Sharp Interface Limit for the Navier–Stokes–Korteweg Equations

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Johannes Daube

(Joint work with H. Abels, C. Kraus and D. Kröner)

11th DFG-CNRS Workshop Micro-Macro Modeling and Simulations of Liquid-Vapour Flows

Paris, March 3, 2016

Organisation

Diffuse Interface Models

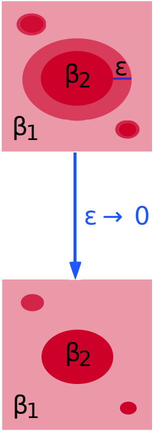
- interface of non-zero thickness (diffuse interface)
- system of PDEs on whole domain describes motion of fluid
- smooth order parameter (here: density) indicates different phases

Sharp Interface Models

- infinitely thin (sharp) interface separates different fluid phases
- system of PDEs in each bulk phase describes motion of fluid
- coupling by boundary conditions at interface
- jump discontinuities at interface

Sharp Interface Limits

- relates diffuse to sharp interface models
- letting in diffuse models the interface thickness tend to zero leads to corresponding sharp interface models



Navier–Stokes–Korteweg Equation (Diffuse Interface Model)

- extension of the compressible Navier–Stokes equations (Korteweg 1901)
- diffuse interface model for liquid–vapour flows which allows for phase transitions
- a small parameter $\varepsilon > 0$ represents the thickness of an interfacial area, where phase transitions occur
- phase field like scaling (Hermsdörfer, Kraus and Kröner 2011)

$$
(\mathrm{DIM})_\varepsilon \quad \begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v_\varepsilon) = 0 \quad \text{in } \Omega \times [0,T], \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \operatorname{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \frac{1}{\varepsilon}\nabla p(\rho_\varepsilon) = 2\operatorname{div}(\mu(\rho_\varepsilon) D v_\varepsilon) + \varepsilon\gamma\rho_\varepsilon \nabla\Delta\rho_\varepsilon \quad \text{in } \Omega \times [0,T], \end{cases}
$$

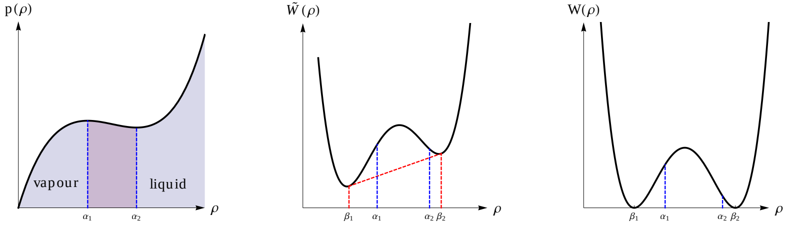
where

- ρ_ε density,
- v_ε velocity,
- $p(\rho_\varepsilon)$ (non-monotone) pressure,
- $\mu(\rho_\varepsilon)$ viscosity.

Boundary conditions: $\nabla\rho_\varepsilon \cdot v = 0$ and $v_\varepsilon = 0$ on $\partial\Omega \times [0,T]$.

Initial conditions: $\rho_\varepsilon(\cdot,0) = \rho_\varepsilon^{(i)}$ and $v_\varepsilon(\cdot,0) = v_\varepsilon^{(i)}$ in Ω.

- non-monotone pressure function $p(\rho) = \rho \widetilde{W}'(\rho) - \widetilde{W}(\rho)$
- (normalized) double-well potential $W(\rho) = \widetilde{W}(\rho) - l(\rho)$, where l Maxwell line and $0 < \beta_1 < \beta_2$ Maxwell points
- Note: $p'(\rho) = \rho \widetilde{W}''(\rho) = \rho W''(\rho) \Longrightarrow \nabla p(\rho) = p'(\rho)\nabla\rho = \rho W''(\rho)\nabla\rho$



Growth of the Double-Well Potential. For $a = \frac{\beta_1+\beta_2}{2}$ and $b = \frac{\beta_2-\beta_1}{2}$, there exist constants $C_1 > 0$, $C_2 \in (0,b)$ and $p > 2$ such that

$$W''(z) \geq C_1 \, |z-a|^{p-2} \text{ for all } z \in \mathbb{R} \text{ with } |z-a| \geq b - C_2.$$

Prototypical example ($p = 4$): $W(z) = (z-\beta_1)^2(z-\beta_2)^2$.

Static Case

Motivation for phase field like scaling of the Navier–Stokes–Korteweg equations.

Static case: velocity $v_\varepsilon \equiv 0$, time independent; $\gamma = 1$ for simplicity.

In the static case $(\mathrm{DIM})_\varepsilon$ reduces to

$$\frac{1}{\varepsilon}\nabla p(\rho_\varepsilon) = \varepsilon\rho_\varepsilon\nabla\Delta\rho_\varepsilon$$

$$(p'(\rho) = \rho W''(\rho)) \quad \Longrightarrow \quad \varepsilon\rho_\varepsilon\nabla\Delta\rho_\varepsilon = \frac{1}{\varepsilon}p'(\rho_\varepsilon)\nabla\rho_\varepsilon = \frac{1}{\varepsilon}\rho_\varepsilon\nabla W'(\rho_\varepsilon)$$

$$(\text{assume } \rho_\varepsilon > 0) \quad \Longrightarrow \quad \nabla\left(\frac{1}{\varepsilon}W'(\rho_\varepsilon) - \varepsilon\Delta\rho_\varepsilon\right) = 0.$$

Hence, there is a constant $l_\varepsilon \in \mathbb{R}$ such that

$$\frac{1}{\varepsilon}W'(\rho_\varepsilon) - \varepsilon\Delta\rho_\varepsilon = l_\varepsilon.$$

Euler–Lagrange equation with Lagrange multiplier l_ε of

$$\min\left\{E_\varepsilon(\rho) : \rho \in H^1(\Omega),\ \int_\Omega \rho(x)\,\mathrm{d}x = m\right\}, \quad E_\varepsilon(\rho) = \int_\Omega \frac{1}{\varepsilon}W(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2\,\mathrm{d}x,$$

where $m \in (\beta_1|\Omega|, \beta_2|\Omega|)$ (prescribed total mass).

Heuristic picture from minimization problem

$$\min\left\{\int_\Omega \frac{1}{\varepsilon}W(\rho) + \frac{\varepsilon}{2}\left|\nabla\rho\right|^2 \mathrm{d}x : \rho \in H^1(\Omega),\ \int_\Omega \rho(x)\,\mathrm{d}x = m\right\}.$$

$$\int_\Omega \frac{1}{\varepsilon}W(\rho)\,\mathrm{d}x$$

- favours configurations “close to pure phases” β_1 and β_2 due to the double-well shape of W.
- minimizing $\int_\Omega \frac{1}{\varepsilon}W(\rho)\,\mathrm{d}x$ leads to phase separation.

$$\int_\Omega \frac{1}{2}\varepsilon\left|\nabla\rho\right|^2 \mathrm{d}x$$

- penalizes occurrence of large interfaces, i.e., rapid transitions between

$$\{\rho \approx \beta_1\} \quad \text{and} \quad \{\rho \approx \beta_2\}.$$

Aim

$$(\mathrm{DIM})_\varepsilon \quad \begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}(\rho_\varepsilon v_\varepsilon) = 0 \quad \text{in } \Omega \times [0,T], \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \mathrm{div}(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \frac{1}{\varepsilon}\nabla p(\rho_\varepsilon) = 2\,\mathrm{div}(\mu(\rho_\varepsilon) D v_\varepsilon) + \varepsilon\gamma\rho_\varepsilon \nabla\Delta\rho_\varepsilon \quad \text{in } \Omega \times [0,T], \end{cases}$$

Sharp interface limit for $(\mathrm{DIM})_\varepsilon$:

$$(\mathrm{DIM})_\varepsilon \quad \xrightarrow{\varepsilon \to 0} \quad \text{Sharp interface model}$$

Main steps

1. Convergence/compactness results for solutions $(\rho_\varepsilon, v_\varepsilon)$ of $(\mathrm{DIM})_\varepsilon$
2. Introduction/Justification of the sharp interface model
3. Limit functions of $(\rho_\varepsilon, v_\varepsilon)$ are solutions of the sharp interface model

$(\mathrm{DIM})_\varepsilon$: Notion of Weak Solutions

A couple $(\rho_\varepsilon, v_\varepsilon) \in \big(L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^p(\Omega))\big) \times L^2(0,T;H^1_0(\Omega)^n)$ is a weak solution to $(\mathrm{DIM})_\varepsilon$ w.r.t. prescribed initial values $(\rho_\varepsilon^{(i)}, v_\varepsilon^{(i)})$, if

Conservation of mass: For all $\varphi \in C^\infty_{(0)}(\overline{\Omega} \times [0,T))$, it holds

$$\int_0^T \int_\Omega \rho_\varepsilon \partial_t \varphi + \rho_\varepsilon v_\varepsilon \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_\varepsilon^{(i)} \varphi(0) \, \mathrm{d}x = 0.$$

Conservation of momentum: For all $\psi \in C^\infty_{(0)}(\Omega \times [0,T))^n$, it holds

$$\int_0^T \int_\Omega \rho_\varepsilon v_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon v_\varepsilon \otimes v_\varepsilon : \nabla \psi + \tfrac{1}{\varepsilon} p(\rho_\varepsilon) \operatorname{div}(\psi) - 2\mu(\rho_\varepsilon) D v_\varepsilon : D\psi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_\varepsilon^{(i)} v_\varepsilon^{(i)} \cdot \psi(0) \, \mathrm{d}x$$
$$= -\gamma\varepsilon \int_0^T \int_\Omega \nabla \rho_\varepsilon \otimes \nabla \rho_\varepsilon : \nabla \psi + \tfrac{1}{2} |\nabla \rho_\varepsilon|^2 \operatorname{div}(\psi) + \rho_\varepsilon \nabla \rho_\varepsilon \cdot \nabla \operatorname{div}(\psi) \, \mathrm{d}x \, \mathrm{d}t.$$

Energy inequality: Let

$$E_\varepsilon^{\mathrm{tot}}(t) = \int_\Omega \tfrac{1}{\varepsilon} W(\rho_\varepsilon(t)) + \tfrac{\varepsilon}{2} \gamma |\nabla \rho_\varepsilon(t)|^2 + \tfrac{1}{2} \rho_\varepsilon(t) |v_\varepsilon(t)|^2 \, \mathrm{d}x.$$

For almost every $t \in (0,T)$, it holds

$$E_\varepsilon^{\mathrm{tot}}(t) + 2 \int_0^t \int_\Omega \mu(\rho_\varepsilon) |D v_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}s \le E_\varepsilon^{\mathrm{tot}}(0) \overset{\text{assumption}}{\le} C.$$

$(\mathrm{DIM})_\varepsilon$: Compactness of Weak Solutions

Family of weak solutions $(\rho_\varepsilon, \varepsilon_\varepsilon)_{\varepsilon\in(0,1)}$ to $(\mathrm{DIM})_\varepsilon$ with uniformly bounded initial energy.

Theorem (Compactness of weak solutions to $(\mathrm{DIM})_\varepsilon$)

There are functions

$$\rho_0 \in C^{0,\frac{1}{29}}([0,T];L^2(\Omega)) \cap L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\})) \quad \textit{and} \quad v_0 \in L^2(0,T;H_0^1(\Omega)^n)$$

such that (up to subsequences)

$$\rho_\varepsilon \to \rho_0 \textit{ in } C^{0,\frac{1}{29}}([0,T];L^2(\Omega)) \quad \textit{and} \quad v_\varepsilon \rightharpoonup v_0 \textit{ weakly in } L^2(0,T;H^1(\Omega)^n).$$

Limiting sharp interface: $\rho_0(t) \in BV(\Omega,\{\beta_1,\beta_2\})$ indicates

$\longrightarrow$ phases $\Omega^-(t) = \{\rho_0(t) = \beta_1\}$ and $\Omega^+(t) = \{\rho_0(t) = \beta_2\}$ of finite perimeter,

$\longrightarrow$ interface $\Gamma(t) = \partial^*(\Omega^-(t)) \cap \Omega$.

Convergence to a reasonable sharp interface model?

$(\mathrm{SIM})_0$: Two-phase Incompressible Navier–Stokes Equation with Surface Tension

Sharp interface model is free boundary problem with unknown

- interface $\Gamma : [0,T] \to \mathbb{R}^n$ such that $\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t)$,
- velocity function $v(t) : \Omega \setminus \Gamma(t) \to \mathbb{R}^n$,
- pressure function $p(t) : \Omega \setminus \Gamma(t) \to \mathbb{R}$

satisfying

$$
(\mathrm{SIM})_0 \quad \begin{cases}
\beta_1 \partial_t v + \beta_1 (v \cdot \nabla) v + \nabla p = 2\,\mathrm{div}(\mu(\beta_1) Dv) & \text{in } \Omega^-(t), & \text{(Navier–Stokes)} \\
\beta_2 \partial_t v + \beta_2 (v \cdot \nabla) v + \nabla p = 2\,\mathrm{div}(\mu(\beta_2) Dv) & \text{in } \Omega^+(t), & \text{(Navier–Stokes)} \\
\mathrm{div}(v) = 0 & \text{in } \Omega \setminus \Gamma(t), & \text{(incompressible phases)} \\
[v]_-^+ = 0 & \text{on } \Gamma(t), & \text{(continuous velocity)} \\
V = v \cdot \nu^- & \text{on } \Gamma(t), & \text{(pure transport of interface)} \\
[T]_-^+ \nu^- = -2\sigma\kappa\nu^- & \text{on } \Gamma(t). & \text{(Young–Laplace law)}
\end{cases}
$$

- $T^-(v,p) = 2\mu(\beta_1)Dv - pI$ and $T^+(v,p) = 2\mu(\beta_2)Dv - pI$ stress tensors,
- $[\cdot]_-^+$ jump across $\Gamma(t)$,
- V normal velocity of interface,
- σ surface tension constant,
- κ mean curvature.

+ suitable boundary conditions on $\partial\Omega$ and initial conditions

$(\mathrm{SIM})_0$: Weak Solutions to Sharp Interface Model

For $(\rho, v) \in L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\})) \times \left(L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^n)\right)$ let

$$\Omega^-(t) = \{\rho(t) = \beta_1\} \quad \text{and} \quad \Gamma(t) = \partial^*(\Omega^-(t)) \cap \Omega$$ (density determines interface).

Then (ρ, v) is a weak solution of $(\mathrm{SIM})_0$, if

Transport equation: characteristic function $\chi(t) = \frac{\rho(t)-\beta_2}{\beta_1-\beta_2}$ of $\Omega^-(t)$ is a distributional solution of $\partial_t \chi + v \cdot \chi = 0$ in $\Omega \times (0,T)$.

Energy inequality: For almost every $t \in (0,T)$, it holds

$$\frac{1}{2}\int_\Omega \rho(t)\,|v(t)|^2\,\mathrm{d}x + 2\sigma H^{n-1}(\Gamma(t)) + 2\int_0^t\int_\Omega \mu(\rho)\,|Dv|^2\,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{2}\int_\Omega \rho^{(i)}\left|v^{(i)}\right|^2\mathrm{d}x + 2\sigma H^{n-1}(\Gamma(0)).$$

Variational formulation: For all divergence free test functions $\psi \in C_0^\infty([0,T);C^\infty_{0,\sigma}(\Omega))$, it holds

$$\int_0^T\int_\Omega \rho v\cdot\partial_t\psi + \rho v\otimes v : \nabla\psi - 2\mu(\rho)Dv : D\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x = -2\sigma\int_0^T\int_{\Gamma(t)} v^-\otimes v^- : \nabla\psi\,\mathrm{d}H^{n-1}(x)\,\mathrm{d}t.$$

Variational formulation

$$\int_0^T \int_\Omega \rho v \cdot \partial_t \psi + \rho v \otimes v : \nabla \psi - 2\mu(\rho) D v : D\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho^{(i)} v^{(i)} \cdot \psi(0)\,\mathrm{d}x$$
$$= -2\sigma \int_0^T \int_{\Gamma(t)} \nu^- \otimes \nu^- : \nabla \psi \,\mathrm{d}H^{n-1}(x)\,\mathrm{d}t$$

uses divergence free test functions

$$\psi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega)) = \left\{ \psi|_{\Omega \times [0,T)} : \psi \in C_0^\infty(\Omega \times (-1,T))^n,\ \mathrm{div}(\psi) = 0 \right\}.$$

+: Removes pressure function from weak formulation.
Consequence : No passage to the limit of pressure term.

-: Reconstruction of pressure function from weak formulation.

$(\mathrm{SIM})_0$: Reconstruction of Pressure (for Smooth Solutions)

Additional smoothness assumptions:

- interface: $\Gamma = \Gamma(t)$ is space-time interface of class $C^{3,1}$.
- velocity: all derivatives of v exist in $L^2(\Omega^-)$ and $L^2(\Omega^+)$, where

$$\bigcup_{t\in(0,T)} (\Omega^-(t) \times \{t\}) \quad \text{and} \quad \bigcup_{t\in(0,T)} (\Omega^+(t) \times \{t\}) .$$

Note: No additional regularity across interface is assumed!

Theorem (Reconstruction of pressure)

For a sufficiently regular weak solution (ρ, v) to $(\mathrm{SIM})_0$, there exists a unique $p \in L^2(\Omega \times (0,T))$ with the following properties.

1. $p|_{\Omega^\pm} \in L^2(0,T;W^{1,2}(\Omega^\pm(t)))$.
2. *For almost all $t \in [0,T)$ it holds* $\int_\Omega p(t)\,\mathrm{d}x = 0$.
3. $\nabla p = -\beta_1 \partial_t v + 2\mu(\beta_1)\operatorname{div}(Dv) - \beta_1 (v\cdot\nabla)v$ *a.e. in* Ω^-.
4. $\nabla p = -\beta_2 \partial_t v + 2\mu(\beta_2)\operatorname{div}(Dv) - \beta_2 (v\cdot\nabla)v$ *a.e. in* Ω^+.
5. $[p]_-^+ = 2\left[\mu(\rho) Dv\, \nu^-\right]_-^+ \cdot \nu^- + 2\sigma\kappa$ *in* $L^2(0,T;H^{\frac{1}{2}}(\Gamma(t)))$.

Jusitfies notion of weak solutions to $(\mathrm{SIM})_0$

UNI FREIBURG

Aim

Weak solutions $(\rho_\varepsilon, v_\varepsilon)_{\varepsilon\in(0,1)}$ of $(\mathrm{DIM})_\varepsilon$ $\overset{\varepsilon\to 0}{\longrightarrow}$ weak solution (ρ_0, v_0) of $(\mathrm{SIM})_0$.

Recall: Compactness properties of weak solutions:

$$\rho_\varepsilon \to \rho_0 \text{ in } C^{0,\frac{1}{29}}([0,T], L^2(\Omega)) \quad \text{with} \quad \rho_0 \in C^{0,\frac{1}{29}}([0,T]; L^p(\Omega)) \cap L^\infty(0,T; BV(\Omega, \{\beta_1, \beta_2\}))$$

$$v_\varepsilon \rightharpoonup v_0 \text{ weakly in } L^2(0,T; H^1(\Omega)^n) \quad \text{with} \quad v_0 \in L^2(0,T; H^1_0(\Omega)^n)$$

Question: (ρ_0, v_0) weak solution of $(\mathrm{SIM})_0$?

$(\mathrm{DIM})_\varepsilon \to (\mathrm{SIM})_0$: Mathematical Main Difficulties

Task: Passing to the limit in the weak formulation.

Momentum balance: Recall variational formulation

$$\int_0^T \int_\Omega \rho_\varepsilon v_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon v_\varepsilon \otimes v_\varepsilon : \nabla \psi + \frac{1}{\varepsilon} p(\rho_\varepsilon) \operatorname{div}(\psi) - 2\mu(\rho_\varepsilon) D v_\varepsilon : D\psi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_\varepsilon^{(i)} v_\varepsilon^{(i)} \cdot \psi(\cdot, 0) \, \mathrm{d}x$$

$$= -\gamma\varepsilon \int_0^T \int_\Omega \nabla\rho_\varepsilon \otimes \nabla\rho_\varepsilon : \nabla\psi + \tfrac{1}{2} |\nabla\rho_\varepsilon|^2 \operatorname{div}(\psi) + \rho_\varepsilon \nabla\rho_\varepsilon \cdot \nabla \operatorname{div}(\psi) \, \mathrm{d}x \, \mathrm{d}t.$$

for any $\psi \in C^\infty_{(0)}(\Omega \times [0, T))^n$.

Challenging terms in the passage to the limit $\varepsilon \to 0$:

$$\frac{1}{\varepsilon} \int_0^T \int_\Omega p(\rho_\varepsilon) \operatorname{div}(\psi) \, \mathrm{d}x \, \mathrm{d}t \quad \text{and} \quad \varepsilon \int_0^T \int_\Omega \nabla\rho_\varepsilon \otimes \nabla\rho_\varepsilon : \nabla\psi \, \mathrm{d}x \, \mathrm{d}t.$$

$(\mathrm{DIM})_\varepsilon \to (\mathrm{SIM})_0$: Treatment of Pressure Term

Weak formulation for $(\mathrm{SIM})_0$ takes into account divergence-free test functions

$$\psi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega)) \quad \Longrightarrow \quad \frac{1}{\varepsilon}\int_0^T\int_\Omega p(\rho_\varepsilon)\,\mathrm{div}(\psi)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Resolves problem of passing pressure term to the limit!

Variational formulation simplifies to

$$\int_0^T\int_\Omega \rho_\varepsilon v_\varepsilon \cdot \partial_t\psi + \rho_\varepsilon v_\varepsilon \otimes v_\varepsilon : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_\varepsilon^{(i)} v_\varepsilon^{(i)} \cdot \psi(\cdot,0)\,\mathrm{d}x$$
$$= \int_0^T\int_\Omega 2\mu(\rho_\varepsilon) D v_\varepsilon : D\psi - \gamma\varepsilon\nabla\rho_\varepsilon \otimes \nabla\rho_\varepsilon : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t$$

for any $\psi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$.

Due to energy inequality, $\frac{1}{\varepsilon}W(\rho_\varepsilon) + \frac{\varepsilon}{2}\gamma|\nabla\rho_\varepsilon|^2$ weak-$*$ pre-compact in $L^1(0,T;C_0(\overline{\Omega}))^*$.

Problem: Identification of limit function.

Solution: If $\frac{1}{\varepsilon}W(\rho_\varepsilon) + \frac{\varepsilon}{2}\gamma|\nabla\rho_\varepsilon|^2$ converges to appropriate surface measure. Then control of

$$\varepsilon\gamma\int_0^T\int_\Omega \nabla\rho_\varepsilon \otimes \nabla\rho_\varepsilon : \nabla\psi \,\mathrm{d}x\,\mathrm{d}t.$$

$(\mathrm{DIM})_\varepsilon \to (\mathrm{SIM})_0$: The Sharp Interface Limit (Solving the Sharp Interface Model)

UNI FREIBURG

Assumption: For every $\varphi \in L^1(0,T;C_0(\overline{\Omega}))$ it holds

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \left(\frac{1}{\varepsilon} W(\rho_\varepsilon) + \frac{\varepsilon}{2}\gamma |\nabla \rho_\varepsilon|^2 \right) \varphi \,\mathrm{d}x\,\mathrm{d}t = 2\sigma \int_0^T \int_{\Gamma(t)} \varphi \,\mathrm{d}H^{n-1}(x)\,\mathrm{d}t$$

with $\sigma = \int_{\beta_1}^{\beta_2} \sqrt{\frac{\gamma W(r)}{2}}\,\mathrm{d}r$ and $\Gamma(t) \subset\subset \Omega$.

Theorem (Convergence of capillary term)

For every $\psi \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$ it holds

$$\lim_{\varepsilon\to 0} \varepsilon\gamma \int_0^T \int_\Omega \nabla\rho_\varepsilon \otimes \nabla\rho_\varepsilon : \nabla\psi \,\mathrm{d}x\,\mathrm{d}t = 2\sigma \int_0^T \int_{\Gamma(t)} \nu^- \otimes \nu^- : \nabla\psi \,\mathrm{d}H^{n-1}(x)\,\mathrm{d}t.$$

Theorem (Solving the sharp interface model)

The couple (ρ_0, v_0) is a weak solution of $(\mathrm{SIM})_0$.

Conclusion

- Navier–Stokes–Korteweg equations
  (diffuse interface model)
  - notion of weak solution
  - compactness properties
- Two-phase incompressible Navier–Stokes equations with surface tension
  (sharp interface model)
  - notion of weak solution
  - consistency result
- Sharp interface limit
  - sufficient condition: convergence of energy to associated surface measure
  - open question: possible to weaken/remove additional assumption?

References

- Abels, H.: On generalized solutions of two-phase flows for viscous incompressible fluids. Interfaces Free Bound. 9, No. 1, 31–65 (2007).
- Chen, X.: Global asymptotic limit of solutions of the Cahn–Hilliard equation. J. Differ. Geom. 44, No. 2, 262–311 (1996).
- Hermsdörfer, K., Kraus, C., Kröner, D.: Interface conditions for limits of the Navier–Stokes–Korteweg model. Interfaces Free Bound. 13, No. 2, 239–254 (2011).
- Korteweg, D. J.: Sur la forme que prennent les équations du mouvement des fluids si l'on tient compte des forces capillaires causées par les variations de densite mais continues et sur la théorie de la capillarité dans l'hypothèse d'une variation continue de la densité. Arch. Néerl. (2) 6, 1–24 (1901).

Thank you for your attention!