All-regime Lagrangian-Remap numerical schemes for the gas dynamics equations. Applications to the large friction and low Mach coefficients

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Joint works with M. Girardin and S. Kokh
Outline

1. Introduction
2. Large friction and low Mach regimes
3. Numerical strategy
4. Numerical results
Outline

1. Introduction
2. Large friction and low Mach regimes
3. Numerical strategy
4. Numerical results
Motivation: numerical study of two-phase flows in nuclear reactors

We consider the following model

\[
\begin{align*}
    \partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
    \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= 0 \\
    \partial_t (\rho E) + \nabla \cdot [(\rho E + p) u] &= 0
\end{align*}
\]

where \( \rho, u = (u, v)^t, E \) denote respectively the density, the velocity vector and the total energy of the fluid.

Let \( e = E - \frac{|u|^2}{2} \) be the specific and \( \tau = 1/\rho \) the covolume.
We are especially interested in the design of numerical schemes when the model depends on a parameter \( \epsilon > 0 \).

The following three flow regimes are of interest

Classical regime : \( \epsilon = O(1) \)
Low \( \epsilon \) regime : \( \epsilon \ll 1 \)
Limit regime : \( \epsilon \to 0 \)
Introduction

Large friction and low Mach regimes

Numerical strategy

Numerical results
Large friction regime

We consider the following model with friction and gravity

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= \rho (g - \alpha u) \\
\partial_t (\rho E) + \nabla \cdot [(\rho E + p)u] &= \rho u.(g - \alpha u)
\end{align*}
\]

where \( g, \alpha \) denote the gravity field and the friction coefficient.

The large friction regime is obtained by replacing \( \alpha \) with \( \frac{\alpha}{\epsilon} \)

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p &= \rho (g - \frac{\alpha}{\epsilon} u) \\
\partial_t (\rho E) + \nabla \cdot [(\rho E + p)u] &= \rho u.(g - \frac{\alpha}{\epsilon} u)
\end{align*}
\]

with \( \epsilon \ll 1 \)
Remark 1. Setting $u = u_0 + \epsilon u_1 + O(\epsilon^2)$, the long time behaviour of the solutions is given by

$$u_0 = 0$$
$$\partial_t \rho + \nabla \cdot (\rho u_1) = 0$$
$$\nabla p = \rho(g - \alpha u_1)$$
$$\partial_t (\rho e) + \nabla \cdot [(\rho e + p)u_1] = \rho u_1.(g - \alpha u_1)$$

see Hsiao-Liu, Nishihara, Junca-Rascle, Lin-Coulombel, Coulombel-Goudon, Marcati-Milani... for rigorous proofs

Remark 2. This system will not be considered in the design of the numerical strategy
Introducing the characteristic and non-dimensional quantities:

\[ x = \frac{x}{L}, \quad t = \frac{t}{T}, \quad \rho = \frac{\rho}{\rho_0}, \quad u = \frac{u}{u_0}, \]

\[ v = \frac{v}{v_0}, \quad e = \frac{e}{e_0}, \quad p = \frac{p}{p_0}, \quad c = \frac{c}{c_0} \]

with \( u_0 = v_0 = \frac{L}{T}, \ e_0 = p_0 \rho_0 \) and \( p_0 = \rho_0 c_0^2 \), the non-dimensional system is (no gravity, no friction)

\[ \partial_t \rho + \nabla \cdot (\rho u) = 0 \]

\[ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \frac{1}{M^2} \nabla p = 0 \]

\[ \partial_t (\rho e) + \nabla \cdot [(\rho e + p)u] + \frac{M^2}{2} \left( \partial_t (\rho u \cdot u) + \nabla \cdot (\rho u \cdot uu) \right) = 0 \]

where \( M = \frac{u_0}{c_0} \) denotes the Mach number and plays the role of \( \epsilon \).
Low Mach regime

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p &= 0 \\
\partial_t (\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2} \left( \partial_t (\rho \mathbf{u} \cdot \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) \right) &= 0
\end{align*}
\]

**Definition.** The flow is said to be in the low Mach regime if \( M \ll 1 \) and \( \nabla p = O(M^2) \)

**Remark 1.** Using asymptotic expansions in powers of \( M \) in the governing equations of \( \rho, \mathbf{u}, p \) and boundary conditions leads to

\[
\begin{align*}
\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{u}_0) &= 0 \\
\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \frac{1}{\rho_0} \nabla p_2 &= 0 \\
\nabla \cdot \mathbf{u}_0 &= 0
\end{align*}
\]

**Remark 2.** This system will not be considered in the design of the numerical strategy.
Numerical issue in the Low Mach regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(M)$
- a time step $\Delta t = O(hM)$

which is out of reach in practice.

More details can be found in the large body of literature on this subject: A. Majda, E. Turkel, H. Guillard, C. Viozat, B. Thornber, S. Dellacherie, P. Omnes, P-A. Raviart, F. Rieper, Y. Penel, P. Degond, S. Jin, J.-G. Liu, P. Colella, K. Pao, E. Turkel, R. Klein, J-P Vila, M.G., B. Després, M. Ndjinga, J. Jung, M. Sun, ...

**General cure**: change the treatment of acoustic waves in the low Mach regime by centering the pressure gradient.
Numerical issue in the large friction regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(\epsilon)$
- a time step $\Delta t = O(\epsilon)$

which is out of reach in practice


General cure: upwinding of the source terms at interfaces (USI)
A couple of definitions

**Uniform stability**
A scheme is said to be stable in the uniform sense *if the CFL condition is uniform* with respect to $\epsilon$
This avoids stringent CFL restrictions $\Delta t = O(hM)$ or $\Delta t = O(\epsilon)$

**Uniform consistency**
A scheme is said to be consistent in the uniform sense *if the truncation error is uniform* with respect to $\epsilon$
This avoids large numerical diffusion and mesh size restrictions $h = o(M)$ or $h = O(\epsilon)$

**All-regime scheme**
A scheme is said to be all-regime if it is able to compute accurate solutions with a mesh size $h$ and a time step $\Delta t$ independent of $\epsilon$
Our objective is to propose a numerical scheme that is

- all-regime: uniform stability and uniform consistency w.r.t. $\epsilon$
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

How to do that...
Outline

1 Introduction
2 Large friction and low Mach regimes
3 Numerical strategy
4 Numerical results
How to reach these objectives

How to get the uniform stability?
- implicit treatment of the fast phenomenon
- explicit treatment of the slow phenomenon (sake of accuracy)
→ Lagrange-Projection strategy  Coquel-Nguyen-Postel-Tran

How to get the uniform consistency?
- modify the numerical fluxes to reduce the numerical diffusion
→ Truncation errors in equivalent equations

How to deal with any (possibly strongly nonlinear) pressure law $p$?
- overcome the non linearities, ”linearization”
→ Relaxation strategy  Suliciu, Jin-Xin, Bouchut, C.-Coquel, C.-Coulombel

How to deal with unstructured meshes in multi-D?
- work on a fixed mesh (no need to deform unstructured meshes)
→ Operator splitting strategy and rotational invariance
Lagrange-Projection strategy

Let us first focus on the 1D system
\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} \rho u &= 0 \\
\frac{\partial}{\partial t} \rho u + \frac{\partial}{\partial x} (\rho u^2 + p) &= 0 \\
\frac{\partial}{\partial t} (\rho E) + \frac{\partial}{\partial x} (\rho E u + pu) &= 0
\end{align*}
\]

Using chain rule arguments, we also have
\[
\begin{align*}
\frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + \rho \frac{\partial}{\partial x} u &= 0 \\
\frac{\partial}{\partial t} \rho u + u \frac{\partial}{\partial x} \rho u + \rho u \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} p &= 0 \\
\frac{\partial}{\partial t} \rho E + u \frac{\partial}{\partial x} \rho E + \rho E \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} pu &= 0
\end{align*}
\]

so that splitting the transport part leads to
\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \rho \frac{\partial}{\partial x} u &= 0 \\
\frac{\partial}{\partial t} \rho u + \rho u \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} p &= 0 \\
\frac{\partial}{\partial t} \rho E + \rho E \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} pu &= 0 \\
\frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho &= 0 \\
\frac{\partial}{\partial t} \rho u + u \frac{\partial}{\partial x} \rho u &= 0 \\
\frac{\partial}{\partial t} \rho E + u \frac{\partial}{\partial x} \rho E &= 0
\end{align*}
\]

Lagrangian-step

Transport-step
Lagrange-Projection strategy

The Lagrangian-step

\[
\begin{align*}
\partial_t \rho + \rho \partial_x u &= 0 \\
\partial_t \rho u + \rho u \partial_x u + \partial_x p &= 0 \\
\partial_t \rho E + \rho E \partial_x u + \partial_x pu &= 0
\end{align*}
\]

also writes

\[
\begin{align*}
\partial_t \tau - \partial_m u &= 0 \\
\partial_t u + \partial_m p &= 0 \\
\partial_t E + \partial_m pu &= 0
\end{align*}
\]

with \( \tau = 1/\rho \) and \( \tau \partial_x = \partial_m \).

- Eigenvalues are given by \(-\rho c, 0, \rho c\)
- Usual CFL conditions for time-explicit schemes write

\[
\frac{\Delta t}{h} \max(\rho c) \leq \frac{1}{2}
\]

The idea is to propose a time-implicit scheme to avoid this time-step restriction (\( \Delta t = O(hM) \) in the low Mach regime)
Lagrange-Projection strategy

The Transport-step is

\[
\begin{align*}
\partial_t \rho + u \partial_x \rho &= 0 \\
\partial_t \rho u + u \partial_x \rho u &= 0 \\
\partial_t \rho E + u \partial_x \rho E &= 0
\end{align*}
\]

also writes

\[
\begin{align*}
\partial_t \rho + \partial_x \rho u - \rho \partial_x u &= 0 \\
\partial_t \rho u + \partial_x \rho u^2 - \rho u \partial_x u &= 0 \\
\partial_t \rho E + \partial_x \rho Eu - \rho E \partial_x u &= 0
\end{align*}
\]

- Eigenvalues are given by \( u \)
- Usual CFL conditions for time-explicit schemes write

\[
\frac{\Delta t}{h} \max(|u|) \leq \frac{1}{2}
\]

The idea is then to propose a standard time-explicit scheme to keep accuracy on the slow phenomenon (\( \Delta t = O(h) \) in all regime)
Operator splitting strategy

We will consider the following three-step numerical scheme:

**First step** \((t^n \rightarrow t^{Lag})\): solve **implicitly** the acoustic system with the solution at time \(t^n\) as initial solution

**Second step** \((t^{Lag} \rightarrow t^{n+1^-})\) solve **implicitly** the source terms when present with the solution at time \(t^{Lag}\) as initial solution

**Third step** \((t^{n+1^-} \rightarrow t^{n+1})\) solve **explicitly** the transport system with the solution at time \(t^{n+1^-}\) as initial solution

Solving implicitly the source terms avoid the time-step restriction \(\Delta t = O(\epsilon)\) when \(\epsilon \ll 1\) \((\Delta t = O(h)\) in all regime\)
A few words about the relaxation approach

The gas dynamics equations in Lagrangian coordinates:

\[
\begin{align*}
\partial_t \tau - \partial_m u &= 0 \\
\partial_t u + \partial_m p &= 0 \\
\partial_t E + \partial_m pu &= 0
\end{align*}
\]

with \( p = p(\tau, e) \) and \( e = E - \frac{1}{2} u^2 \)

Due to the nonlinearities of \( p \), the Riemann problem is difficult to solve. The relaxation strategy:

- **Idea**: to deal with a larger but simpler system
- **Design principle**: to understand \( p(\tau, e) \) as a new unknown that we denote \( \Pi \)
A few words about the relaxation approach

The proposed relaxation system is

\[
\begin{align*}
\partial_t \tau - \partial_m u &= 0 \\
\partial_t u + \partial_m \Pi &= 0 \\
\partial_t E + \partial_m \Pi u &= 0 \\
\partial_t \Pi + a^2 \partial_m u &= \lambda (p - \Pi)
\end{align*}
\]

At least formally, observe that

\[
\lim_{\lambda \to +\infty} \Pi = p \quad (\text{if} \quad a > \rho c(\tau, e))
\]

(see e.g. Chalons-Coulombel for a rigorous proof)

Why is it interesting? The characteristic fields are linearly degenerate whatever the pressure law is!
A few words about the relaxation approach

The time-explicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

\[
\begin{align*}
\tau_{\text{Lag}}^n_j &= \tau_j^n + \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*), \\
u_{\text{Lag}}^n_j &= u_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* - p_{j-1/2}^*), \\
\Pi_{\text{Lag}}^n_j &= \Pi_j^n - a^2 \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*), \\
E_{\text{Lag}}^n_j &= E_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* u_{j+1/2}^* - p_{j-1/2}^* u_{j-1/2}^*)
\end{align*}
\]

with \( \Pi_j^n = p(\tau_j^n, e_j^n) \) and

\[
\begin{align*}
u_{j+1/2}^* &= \frac{1}{2} (u_j^n + u_{j+1}^n) - \frac{1}{2a} (\Pi_{j+1}^n - \Pi_j^n), \\
p_{j+1/2}^* &= \frac{1}{2} (\Pi_j^n + \Pi_{j+1}^n) - \frac{a}{2} (u_{j+1}^n - u_j^n)
\end{align*}
\]
A few words about the relaxation approach

The time-implicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

\[
\begin{align*}
\tau_j^{\text{Lag}} &= \tau_j^n + \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*) \\
u_j^{\text{Lag}} &= u_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* - p_{j-1/2}^*) \\
\Pi_j^{\text{Lag}} &= \Pi_j^n - a^2 \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*) \\
E_j^{\text{Lag}} &= E_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* u_{j+1/2}^* - p_{j-1/2}^* u_{j-1/2}^*)
\end{align*}
\]

with \( \Pi_j^n = p(\tau_j^n, e_j^n) \) and

\[
\begin{align*}
u_{j+1/2}^* &= \frac{1}{2} \left( u_j^{\text{Lag}} + u_{j+1}^{\text{Lag}} \right) - \frac{1}{2a} (\Pi_{j+1}^{\text{Lag}} - \Pi_j^{\text{Lag}}) \\
p_{j+1/2}^* &= \frac{1}{2} \left( \Pi_j^{\text{Lag}} + \Pi_{j+1}^{\text{Lag}} \right) - \frac{a}{2} \left( u_{j+1}^{\text{Lag}} - u_j^{\text{Lag}} \right)
\end{align*}
\]
A few words about the relaxation approach

The **time-implicit** scheme
- deals with (possibly strongly nonlinear) pressure laws
- is free of CFL restriction!
- is **not expensive** in the sense that only a **linear** problem w.r.t. $u$ and $\Pi$ has to be solved. Thanks to the relaxation strategy!
Formulation on unstructured meshes

On unstructured meshes, the time-explicit ($\# = n$) and time-implicit ($\# = Lag$) schemes write

\[
\begin{align*}
\mathbf{u}_j^{Lag} & = \mathbf{u}_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} \Pi_{jk}^* \mathbf{n}_{jk} \\
\Pi_j^{Lag} & = \Pi_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} (a_{jk})^2 u_{jk}^* \\
\tau_j^{Lag} & = \tau_j^n + \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \\
E_j^{Lag} & = E_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} p_{jk}^* u_{jk}^* \\
u_{jk}^* & = \frac{1}{2} \mathbf{n}_{jk}^T (\mathbf{u}_j^\# + \mathbf{u}_k^\#) - \frac{1}{2a_{jk}} (\Pi_k^\# - \Pi_j^\#), \quad p_{jk}^* = \frac{1}{2} (\Pi_j^\# + \Pi_k^\#) - \frac{a_{jk}}{2} \mathbf{n}_{jk}^T (\mathbf{u}_k^\# - \mathbf{u}_j^\#)
\end{align*}
\]
The **time-implicit** point-wise scheme for the gravity terms and external forces writes

\[
\begin{align*}
\tau_j^{n+1-} &= \tau_j^{\text{Lag}} \\
\mathbf{u}_j^{n+1-} &= \mathbf{u}_j^{\text{Lag}} + \Delta t (\mathbf{g} - \alpha \mathbf{u}_j^{n+1-}) \\
E_j^{n+1-} &= E_j^{\text{Lag}} + \Delta t \mathbf{u}_j^{n+1-} \cdot (\mathbf{g} - \alpha \mathbf{u}_j^{n+1-})
\end{align*}
\]

It is free of CFL restriction
In order to approximate the solutions of the transport step:

\[
\begin{align*}
\partial_t \rho + (\mathbf{u} \cdot \nabla) \rho &= 0 \\
\partial_t (\rho \mathbf{u}) + (\mathbf{u} \cdot \nabla) \rho \mathbf{u} &= 0 \iff \\
\partial_t (\rho E) + (\mathbf{u} \cdot \nabla) \rho E &= 0
\end{align*}
\]

we simply use the time-explicit upwind finite-volume scheme:

\[
\varphi_j^{n+1} = \varphi_j^{n+1-} - \Delta t \sum_{k \in N(j)} \frac{\Gamma_{jk}}{|\Omega_j|} u_{jk}^* \varphi_j^{n+1-} + \Delta t \varphi_j^{n+1-} \sum_{k \in N(j)} \frac{\Gamma_{jk}}{|\Omega_j|} u_{jk}^*
\]

where \( \varphi = \rho, \rho \mathbf{u}, \rho E \) and \( \varphi_{jk}^{n+1-} = \begin{cases} 
\varphi_j^{n+1-} & \text{if } u_{jk}^* > 0 \\
\varphi_j^{n+1-} & \text{if } u_{jk}^* \leq 0
\end{cases} \)

This scheme is stable under a material CFL condition (\( \Delta t = O(h) \)).
Our objective was to propose a numerical scheme that is

- all-regime: uniform stability and uniform consistency w.r.t. $\epsilon$
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

What about the uniform consistency?
Let us first recall that the Lagrangian-step in 1D writes:

\[ \tau_{j+1}^n = \tau_j^n + \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*) \]

\[ u_{j+1}^n = u_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* - p_{j-1/2}^*) \]

\[ E_{j+1}^n = E_j^n - \frac{\Delta t}{\Delta m} ((pu)^*_{j+1/2} - (pu)^*_{j-1/2}) \]

with

\[ u_{j+1/2}^* = \frac{1}{2} (u_j + u_{j+1}) - \frac{1}{2a} (p_{j+1} - p_j) \]

\[ p_{j+1/2}^* = \frac{1}{2} (p_j + p_{j+1}) - \frac{a}{2} (u_{j+1} - u_j) \]
Uniform consistency in the low Mach regime

In dimensionless form we get

\[ \tau^{n+1}_j = \tau^n_j + \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*) \]
\[ u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* - p_{j-1/2}^*) \]
\[ E^{n+1}_j = E^n_j - \frac{\Delta t}{\Delta m} ((pu)_{j+1/2}^* - (pu)_{j-1/2}^*) \]

with, since \( p_{j+1} - p_j = \mathcal{O}(\Delta m M^2) \),

\[ u_{j+1/2}^* = \frac{u_j + u_{j+1}}{2} - \frac{M}{2a} \frac{(p_{j+1} - p_j)}{M^2} = \frac{u_j + u_{j+1}}{2} + \mathcal{O}(M \Delta m) \]

\[ p_{j+1/2}^* = \frac{p_j + p_{j+1}}{2M^2} - \frac{a}{2M} (u_{j+1} - u_j) = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O} \left( \frac{\Delta m}{M} \right) \]
Uniform consistency in the low Mach regime

The problem comes from the numerical diffusion in $p_{j+1/2}^*$

To obtain the uniform consistency w.r.t. $M$ we introduce the parameter $\theta_{j+1/2}$ and simply consider the new definition of $p_{j+1/2}^*$

$$p_{j+1/2}^* = \frac{1}{2}(p_j^n + p_{j+1}^n) - \theta_{j+1/2} \frac{a}{2} (u_{j+1}^n - u_j^n)$$

Then we get $p_{j+1/2}^* = \frac{p_j + p_{j+1}}{2M^2} + O\left(\frac{\theta_{j+1/2} \Delta m}{M}\right)$

Which gives the uniform consistency provided that $\theta_{j+1/2} = O(M)$

The modification is extremely simple and applies directly in multi-D
Uniform consistency in the large friction regime

Let us first recall that the first two steps write

\[
\begin{align*}
\tau_j^{n+1} &= \tau_j^n + \frac{\Delta t}{\Delta m} (u_{j+1/2}^* - u_{j-1/2}^*) \\
u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta m} (p_{j+1/2}^* - p_{j-1/2}^*) + \Delta t (g - \frac{\alpha}{\epsilon} u_j^{n+1}) \\
E_j^{n+1} &= E_j^n - \frac{\Delta t}{\Delta m} ((pu)_{j+1/2}^* - (pu)_{j-1/2}^*) + \Delta t u_j^{n+1} (g - \frac{\alpha}{\epsilon} u_j^{n+1})
\end{align*}
\]

with

\[
\begin{align*}
u_{j+1/2}^* &= \frac{1}{2} (u_j + u_{j+1}) - \frac{1}{2a} (p_{j+1} - p_j) \\
p_{j+1/2}^* &= \frac{1}{2} (p_j + p_{j+1}) - \frac{a}{2} (u_{j+1} - u_j)
\end{align*}
\]
Uniform consistency in the large friction regime

\[
\begin{align*}
\tau_{j}^{n+1^-} &= \tau_{j}^{n} + \Delta t \left( u_{j+1/2}^* - u_{j-1/2}^* \right) \\
u_{j}^{n+1^-} &= u_{j}^{n} - \frac{\Delta t}{\Delta m} \left( p_{j+1/2}^* - p_{j-1/2}^* \right) + \Delta t \left( g - \frac{\alpha}{\epsilon} u_{j}^{n+1^-} \right) \\
u_{j+1/2}^* &= \frac{1}{2} \left( u_{j}^{n} + u_{j+1}^{n} \right) - \frac{1}{2a} \left( p_{j+1}^{n} - p_{j}^{n} \right) \\
p_{j+1/2}^* &= \frac{1}{2} \left( p_{j}^{n} + p_{j+1}^{n} \right) - \frac{a}{2} \left( u_{j+1}^{n} - u_{j}^{n} \right)
\end{align*}
\]

Numerical asymptotic analysis. \( u_{j} = u_{j}^{(0)} + \epsilon u_{j}^{(1)} + \mathcal{O}(\epsilon^2) \)

- Multiply the second equation by \( \epsilon \) and let \( \epsilon \to 0 \) : \( u_{j}^{(0)} = 0 \)
- Let \( \epsilon \to 0 \) in the second equation : \( \frac{p_{j+1} - p_{j-1}}{2\Delta m} = \left( g - \alpha u_{j}^{(1)} \right) \)
- It remains to insert \( u_{j} = 0 + \epsilon u_{j}^{(1)} + \mathcal{O}(\epsilon^2) \) in the first equation
Let us insert $u_j = 0 + \epsilon u_j^{(1)} + O(\epsilon^2)$ in the first equation, we immediately get

$$
\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m} \epsilon (u_{j+1/2}^{(1)} - u_{j-1/2}^{(1)}) + O(\epsilon^2)
$$

with

$$
u_{j+1/2}^{(1)} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} - \frac{1}{\epsilon} \frac{p_{j+1} - p_j}{2a} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} + O\left(\frac{\Delta m}{\epsilon}\right)
$$

which is clearly not consistent with $\partial_t \tau - \epsilon \partial_m u_1 = O(\epsilon^2)$
Uniform consistency in the large friction regime

The problem comes from the numerical diffusion in $u_{j+1/2}^*$

To obtain an uniform consistency w.r.t. $\epsilon$ we introduce the parameter $\theta_{j+1/2}$ and simply consider the following definition of $u_{j+1/2}^*$

$$u_{j+1/2}^* = \frac{1}{2} (u_j^n + u_{j+1}^n) - \frac{\theta_{j+1/2}}{2a} (p_{j+1}^n - p_j^n)$$

Then we get $u_{j+1/2}^{(1)} = \frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} + O(\frac{\theta_{j+1/2} \Delta m}{\epsilon})$

Which gives the uniform consistency provided that $\theta_{j+1/2} = O(\epsilon)$

The modification is extremely simple and applies directly in multi-D
All the objectives are reached!

How does the modifications affect the stability properties?
- conservative (with no source terms and external forces)
- positive
- unconditionally entropy satisfying for all $\theta \geq 0$ in the linear case
- conditionally entropy satisfying in the non linear case. $\theta = 0$ is also possible in practice! (numerical diffusion in the transport step)

Interestingly, operator-splitting techniques are compatible with the all-regime property. **USI approach not mandatory**

High-order extension under progress using DG methods, as well as shallow-water equations and diffusion terms
Outline

1. Introduction
2. Large friction and low Mach regimes
3. Numerical strategy
4. Numerical results
Numerical results

We want to assess the following properties of the numerical scheme:

- Accuracy of the numerical scheme in the large friction regime if $\tilde{\theta} = O(\epsilon)$
- Accuracy of the numerical scheme in the low Mach regime if $\theta = O(M)$
- Robustness of the numerical scheme with respect to the choice of $\theta$ (resp. $\tilde{\theta}$) in and outside the low Mach regime (resp. large friction regime)
- Performance in terms of CPU time of the mixed implicit-explicit numerical scheme
Large friction modification

Large friction modification
The fluid is equipped with a perfect gas equation of state

\[ p = (\gamma - 1) \rho e, \quad \gamma = 1.4 \]

We consider the domain \( \Omega = (0, 1) \).

The initial condition is given by

\[
\begin{cases}
(\rho, u, p) = (1.0, 0, 10000.0), & \text{if } x \in [0, 0.35] \cap [0.65, 1], \\
(\rho, u, p) = (2.0, 0, 26390.2), & \text{if } x \in [0.35, 0.65].
\end{cases}
\]

We impose periodic boundary conditions.

The friction parameter is given by \( \alpha = 10^6 \, s^{-1} \), so that we are in the large friction regime.
test case : sensitivity w.r.t. the space step

We compute approximate solutions with a 100-cell, 1000-cell and a 10 000-cell grid, with $\beta = n$

$$\tilde{\Theta} = 1$$

$$\tilde{\Theta} = \min \left( \frac{2a}{\alpha \Delta x}, 1 \right)$$
test case: sensitivity w.r.t. the space step

We plot convergence curves in $L^1$ norm for

\[ \tilde{\theta} = 1 \text{ (black), } \tilde{\theta} = \min \left( \frac{2a}{\alpha \Delta x}, 1 \right) \text{ (blue), } \tilde{\theta} = \frac{1}{\alpha} \text{ (red)} \]
Low Mach modification
Vortex in a box: test case

The fluid is equipped with a perfect gas equation of state

\[ p = (\gamma - 1)\rho e, \quad \gamma = 1.4 \]

We consider the domain \( \Omega = (0, 1)^2 \).

The initial condition is given by

\[
\begin{aligned}
\rho_0(x, y) &= 1 - \frac{1}{2} \tanh \left( y - \frac{1}{2} \right), & u_0(x, y) &= 2\sin^2(\pi x)\sin(\pi y)\cos(\pi y), \\
p_0(x, y) &= 1000, & v_0(x, y) &= -2\sin(\pi x)\cos(\pi x)\sin^2(\pi y).
\end{aligned}
\]

We impose a no-slip boundary condition.

This configuration leads to a Mach number of order 0.026, so that we are in the low Mach regime.
Vortex in a box \((M\#0.026)\): explicit scheme

We plot the flow speed magnitude at time \(T = 0.125s\).

- explicit scheme \((\theta = 1)\)
  - Cartesian Mesh
  - \(50 \times 50\) cells
- explicit scheme \((\theta = 1)\)
  - Cartesian Mesh
  - \(400 \times 400\) cells
- reference solution
  - explicit scheme
  - \((\theta = 1)\)
  - Triangular Mesh
Vortex in a box ($M\#=0.026$): modified explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.

- explicit scheme ($\theta = 1$)
  - Cartesian Mesh
  - 50 * 50 cells

- explicit scheme ($\theta_{ij} = M_{ij}^n$)
  - Cartesian Mesh
  - 50 * 50 cells

- reference solution
  - explicit scheme ($\theta = 1$)
  - Triangular Mesh
Vortex in a box ($M\#0.026$): modified implicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.

- implicit-explicit scheme ($\theta = 1$)
  - Cartesian Mesh
  - $50 \times 50$ cells

- implicit-explicit scheme ($\theta_{ij} = M_{ij}^n$)
  - Cartesian Mesh
  - $50 \times 50$ cells

- reference solution
  - explicit scheme ($\theta = 1$)
  - Triangular Mesh
**Vortex in a box (M#0.026) : CPU Time**

EX : $\beta = n$,  \quad IMEX : $\beta = \text{Lag}$.

<table>
<thead>
<tr>
<th>Numerical scheme</th>
<th>EX($\theta = 1$) (Mesh 400 * 400)</th>
<th>EX($\theta = 1$) (Mesh 50 * 50)</th>
<th>EX($\theta_{ij} = M_{ij}$) (Mesh 50 * 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>18 457</td>
<td>2 306</td>
<td>2 305</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>9 263.04 (2h34min)</td>
<td>17.14</td>
<td>19.3</td>
</tr>
</tbody>
</table>

**Speed up ($\theta = 1 \rightarrow \theta_{ij} = M_{ij}$) = 480**

<table>
<thead>
<tr>
<th>Numerical scheme</th>
<th>IMEX($\theta = 1$) (Mesh 50 * 50)</th>
<th>IMEX($\theta_{ij} = M_{ij}$) (Mesh 50 * 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>43</td>
<td>56</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>3.75</td>
<td>5.77</td>
</tr>
</tbody>
</table>

**Speed up (explicit $\rightarrow$ implicit-explicit) = 3.3**
Vortex in a box ($M = 0.026$): Influence of the cell geometry

We plot a 1D-cut at $x = 0.5$ of the flow speed magnitude at time $T = 0.125s$. 

![Velocity Magnitude - Cartesian Mesh](image1)

![Velocity Magnitude - Triangular Mesh](image2)
2D-Riemann problem: test case

The fluid is equipped with a perfect gas equation of state

\[ p = (\gamma - 1)\rho e, \quad \gamma = 1.4 \]

We consider the domain \( \Omega = (0, 1)^2 \).

The initial condition corresponds to a 2D Riemann problem that consists of 4 shock waves. We impose Neumann boundary conditions.

This configuration leads to a Mach number that ranges from \( 10^{-5} \) to 3.15, so that we have both low Mach and order 1 Mach values.
2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified explicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.

- Explicit scheme ($\theta = 1$), Cartesian Mesh, $50 \times 50$ cells
- Explicit scheme ($\theta = 0$), Cartesian Mesh, $50 \times 50$ cells
- Reference solution, explicit scheme ($\theta = 1$), Triangular Mesh
2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified implicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.

- implicit-explicit scheme ($\theta = 1$)
  Cartesian Mesh
  $50 \times 50 \text{cells}$

- implicit-explicit scheme ($\theta = 0$)
  Cartesian Mesh
  $50 \times 50 \text{cells}$

- reference solution
  explicit scheme ($\theta = 1$)
  Triangular Mesh
2D-Riemann problem $M \in (10^{-5}, 3.15)$: CPU time

<table>
<thead>
<tr>
<th>Numerical scheme</th>
<th>EX($\theta = 1$) (Mesh 50 * 50)</th>
<th>EX($\theta = 0$) (Mesh 50 * 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>323</td>
<td>343</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>2.59</td>
<td>2.79</td>
</tr>
</tbody>
</table>

Speed up ($\theta = 1 \rightarrow \theta = 0$) $\approx 1$

<table>
<thead>
<tr>
<th>Numerical scheme</th>
<th>IMEX($\theta = 1$) (Mesh 50 * 50)</th>
<th>IMEX($\theta = 0$) (Mesh 50 * 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>216</td>
<td>218</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>10.28</td>
<td>10.33</td>
</tr>
</tbody>
</table>

Speed up (explicit $\rightarrow$ implicit-explicit) $= 0.25$
flow in a channel with bump

The fluid is equipped with a mixture of two perfect gas with different adiabatic coefficients equation of state: $\gamma_1 = 2, \gamma_2 = 1.4$.

We consider for the domain a channel with a 20% sinusoidal bump.

The initial condition corresponds to a constant state

$$(\rho, Y, p, u, v) = (7.81, 0, 3124, 0, 0).$$

We impose inlet/outlet and Wall boundary conditions.

This configuration leads to a subsonic flow for $u_{in} = 0.2$ and a transonic flow for $u_{in} = 12$. 
flow in a channel with bump : subsonic flow

We plot the results obtained for the subsonic test case $(u_{in} = 0.2)$ on a $80 \times 20$ quadrangular mesh at time $T = 2s$ with $\beta = Lag$ and $\theta_{ij} = M_{ij}$

Flow speed animation
We plot the results obtained for the transonic test case \((u_{in} = 12)\) on a \(80 \times 20\) quadrangular mesh at time \(T = 2s\) with \(\beta = n\) and \(\theta_{ij} = 0\).


C. Chalons, M. Girardin and S. Kokh, An all-regime Lagrange-Projection like scheme for 2D homogeneous models for two-phase flows on unstructured meshes, submitted to JCP

Numerical strategies

Several approaches can be envisaged to compute accurate solutions when $\epsilon \ll 1$

- Use and discretize the limit model (the nature of which changes)
- Couple the original and limit models at moving interfaces
- Design Asymptotic-Preserving schemes (consistency with the limit model when $\epsilon \to 0$ and with the original model when $\epsilon \to 0$, no coupling)
- Consider all-regime stability and consistency properties ($\epsilon$ is kept constant in order to compute accurate solutions also in intermediate regimes)