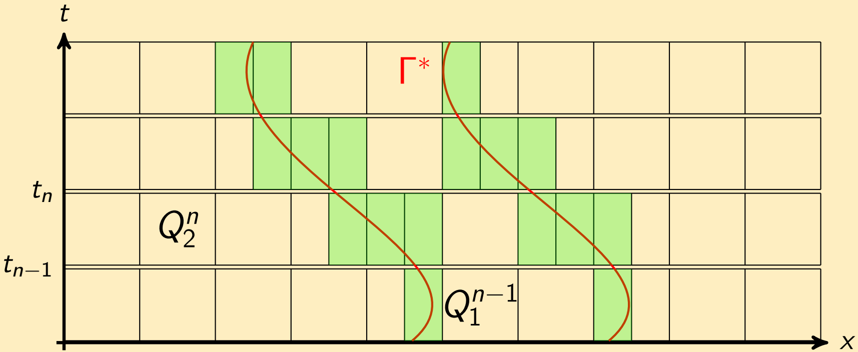
Space-time FE discretization

Space-time FE

$I_n = (t_{n-1}, t_n]$, $Q^n = \Omega \times I_n$. V_n : standard FE space on Ω.

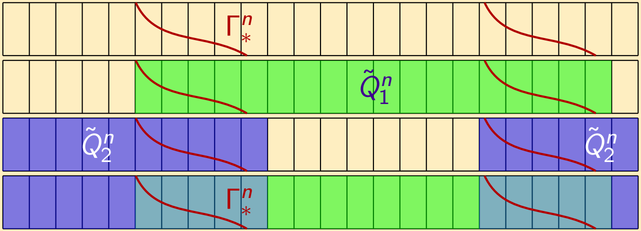
$$W_n := \{\, v : Q^n \to \mathbb{R} \mid v(x,t) = \phi_0(x) + t\phi_1(x), \ \ \phi_0, \phi_1 \in V_n \,\}$$

$$W := \oplus_{n=1}^{N} W_n \quad \text{(space-time FE).}$$



CutFEM for approximating discontinuities

Space-time FE ⇒ Space-time CutFEM



R_i^n : restriction to Q_i^n

$$W_n^{\Gamma_*} := R_1^n W_n \oplus R_2^n W_n,$$

$$W^{\Gamma_*} := \oplus_{n=1}^N W_n^{\Gamma_*} \quad \text{(Space-time CutFE space)}$$

Discretization: DG + CutFEM + Nitsche

Variational form PDE:

$$a^n(u,v) = \sum_{i=1}^{2} \int_{Q_i^n} \left(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i\right)\beta_i v_i + \alpha_i \beta_i \nabla u_i \cdot \nabla v_i \, dx \, dt$$

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Discontinuous Galerkin (DG) w.r.t. time:

$$d^n(u,v) = \int_{\Omega} \beta(\cdot, t_n)[u]^{n-1} v_+^{n-1} \, dt$$

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Discontinuous Galerkin (DG) w.r.t. time:

$$d^n(u,v) = \int_{\Omega} \beta(\cdot, t_n)[u]^{n-1} v_+^{n-1}\, dt$$

Nitsche method for Henry condition:

$$N_{\Gamma_*}^n(u,v) = -\int_{\Gamma_*^n} \{\alpha \nabla u \cdot \mathbf{n}\}_{\Gamma_*} [\beta v]_{\Gamma_*}\, ds - \int_{\Gamma_*^n} \{\alpha \nabla v \cdot \mathbf{n}\}_{\Gamma_*} [\beta u]_{\Gamma_*}\, ds$$
$$+ \lambda h_n^{-1} \int_{\Gamma_*^n} [\beta u]_{\Gamma_*} [\beta v]_{\Gamma_*}\, ds,$$

with $\{\cdot\}_{\Gamma_*}$ suitable area weighted average. $\lambda > 0$: stabilization parameter.

DG-CutFEM-Nitsche variational problem

One time slab problem: $u \in W^{\Gamma_*^n}$ such that

$$a^n(u,v) + d^n(u,v) + N^{\Gamma_*^n}(u,v) = f^n(v) \qquad \forall\ v \in W^{\Gamma_*^n}$$

Time marching method: Solution is obtained time slab by time slab.

DG-CutFEM-Nitsche variational problem

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Global problem (only for analysis purposes)

global bilinearforms:

$$a(u,v) = \sum_{n=1}^{N} a^n(u,v), \quad \text{similarly}: \quad d(u,v),\ N_{\Gamma_*}(u,v).$$

Discrete problem:
Determine $U \in W^{\Gamma_*}$ such that

$$B(U,V) = f(V) \quad \text{for all} \quad V \in W^{\Gamma_*},$$
$$B(U,V) := a(U,V) + d(U,V) + N_{\Gamma_*}(U,V).$$

Error analysis for linear FE

Theorem [Lehrenfeld/AR SINUM 2013]

.

$$\|(u - U)(\cdot, t_N)\|_{L^2(\Omega)} \leq c(h^2 + \Delta t^2).$$

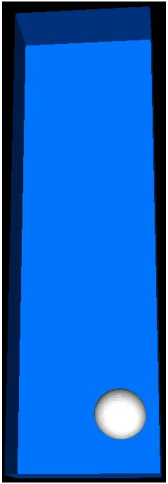
Remarks:

- Error analysis based on inf-sup + Cea-lemma type arguments.
- Second order convergence for transport problem with moving discontinuity.
- Reconstruction of space-time interface Γ_*^n needed.

Numerical experiments

For model problems (analytical solution]: [Lehrenfeld/AR SINUM 2013].

Rising droplet with mass transport



Navier-Stokes equations for fluid dynamics

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Unfitted FEM for surfactant PDE: space-time trace FEM

Surfactant PDE

$\Gamma(0)$ smooth surface in $\mathbb{R}^3$, $\partial\Gamma(t) = \emptyset$.

$$\Gamma(t),\ t \in [0,T], \quad \text{advected by smooth } \mathbf{w} = \mathbf{w}(\mathbf{x},t) \in \mathbb{R}^3.$$

Model for diffusive mass transport on $\Gamma(t)$:

Diffusion equation

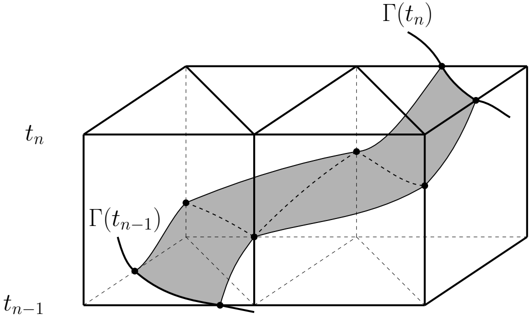
$$\dot{u} + (\mathrm{div}_\Gamma \mathbf{w})u - \Delta_\Gamma u = 0 \qquad \text{on} \quad \Gamma(t), \quad t \in (0,T]$$

with $\dot{u} = \frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u$.
Initial condition $u(\mathbf{x},0) = u_0(\mathbf{x})$ for $\mathbf{x} \in \Gamma(0)$.

Space-time trace FEM

Key ideas:

- Space-time variational formulation.
- Time-stepping by DG.
- Outer standard space-time FE spaces.
- Use trace of these space on space-time manifold.



Error analysis [Olshanskii,AR, SINUM 2014]

Discrete stability

$$\inf_{u \in W^b} \sup_{v \in W^b} \frac{\langle \dot{u}, v \rangle_b + a(u,v) + d(u,v)}{|||v|||_h |||u|||_h} \geq c_s$$

Global stability result. No conditions on Δt.

Error analysis [Olshanskii,AR, SINUM 2014]

Discrete stability

$$\inf_{u\in W^b} \sup_{v\in W^b} \frac{\langle \dot{u}, v\rangle_b + a(u,v) + d(u,v)}{\|v\|_h \|u\|_h} \geq c_s$$

Global stability result. No conditions on Δt.

Discretization error bounds. Assume $\Delta t \sim h$

$$\|u - u_h\|_h \leq ch\big(\|u\|_{H^2(\Gamma_*)} + \sup_{t\in[0,T]} \|u\|_{H^2(\Gamma(t))}\big).$$

$$\|u - u_h\|_{H^{-1}(\Gamma_*)} \leq ch^2\big(\|u\|_{H^2(\Gamma_*)} + \sup_{t\in[0,T]} \|u\|_{H^2(\Gamma(t))}\big).$$

Numerical experiments

For model problems: [Olshanskii,AR, SINUM 2014]

Evolving surface with topological singularity

Domain $\Omega = (-3,3) \times (-2,2)^2$, $t \in [0,1]$.

Prescribed level set function ϕ,

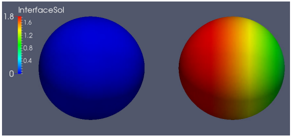
determines $\Gamma(t)$.

Space-time interpolation yields Γ_*^n.

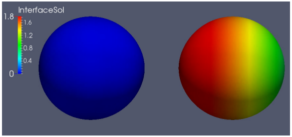
Surfactant transport equation.

$u_0(x) = 3 - x_1$ for $x_1 \geq 0$, zero otherwise.

$h = 1/16, \ \Delta t = 1/128$



$h = 1/16, \ \ \Delta t = 1/4$



Unfitted FEM for time dependent Stokes interface problem (ongoing research)

Time dependent Stokes model problem

Given smooth velocity field:
$\mathbf{w} = \mathbf{w}(x,t) \in \mathbb{R}^d$ on $Q_T = \Omega \times [0,T]$, with $\operatorname{div} \mathbf{w} = 0$.

Interface $\Gamma(t)$ is evolved by $\mathbf{w}$: $V_\Gamma = \mathbf{w} \cdot \mathbf{n}_\Gamma$.

$$\dot{v} := \frac{\partial v}{\partial t} + \mathbf{w} \cdot \nabla v.$$

Determine $\mathbf{u}$, with $\mathbf{u}(\cdot,0) = 0$, $\mathbf{u}_{|\partial\Omega} = 0$, and p such that

$$\begin{cases} \rho\dot{\mathbf{u}} - \operatorname{div}(\mu D(\mathbf{u})) + \nabla p = \rho\mathbf{g} \\ \operatorname{div} \mathbf{u} = 0 \end{cases} \quad \text{in } \Omega_i(t), \quad i = 1,2,$$

$$[(-p\mathbf{I} + \mu D(\mathbf{u}))\mathbf{n}_\Gamma] = -\tau\kappa\mathbf{n}_\Gamma \quad \text{on } \Gamma(t),$$
$$[\mathbf{u}] = 0 \quad \text{on } \Gamma(t),$$

Discontinuity of p and $\nabla\mathbf{u}$ across $\Gamma(t)$.

Space-time FEM components

Generalization of approach used for mass transport equation:

- space-time weak formulation (..well-posedness?)
- time-stepping by DG.
- Use cut of standard space-time FE spaces (for $\mathbf{u}$ and p).
- Use Nitsche for interface conditons (continuity of $\mathbf{u}$)

Many open problems....ongoing research...

Summary and outlook

- Two-phase flow sharp interface model: many numerical challenges:
- Moving discontinuities and PDEs on moving surfaces.

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- Moving discontinuities and PDEs on moving surfaces.
- Unfitted FEM for mass transport problem: DG+CutFEM+Nitsche. Second order accuracy.
- Diffusion on moving surface: space-time trace FEM. Second order accuracy.
- Unfitted space-time FEM for time-dependent Stokes interface problem: ...work in progress.. many open problems......

Summary and outlook

- Two-phase flow sharp interface model: many numerical challenges:
- Moving discontinuities and PDEs on moving surfaces.
- Unfitted FEM for mass transport problem: DG+CutFEM+Nitsche.
  Second order accuracy.
- Diffusion on moving surface: space-time trace FEM.
  Second order accuracy.
- Unfitted space-time FEM for time-dependent Stokes interface problem: ...work in progress.. many open problems......
- Higher order methods?
- Linear algebra issues: preconditioners?