

# Relative entropy method for the diffusive limit of numerical schemes

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# Continuous framework

- Hyperbolic system with source term:

$$\partial_t W + \partial_x f(W) = -R(W). \quad (H)$$

- Diffusive limit  $t \rightarrow +\infty$ :

$$\partial_t \bar{W} - \partial_x (D(\bar{W})\partial_x \bar{W}) = 0. \quad (P)$$

- Rescaled system:

$$\varepsilon \partial_t W_\varepsilon + \partial_x f(W_\varepsilon) = -\frac{1}{\varepsilon} R(W_\varepsilon). \quad (H_\varepsilon)$$

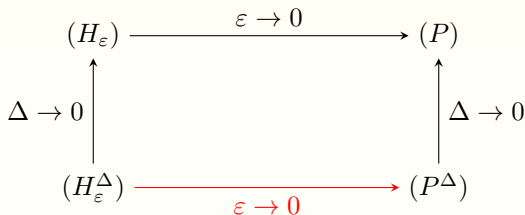
- $(H_\varepsilon) \rightarrow (P)$  as  $\varepsilon \rightarrow 0$ .

- **Lattanzio & Tzavaras** : Euler isentropic,  $p$ -system.  
↔ relative entropy ⇒ convergence rate in  $O(\varepsilon^4)$ .

# Motivations

AP schemes for the diffusive limit:

- Kinetic equations: **Jin, Pareschi & Toscani** , **Naldi & Pareschi**.
- Hyperbolic systems: **Gosse & Toscani** , **Buet & Després**, **CB & Turpault**.



**Aim:** Study the limit  $\varepsilon \rightarrow 0$  in the discrete framework.

- Strategy: relative entropy (**Lattanzio & Tzavaras, 13**).
- Particular system:  $p$ -system with friction.

- 1  $p$ -system with friction
- 2 Convergence rate for the continuous problem
- 3 Convergence rate for a semidiscrete scheme
- 4 Numerical experiments

# From the $p$ -system to the porous media equation

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\sigma u, \end{cases} \quad (H)$$

pressure  $p(s) = s^{-\gamma}$ ,  $\gamma > 1$ .

**Rescaling:** (Naldi & Pareschi, Mei)

$$\begin{aligned} t &\rightarrow t/\varepsilon, & u &\rightarrow \varepsilon u^\varepsilon, & \sigma &\rightarrow \sigma/\varepsilon \\ \Rightarrow \begin{cases} \partial_t \tau^\varepsilon - \partial_x u^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon + \partial_x p(\tau^\varepsilon) = -\sigma u^\varepsilon. \end{cases} & & & & (H_\varepsilon) \end{aligned}$$

**Limit  $\varepsilon \rightarrow 0$ :**

$$\begin{aligned} \tau^\varepsilon &= \bar{\tau} + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \dots \\ u^\varepsilon &= \bar{u} + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \end{aligned}$$

Order  $\varepsilon^0$ :

$$\begin{cases} \partial_t \bar{\tau} + \frac{1}{\sigma} \partial_{xx} p(\bar{\tau}) = 0, \\ \partial_x p(\bar{\tau}) = -\sigma \bar{u}. \end{cases} \quad (P)$$

# Relative entropy

## Entropy-entropy flux pair for $(H_\varepsilon)$

$$\eta^\varepsilon(\tau, u) = \varepsilon^2 \frac{u^2}{2} - P(\tau), \quad \psi(\tau, u) = u p(\tau),$$

where  $P(\tau) = \int_0^\tau p(s) ds$ , satisfying

$$\partial_t \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon) + \partial_x \psi(\tau^\varepsilon, u^\varepsilon) \leq -\sigma(u^\varepsilon)^2.$$

## Relative entropy

$$\begin{aligned} \eta^\varepsilon(\tau, u | \bar{\tau}, \bar{u}) &:= \eta^\varepsilon(\tau, u) - \eta^\varepsilon(\bar{\tau}, \bar{u}) - \nabla \eta^\varepsilon(\bar{\tau}, \bar{u}) \cdot \begin{pmatrix} \tau - \bar{\tau} \\ u - \bar{u} \end{pmatrix} \\ &= \frac{\varepsilon^2}{2} (u - \bar{u})^2 - P(\tau | \bar{\tau}), \end{aligned}$$

où  $P(\tau | \bar{\tau}) := P(\tau) - P(\bar{\tau}) - p(\bar{\tau})(\tau - \bar{\tau})$ .

Associated relative entropy flux:  $\psi(\tau, u | \bar{\tau}, \bar{u}) = (u - \bar{u})(p(\tau) - p(\bar{\tau}))$ .

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# Convergence rate

$$\phi^\varepsilon(t) := \int_{\mathbb{R}} \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon | \bar{\tau}, \bar{u}) dx.$$

## Theorem (Lattanzio and Tzavaras, 2013)

Let  $(\bar{\tau}, \bar{u})$  be a smooth solution of  $(P)$  such that

- $\bar{\tau} \geq c > 0$ ,
- $\|\partial_{xx} p(\bar{\tau})\|_{L^\infty(\mathbb{R} \times [0, T])} \leq K < +\infty$ ,
- $\|\partial_{xt} p(\bar{\tau})\|_{L^2(\mathbb{R} \times [0, T])} \leq K < +\infty$ .

Then

$$\phi^\varepsilon(t) \leq C (\phi^\varepsilon(0) + \varepsilon^4).$$



# Sketch of proof (1/2)

- **1st step:** write an equation verified by the relative entropy.

$$\begin{aligned} \partial_t \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon | \bar{\tau}, \bar{u}) + \partial_x \psi(\tau^\varepsilon, u^\varepsilon | \bar{\tau}, \bar{u}) = \\ - \sigma (u^\varepsilon - \bar{u})^2 + \frac{1}{\sigma} \partial_{xx} p(\bar{\tau}) p(\tau^\varepsilon | \bar{\tau}) + \frac{\varepsilon^2}{\sigma} \partial_{xt} p(\bar{\tau}) (u^\varepsilon - \bar{u}). \end{aligned}$$

- **2nd step:** integrate on  $Q_t = \mathbb{R} \times [0, t)$ .

$$\begin{aligned} \phi^\varepsilon(t) - \phi^\varepsilon(0) \leq - \sigma \int_{Q_t} (u^\varepsilon - \bar{u})^2 + \underbrace{\frac{1}{\sigma} \int_{Q_t} \partial_{xx} p(\bar{\tau}) p(\tau^\varepsilon | \bar{\tau})}_{T_1} \\ + \underbrace{\frac{\varepsilon^2}{\sigma} \int_{Q_T} \partial_{xt} p(\bar{\tau}) (u^\varepsilon - \bar{u})}_{T_2}. \end{aligned}$$

## Sketch of proof (2/2)

- **Control of  $T_1$ :**  $\exists C > 0$  such that  $\forall \tau, \bar{\tau} \geq c > 0, |p(\tau|\bar{\tau})| \leq -C P(\tau|\bar{\tau})$

$$\begin{aligned} \Rightarrow \frac{1}{\sigma} \int_{Q_t} |\partial_{xx} p(\bar{\tau}) p(\tau^\varepsilon|\bar{\tau})| dx ds &\leq -\frac{C}{\sigma} \|\partial_{xx} p(\bar{\tau})\|_\infty \int_{Q_t} P(\tau^\varepsilon|\bar{\tau}) dx ds \\ &\leq \frac{C}{\sigma} \int_0^t \phi^\varepsilon(s) ds. \end{aligned}$$

- **Control of  $T_2$ :** Cauchy-Schwarz and Young inequalities

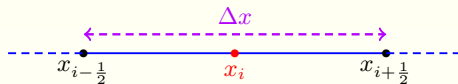
$$\begin{aligned} \Rightarrow \frac{\varepsilon^2}{\sigma} \int_{Q_t} |\partial_{xt} p(\bar{\tau})(u^\varepsilon - \bar{u})| &\leq \frac{\sigma}{2} \int_{Q_t} (u^\varepsilon - \bar{u})^2 + \frac{\varepsilon^4}{2\sigma^3} \int_{Q_t} |\partial_{xt} p(\bar{\tau})|^2 \\ &\leq \frac{\sigma}{2} \int_{Q_t} (u^\varepsilon - \bar{u})^2 dx ds + C \varepsilon^4. \end{aligned}$$

- **Conclusion:** using the Gronwall Lemma,

$$\phi^\varepsilon(t) \leq \phi^\varepsilon(0) + C\varepsilon^4 + \frac{C}{\sigma} \int_0^t \phi^\varepsilon(s) ds \quad \Rightarrow \quad \phi^\varepsilon(t) \leq (\phi^\varepsilon(0) + C\varepsilon^4) e^{CT/\sigma}.$$

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# From the continuous to the semidiscrete framework



- **Finite volume scheme**, with HLL fluxes:

$$\frac{d}{dt}\tau_i = \frac{1}{2\Delta x}(u_{i+1} - u_{i-1}) + \frac{\lambda}{2\Delta x}(\tau_{i+1} - 2\tau_i + \tau_{i-1}), \quad (H_\varepsilon^\Delta)$$

$$\frac{d}{dt}u_i = -\frac{1}{2\varepsilon^2\Delta x}(p(\tau_{i+1}) - p(\tau_{i-1})) + \frac{\lambda}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1}) - \frac{\sigma}{\varepsilon^2}u_i,$$

where  $\lambda = \max_{i \in \mathbb{Z}}(\sqrt{-p'(\tau_i)})$ .

- **Limit scheme** ( $\varepsilon \rightarrow 0$ ):

$$\frac{d}{dt}\bar{\tau}_i = \frac{1}{2\Delta x}(\bar{u}_{i+1} - \bar{u}_{i-1}) + \frac{\lambda}{2\Delta x}(\bar{\tau}_{i+1} - 2\bar{\tau}_i + \bar{\tau}_{i-1}),$$

$$\sigma\bar{u}_i = -\frac{1}{2\Delta x}(p(\bar{\tau}_{i+1}) - p(\bar{\tau}_{i-1})). \quad (P^\Delta)$$

# Convergence rate

$$\eta_i^\varepsilon := \eta^\varepsilon(\tau_i, u_i | \bar{\tau}_i, \bar{u}_i), \quad \phi^\varepsilon(t) := \sum_{i \in \mathbb{Z}} \Delta x \eta_i^\varepsilon(t).$$

## Theorem

We assume that  $(\bar{\tau}_i, \bar{u}_i)_{i \in \mathbb{Z}}$  solution of  $(P^\Delta)$  satisfies:

- $\bar{\tau}_i \geq c > 0$ ,  $\|D_{xx}p(\bar{\tau})\|_\infty, \|D_{xt}p(\bar{\tau})\|_2 \leq K < +\infty$ ,
- $\|\tilde{D}_{xx}\bar{\tau}\|_\infty \leq K$ ,  $\|D_x\bar{\tau}\|_\infty \leq K$ ,  $\|D_{xx}\bar{u}\|_2 \leq K$ .

### ► Définitions

Then

$$\phi^\varepsilon(t) \leq B(\phi^\varepsilon(0) + \varepsilon^4),$$

where  $B$  only depends on  $p$ ,  $T$ ,  $K$  and  $c$ .

# Sketch of proof (1/3)

- **1st step:** write an equation verified by the semidiscrete relative entropy.

$$\begin{aligned} \frac{d\eta_i^\varepsilon}{dt} + \frac{1}{\Delta x}(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}}) = \\ -\sigma(u_i - \bar{u}_i)^2 + \frac{1}{\sigma} \frac{p(\bar{\tau}_{i+2}) - 2p(\bar{\tau}_i) + p(\bar{\tau}_{i-2}))}{(2\Delta x)^2} p(\tau_i | \bar{\tau}_i) \\ + \frac{\varepsilon^2}{\sigma} (u_i - \bar{u}_i) \frac{d}{dt} \left( \frac{p(\bar{\tau}_{i+1}) - p(\bar{\tau}_{i-1}))}{2\Delta x} \right) + R_i^u + R_i^\tau, \end{aligned}$$

where

$$\psi_{i+\frac{1}{2}} := \frac{1}{2}(u_i - \bar{u}_i)(p(\tau_{i+1}) - p(\bar{\tau}_{i+1})) + \frac{1}{2}(p(\tau_i) - p(\bar{\tau}_i))(u_{i+1} - \bar{u}_{i+1}),$$

and

$$\begin{aligned} R_i^u &:= \frac{\lambda \varepsilon^2}{2\Delta x} (u_i - \bar{u}_i)(u_{i+1} - 2u_i + u_{i-1}), \\ R_i^\tau &:= -\frac{\lambda}{2\Delta x} [(p(\tau_i) - p(\bar{\tau}_i))(\tau_{i+1} - 2\tau_i + \tau_{i-1}) \\ &\quad - (\tau_i - \bar{\tau}_i)p'(\bar{\tau}_i)(\bar{\tau}_{i+1} - 2\bar{\tau}_i + \bar{\tau}_{i-1})]. \end{aligned}$$

# Sketch of proof (1/3)

- **1st step:** write an equation verified by the semidiscrete relative entropy.

$$\begin{aligned} \frac{d\eta_i^\varepsilon}{dt} + \frac{1}{\Delta x}(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}}) = \\ - \sigma(u_i - \bar{u}_i)^2 + \frac{1}{\sigma}(\tilde{D}_{xx}p(\bar{\tau}))_i p(\tau_i | \bar{\tau}_i) \\ + \frac{\varepsilon^2}{\sigma}(u_i - \bar{u}_i) \frac{d}{dt}(D_x p(\bar{\tau}))_i + R_i^u + R_i^\tau, \end{aligned}$$

where

$$\psi_{i+\frac{1}{2}} \approx (u - \bar{u})(p(\tau) - p(\bar{\tau}))|_{x_{i+\frac{1}{2}}} = \psi(\tau, u | \bar{\tau}, \bar{u})|_{x_{i+\frac{1}{2}}},$$

and

$$R_i^u := \frac{\lambda \varepsilon^2}{2} \Delta x (u_i - \bar{u}_i) (D_{xx} u)_i,$$

$$R_i^\tau := - \frac{\lambda}{2} \Delta x [(p(\tau_i) - p(\bar{\tau}_i))(D_{xx} \tau)_i - (\tau_i - \bar{\tau}_i) p'(\bar{\tau}_i) (D_{xx} \bar{\tau})_i].$$

# Sketch of proof (2/3)

- **2nd step:** integrate over  $[0, t)$  and sum over  $i \in \mathbb{Z}$ .

$$\begin{aligned}
 \phi^\varepsilon(t) - \phi^\varepsilon(0) &\leq -\sigma \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2(s) ds \\
 &\quad + \underbrace{\frac{1}{\sigma} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x \left( \tilde{D}_{xx} p(\bar{\tau}) \right)_i p(\tau_i | \bar{\tau}_i) ds}_{T_1} \\
 &\quad + \underbrace{\frac{\varepsilon^2}{\sigma} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i) \frac{d}{dt} (D_x p(\bar{\tau}))_i ds}_{T_2} \\
 &\quad + \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds + \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^\tau ds
 \end{aligned}$$

▸  $T_1$  et  $T_2$



## Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds = \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx} u)_i (u_i - \bar{u}_i) ds$$

# Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds = \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx}(u - \bar{u}))_i (u_i - \bar{u}_i) ds$$

$$+ \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx} \bar{u})_i (u_i - \bar{u}_i) ds.$$

# Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\begin{aligned} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds &= \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx}(u - \bar{u}))_i (u_i - \bar{u}_i) ds \\ &\quad + \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx}\bar{u})_i (u_i - \bar{u}_i) ds. \end{aligned}$$

- ▶ **1st term:** we perform a discrete integration by parts.

$$\begin{aligned} &\frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx}(u - \bar{u}))_i (u_i - \bar{u}_i) ds \\ &= -\frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x \left[ (D_x(u - \bar{u}))_{i+\frac{1}{2}} \right]^2 ds \leq 0. \end{aligned}$$

# Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds \leq \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (D_{xx} \bar{u})_i (u_i - \bar{u}_i) ds.$$

- ▶ **2nd term:** we use the Cauchy-Schwarz and Young inequalities.

$$\begin{aligned} \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \sqrt{\Delta x} (D_{xx} \bar{u})_i \sqrt{\Delta x} (u_i - \bar{u}_i) ds &\leq \\ \frac{\lambda \Delta x}{2} \frac{\varepsilon^4}{\theta} \|D_{xx} \bar{u}\|_2^2 + \frac{\lambda \Delta x}{2} \theta \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2. \end{aligned}$$

## Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds \leq C \varepsilon^4 + \frac{\lambda \theta \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2 ds.$$

## Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds \leq C \varepsilon^4 + \frac{\lambda \theta \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2 ds.$$

In the same way,

$$\int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^T ds \leq C \int_0^t \phi^\varepsilon(s) ds.$$

## Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^u ds \leq C \varepsilon^4 + \frac{\lambda \theta \Delta x}{2} \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2 ds.$$

In the same way,

$$\int_0^t \sum_{i \in \mathbb{Z}} \Delta x R_i^T ds \leq C \int_0^t \phi^\varepsilon(s) ds.$$

- **Conclusion:**

$$\phi^\varepsilon(t) \leq \left( -\frac{\sigma}{2} + \frac{\lambda \Delta x \theta}{2} \right) \int_0^t \sum_{i \in \mathbb{Z}} \Delta x (u_i - \bar{u}_i)^2 + \phi^\varepsilon(0) + C \varepsilon^4 + C \int_0^t \phi^\varepsilon(s) ds.$$

We choose  $\theta = \frac{\sigma}{\lambda \Delta x}$ , and we conclude by using the Gronwall Lemma.

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# Temporal discretization

Jin, Pareschi & Toscani, 98 : splitting scheme based on the reformulation

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = \frac{1}{\varepsilon^2} (\sigma u + (1 - \varepsilon^2) \partial_x p(\tau)). \end{cases}$$

## • Step 1: transport

$$\begin{aligned} \tau_i^{n+\frac{1}{2}} &= \tau_i^n + \frac{\Delta t}{2\Delta x} (u_{i+1}^n - u_{i-1}^n) + \frac{\lambda \Delta t}{2\Delta x} (\tau_{i+1}^n - 2\tau_i^n + \tau_{i-1}^n), \\ u_i^{n+\frac{1}{2}} &= u_i^n - \frac{\Delta t}{2\Delta x} (p(\tau_{i+1}^n) - p(\tau_{i-1}^n)) + \frac{\lambda \Delta t}{2\Delta x} (u_{i+1}^n - 2u_i^n + u_{i-1}^n). \end{aligned}$$

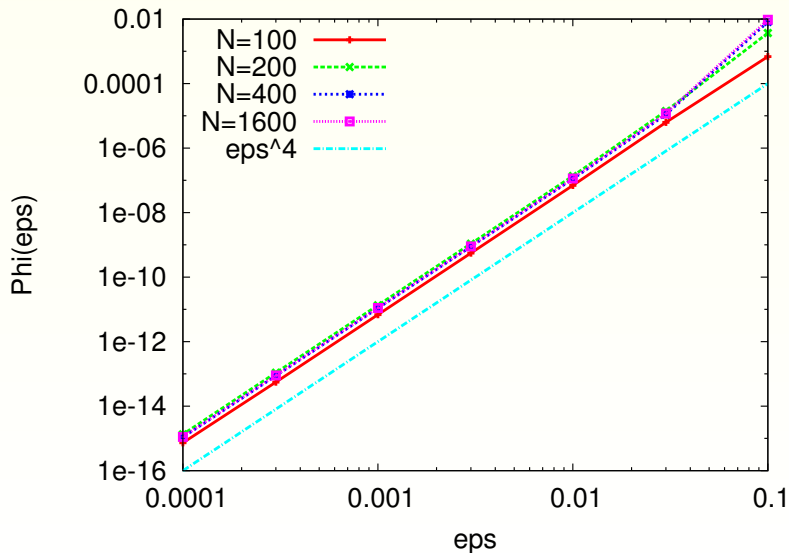
## • Step 2: relaxation

$$\begin{aligned} \tau_i^{n+1} &= \tau_i^{n+\frac{1}{2}}, \\ u_i^{n+1} &= u_i^{n+\frac{1}{2}} - \Delta t \frac{\sigma}{\varepsilon^2} u_i^{n+\frac{1}{2}} - \Delta t \left( \frac{1 - \varepsilon^2}{\varepsilon^2} \right) \frac{p(\tau_{i+1}^{n+\frac{1}{2}}) - p(\tau_{i-1}^{n+\frac{1}{2}})}{2\Delta x}. \end{aligned}$$

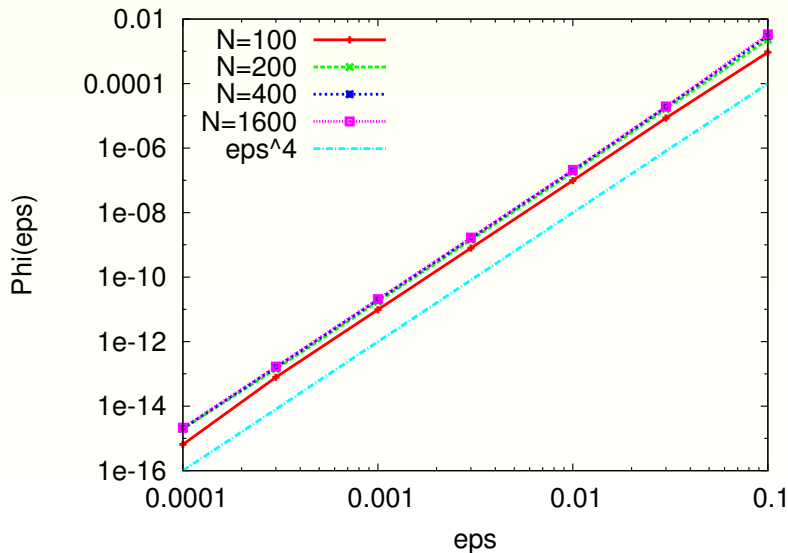
# Data

- Final time:  $T = 10^{-2}$ .
- Spatial domain:  $[-4, 4]$ , cell number = 100, 200, 400, 1600.
- Pressure law:  $p(s) = s^{-1.4}$ .
- $\sigma = 1$ ,  $\varepsilon \in \{10^{-1}, 3 \cdot 10^{-2}, 10^{-2}, 3 \cdot 10^{-3}, 10^{-3}, 3 \cdot 10^{-4}, 10^{-4}\}$ .
- Boundary conditions: homogeneous Neumann.
- Two initial data:
  - ▶ discontinuous:  $\tau_0(x) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$ ,
  - ▶ smooth:  $\tau_0(x) = \exp(-100x^2) + 1$ .

# Discontinuous I.C.



# Smooth I.C.



# Work in progress

- In the continuous framework:
  - ▶ generalization to other models,
  - ▶ general formalism ?
- In the discrete framework, generalization of the method:
  - ▶ for full spatiotemporal discretizations,
  - ▶ for general IMEX schemes,
  - ▶ for high order schemes.

**Thank you for your attention !**

## Definition of the semidiscrete norms

Let  $v : Q_T = \mathbb{R} \times [0, T) \rightarrow \mathbb{R}$  defined by  $v(x, t) = v_i(t)$ ,  $\forall x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ .

- $\|D_x v\|_\infty := \sup_{t \in [0, T)} \sup_{i \in \mathbb{Z}} \left| \frac{v_{i+1} - v_i}{\Delta x} \right|,$
- $\|\tilde{D}_{xx} v\|_\infty := \sup_{t \in [0, T)} \sup_{i \in \mathbb{Z}} \left| \frac{v_{i+2} - 2v_i + v_{i-2}}{(2\Delta x)^2} \right|,$
- $\|D_{xx} v\|_\infty := \sup_{t \in [0, T)} \sup_{i \in \mathbb{Z}} \left| \frac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2} \right|,$
- $\|D_{tx} v\|_2 := \left( \int_0^T \sum_{i \in \mathbb{Z}} \Delta x \left| \frac{d}{dt} \left( \frac{v_{i+1} - v_{i-1}}{2\Delta x} \right) (t) \right|^2 dt \right)^{\frac{1}{2}},$
- $\|D_{xx} v\|_2 := \left( \int_0^T \sum_{i \in \mathbb{Z}} \Delta x \left| \left( \frac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2} \right) (t) \right|^2 dt \right)^{\frac{1}{2}}.$