Numerical Approximation of Hyperbolic Systems Containing an Interface

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11th DFG-CNRS workshop
Micro-Macro Modelling and Simulation of Liquid-Vapour-Flows
Paris
March 3rd, 2016

Framework

We consider a hyperbolic system with an interface at $x = 0$:

$$\begin{cases} \partial_t U + \partial_x f(U, x) = 0, & \forall x \in \mathbb{R} \setminus \{0\}, \forall t \in \mathbb{R}^+, \\ \big(U(t, 0^-), U(t, 0^+)\big) \in \mathcal{G}(t), & \forall t \in \mathbb{R}^+, \\ U(0, x) = U^0(x), & \forall x \in \mathbb{R}. \end{cases}$$

The interface can have a physical meaning:

- grid slowing down a 1-dimensionnal flow;
- modelization of an curve in a duct;
- discontinuous variation of the cross section...

or it can be an artificial interface, introduced by some practical considerations:

- different pressure laws on each side of the interface;
- use of a more elaborate model on one side of the interface...

Is the problem well posed?

In the scalar case ($u \in \mathbb{R}$), with overall conservation of u

$$\begin{cases} \partial_t u + \partial_x f_L(u) = 0, & \forall x < 0, \forall t \in \mathbb{R}^+, \\ \partial_t u + \partial_x f_R(u) = 0, & \forall x > 0, \forall t \in \mathbb{R}^+, \\ f_L(u(t,0^-)) = f_R(u(t,0^+)), & \forall t \in \mathbb{R}^+, \\ u(0,x) = u^0(x), & \forall x \in \mathbb{R}. \end{cases}$$

- There might be infinitely many solutions;
- The difficulty is to propose a criterion to select the relevant solutions (i.e. to extend the notion of Kruskov entropy solution to the discontinuous flux case).

Adimurthi, Mishra and Gowda Optimal entropy solutions for conservation laws with discontinuous flux-functions Journal of Hyperbolic Differential Equations

Andreianov, Karlsen et Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. Arch. Ration. Mech. Anal.

Is the problem well posed?

In the system case ($U \in \mathbb{R}^n$), with general interface conditions

$$\begin{cases} \partial_t U + \partial_x f(U) = 0, & \forall x \in \mathbb{R} \setminus \{0\},\ \forall t \in \mathbb{R}^+, \\ \big(U(t,0^-), U(t,0^+)\big) \in \mathcal{G}(t), & \forall t \in \mathbb{R}^+, \\ U(0,x) = U^0(x), & \forall x \in \mathbb{R}. \end{cases}$$

- There is no general theory of well-posedness;
- The interface conditions must be considered in a weak sense;
- It is often possible to prove local well posedness when the initial data is a small perturbation of an equilibrium.

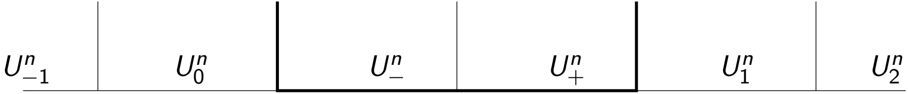
Dubois and LeFloch Boundary conditions for nonlinear hyperbolic systems of conservation laws Journal of Differential Equations

Borsche, Colombo and Garavello Mixed Systems: ODEs – Balance Laws Journal of Differential Equations

Finite volumes schemes

We want to approximate the solution of

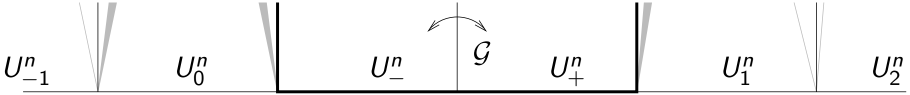
$$\begin{cases} \partial_t U + \partial_x f(U) = 0, & \forall x \in \mathbb{R} \setminus \{0\}, \forall t \in \mathbb{R}^+, \\ \big(U(t,0^-), U(t,0^+)\big) \in \mathcal{G}, & \forall t \in \mathbb{R}^+, \\ U(0,x) = U^0(x), & \forall x \in \mathbb{R}. \end{cases}$$



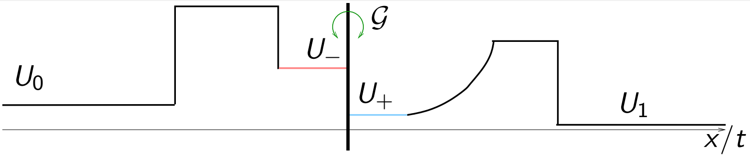
We look for U^n_- the value of the solution at the left entrance of the interface and U^n_+ its value at the right exit, and use these values to update U on cells 0 and 1.

$$\begin{cases} U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(g(U_j^n, U_{j+1}^n) - g(U_{j-1}^n, U_j^n)), & j \notin \{0,1\}, \\ U_0^{n+1} = U_0^n - \frac{\Delta t}{\Delta x}(g(U_0^n, U_-^n) - g(U_{-1}^n, U_0^n)), \\ U_1^{n+1} = U_1^n - \frac{\Delta t}{\Delta x}(g(U_0^n, U_1^n) - g(U_+^n, U_1^n)). \end{cases}$$

Choice of U_-^n and U_+^n



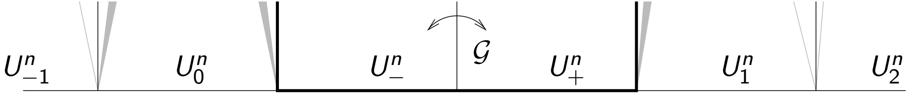
The natural idea is to chose (U_-^n, U_+^n) as the exact traces around the interface of the solution to the Riemann problem with interface, with left state U_0^n and right state U_1^n.



- The Riemann problem with left state U_0 and right state U_- has only waves going to the left.
- The Riemann problem with left state U_+ and right state U_1 has only waves going to the right.
- U_- and U_+ verify the interface conditions.

- $g_{God}(U_0, U_-) = f(U_-)$.
- $g_{God}(U_+, U_1) = f(U_+)$.
- $(U_-, U_+) \in \mathcal{G}$.

Use of another numerical flux

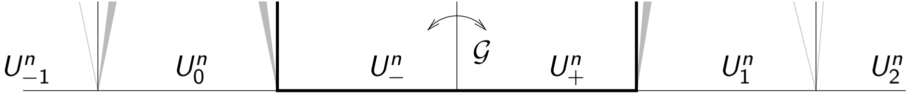


We would like to define (U_-, U_+) as the solution of

$$g_{God}(U_0, U_-) = f(U_-),\ g_{God}(U_+, U_1) = f(U_+) \ \text{ and } \ (U_-, U_+) \in \mathcal{G}.$$

With the Godunov numerical flux, this system is as hard to solve than finding the solution to the Riemann problem.

Use of another numerical flux



We would like to define (U_-, U_+) as the solution of

$$g_{num}(U_0, U_-) = f(U_-), \; g_{num}(U_+, U_1) = f(U_+) \;\text{ and }\; (U_-, U_+) \in \mathcal{G}.$$

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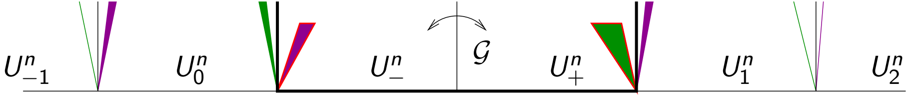
With another numerical flux, this system is overconstrained.

Use of another numerical flux



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With the Godunov numerical flux, this system is as hard to solve than finding the solution to the Riemann problem.
With another numerical flux, this system is overconstrained.

We define (U^n_-, U^n_+) as the solution of

$$g_{num}(U^n_0, U^n_-) - f(U^n_-) + f(U^n_+) - g_{num}(U^n_+, U^n_1) = 0 \ \text{ and } \ (U^n_-, U^n_+) \in \mathcal{G}.$$

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\Big(\underbrace{g(U^n_j, U^n_{j+1}) - f(U^n_j)}_{\text{fluctuations due to the right interface}} + \underbrace{f(U^n_j) - g(U^n_{j-1}, U^n_j)}_{\text{fluctuations due to the left interface}} \Big)$$

First properties of the scheme

$$\begin{cases} \partial_t U + \partial_x f(U) = 0, \\ (U(0^-), U(0^+)) \in \mathcal{G}, \\ U(0,x) = U^0(x), \end{cases} \qquad \begin{cases} U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(g(U_j^n, U_{j+1}^n) - g(U_{j-1}^n, U_j^n)), \\ U_0^{n+1} = U_0^n - \frac{\Delta t}{\Delta x}(g(U_0^n, U_-^n) - g(U_{-1}^n, U_0^n)), \\ U_1^{n+1} = U_1^n - \frac{\Delta t}{\Delta x}(g(U_0^n, U_1^n) - g(U_+^n, U_1^n)), \\ g(U_0^n, U_-^n) - g(U_+^n, U_1^n) = f(U_-^n) - f(U_+^n), \\ (U_-^n, U_+^n) \in \mathcal{G}. \end{cases}$$

Preservation of equilibrium

If $U^0(x) = U_L \mathbf{1}_{x<0} + U_R \mathbf{1}_{x>0}$ with $(U_L, U_R) \in \mathcal{G}$, then this equilibrium is preserved by the scheme:

$$\forall n \geq 0, \; U_j^n = U_L \;\text{ if }\; j \leq 0 \text{ and } \; U_j^n = U_R \;\text{ if }\; j \geq 1.$$

Numerical conservation of conservation interface conditions

If one of the interface condition states that U_i is conserved, than the numerical scheme preserves it as well:

$$f_i(U(0^-)) = f_i(U(0^+)) \qquad \Longrightarrow \qquad g(U_0^n, U_-^n) = g(U_+^n, U_1^n).$$

Questions

If $U^0(x) = U_L \mathbf{1}_{x<0} + U_R \mathbf{1}_{x>0}$ with $(U_L, U_R) \in \mathcal{G}$, then this is preserved by the scheme: $\forall n \geq 0,\ U_j^n = U_L$ if $j \leq 0$ and $U_j^n = U_R$ if $j \geq 1$.

Does the converse hold? Is there some spurious numerical equilibrium?

The states (U_-, U_+) are defined as a solution to the system

$$g(U_0, U_-) - g(U_+, U_1) = f(U_-) - f(U_+) \qquad (U_-, U_+) \in \mathcal{G}.$$

Is there one and only one solution? If not, can we select the relevant one?

Classical case $\mathcal{G}_{\text{cla}}$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + c^2 \rho) = 0, \\ (\rho u)_- = (\rho u)_+, \\ (\rho u^2 + c^2\rho)_- = (\rho u^2 + c^2 \rho)_+, \\ \text{FEntrop}(\rho_-, u_-) \leq \text{FEntrop}(\rho_+, u_+). \end{cases}$$

Fluid/particle interaction $\mathcal{G}_{\text{part}}(\lambda)$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + c^2 \rho) = \lambda \rho u \delta_0, \\ (\rho u)_- = (\rho u)_+, \\ (\rho u^2 + c^2\rho)_- - (\rho u^2 + c^2 \rho)_+ = \lambda(\rho u)_-, \\ 0 \leq u_- \leq c \Rightarrow 0 \leq u_+ \leq c, \\ -c \leq u_+ \leq 0 \Rightarrow -c \leq u_- \leq 0. \end{cases}$$

Numerical equilibrium

Godunov flux: existence of spurious numerical equilibrium

Suppose that g is the Godunov flux. Suppose that $\lambda > 0$. Then there exists (U_0, U_1) and (U_-, U_+) such that

- $(U_0, U_1) \notin \mathcal{G}_{\text{part}}(\lambda)$, $(U_-, U_+) \in \mathcal{G}_{\text{part}}(\lambda)$;
- $g(U_0, U_-) - g(U_+, U_1) = f(U_-) - f(U_+)$;
- $g(U_0, U_-) = f(U_0)$, thus $U_0 - \frac{\Delta t}{\Delta x}(g(U_0, U_-) - g(U_0, U_0)) = U_0$;
- $g(U_+, U_1) = f(U_1)$, thus $U_1 - \frac{\Delta t}{\Delta x}(g(U_1, U_1) - g(U_+, U_1)) = U_1$.

Fluid/particle interaction $\mathcal{G}_{\text{part}}(\lambda)$

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + c^2 \rho \right) = \lambda q \delta_0, \\ q_- = q_+, \\ \left(\frac{q_-^2}{\rho_-} + c^2 \rho_- \right) - \left(\frac{q_+^2}{\rho_+} + c^2 \rho_+ \right) = \lambda q_-, \\ \text{subsonic} \Rightarrow \text{subsonic}. \end{cases}$$

$U_- = (\rho_-, q_-)$, $U_+ = (\rho_+, q_+)$,
$q_- = q_+ := q_{\text{in}}$,
$\left(\frac{q_{\text{in}}^2}{\rho_-} + c^2 \rho_- \right) - \left(\frac{q_{\text{in}}^2}{\rho_+} + c^2 \rho_+ \right) = \lambda q_{\text{in}}$,

$U_0 = (\rho_0, q_0)$, $U_1 = (\rho_1, q_1)$,
$q_0 = q_1 := q_{\text{out}}$,
$\left(\frac{q_{\text{out}}^2}{\rho_0} + c^2 \rho_0 \right) - \left(\frac{q_{\text{out}}^2}{\rho} + c^2 \rho \right) = \lambda q_{\text{in}}$.

Numerical equilibrium

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- $(U_0, U_1) \notin \mathcal{G}_{\text{part}}(\lambda)$, $(U_-, U_+) \in \mathcal{G}_{\text{part}}(\lambda)$,
- $g(U_0, U_-) - g(U_+, U_1) = f(U_-) - f(U_+)$,
- $g(U_0, U_-) = f(U_0)$, thus $U_0 - \frac{\Delta t}{\Delta x}(g(U_0, U_-) - g(U_0, U_0)) = U_0$
- $g(U_+, U_1) = f(U_1)$, thus $U_1 - \frac{\Delta t}{\Delta x}(g(U_1, U_1) - g(U_+, U_1)) = U_1$

Rusanov flux: numerical equilibrium are genuine equilibrium

Let f be an hyperbolic system with interface conditions $\mathcal{G}$ at $x = 0$. Take g the Rusanov flux $g(U_L, U_R) = \frac{f(U_L)+f(U_R)}{2} - \frac{A}{2}(U_R - U_L)$ with A "large enough" (subcharacteristic condition). Suppose that the scheme preserves the initial data $U_j^0 = U_L \mathbf{1}_{j\leq 0} + U_R \mathbf{1}_{j \geq 1}$. Then $(U_L, U_R) \in \mathcal{G}$.

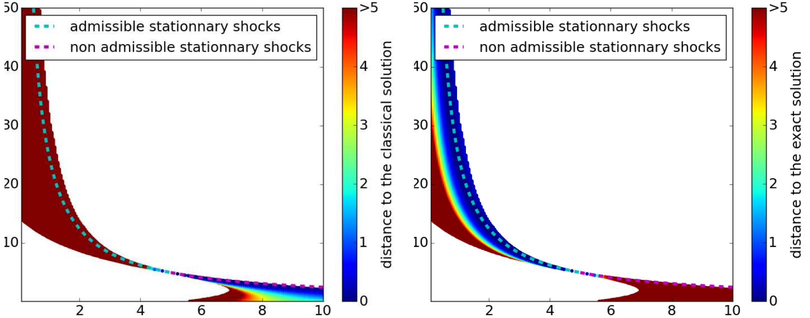
Uniqueness / selection of the solution

Consider the Rusanov flux $g(U_L, U_R) = \frac{f(U_L)+f(U_R)}{2} - \frac{A}{2}(U_R - U_L)$, the coupling $\mathcal{G}_{\text{cla}}$ and the isothermal Euler equations. The system

$$g(U_0, U_-) - g(U_+, U_1) = f(U_-) - f(U_+)$$

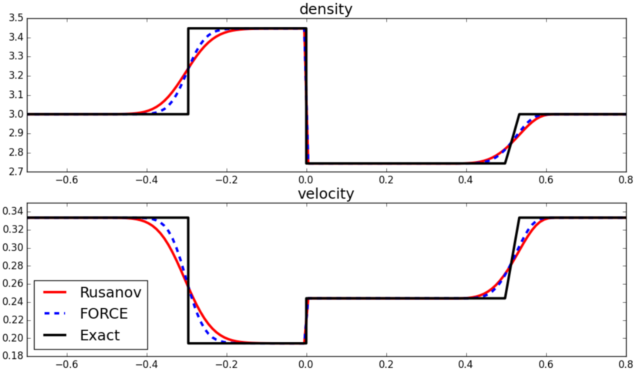
has between one and three solutions.

- the middle one is always entropy satisfying and yields a scheme that verifies a discrete entropy inequality;
- another one does not dissipate the entropy;
- the last one allows to preserve stationnary shocks.



Isothermal fluid/particle coupling

$$\begin{cases} \partial_t \rho + \partial_x(\rho u), = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + c^2\rho) = \lambda \rho u \delta_0, \\ (\rho u)_- = (\rho u)_+, \\ (\rho u^2 + c^2\rho)_- - (\rho u^2 + c^2\rho)_+ = \lambda(\rho u)_-, \\ 0 \le u_- \le c \Rightarrow 0 \le u_+ \le c, \\ -c \le u_+ \le 0 \Rightarrow -c \le u_- \le 0. \end{cases}$$



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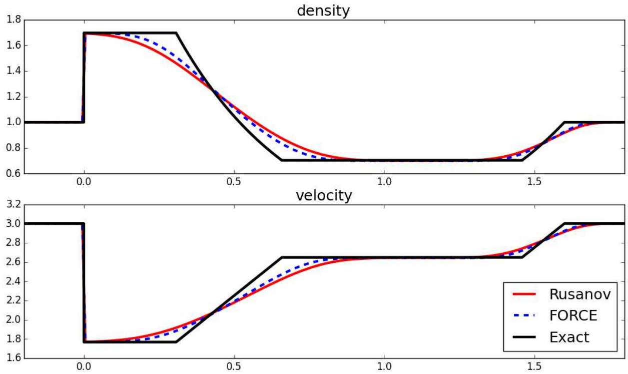

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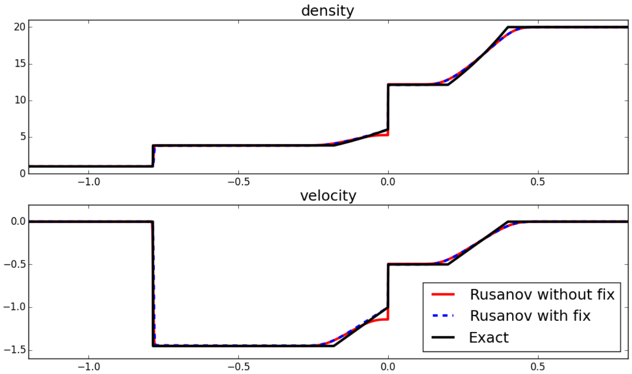
Requires a fix to impose the last two conditions

$$0 \le u_- \le c \Rightarrow 0 \le u_+ \le c,$$
$$-c \le u_+ \le 0 \Rightarrow -c \le u_- \le 0.$$



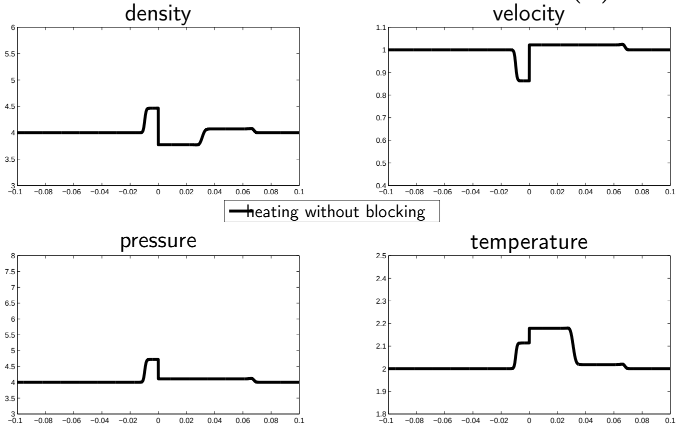
Fluid/ particle coupling with heat exchange

pressure law $p = \rho e(\gamma - 1)$.

$\lambda,\ \mu,\ s_P \geq 0$ parameters, $s = e\rho^{1-\gamma}$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(u(E+p)) = 0 \end{cases}$$

$$\begin{cases} (\rho u)_- = (\rho u)_+ = q \\ \left(\frac{q^2}{\rho_-} + p_-\right) - \left(\frac{q^2}{\rho_+} + p_+\right) = q \\ s_+ - s_P = \exp\left(\frac{\mu}{q}\right)(s_- - s_P) \end{cases}$$



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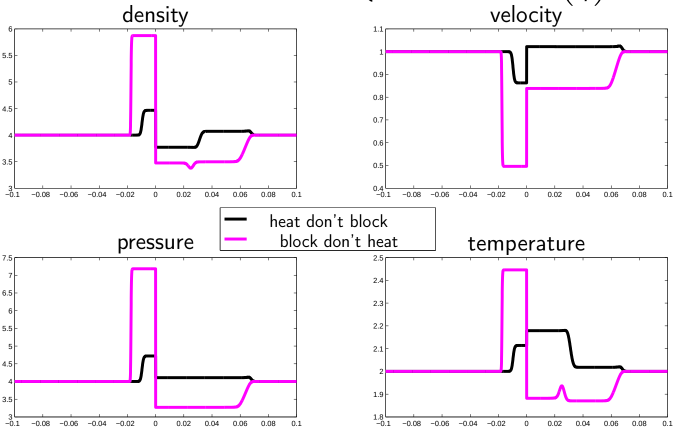

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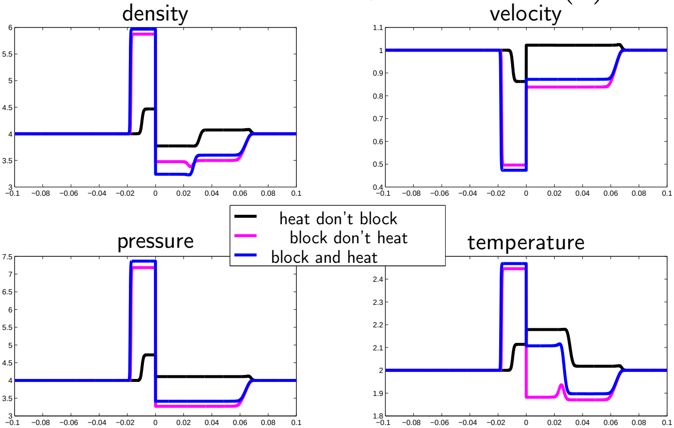
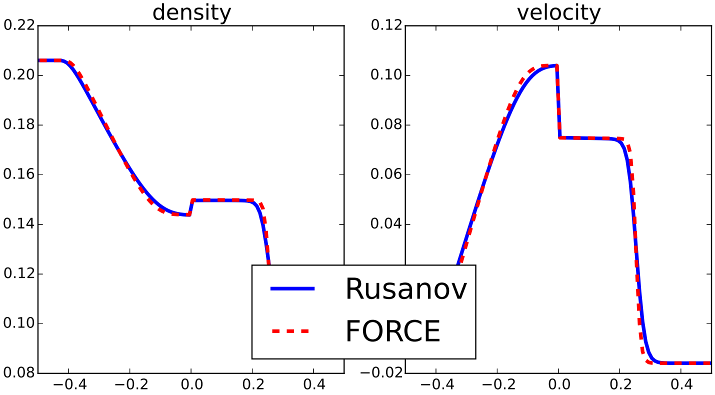


$$\begin{cases} \partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0, \\ \partial_t(\alpha\rho u) + \partial_x(\alpha\rho u^2 + \alpha p(\tau)), \\ \qquad = -p(\tau)\partial_x\alpha, \\ \partial_t E + \partial_x(u(E+p)) = 0. \end{cases}$$

cross section $\alpha = \alpha_L \mathbf{1}_{x<0} + \alpha_R \mathbf{1}_{x\geq 0}$;
specific volume $\tau = \frac{1}{\rho}$;
Interface conditions: continuity of $\alpha\rho u$ and $\frac{w^2}{2} - \int^\tau p'(t)dt + \tau p(\tau)$.



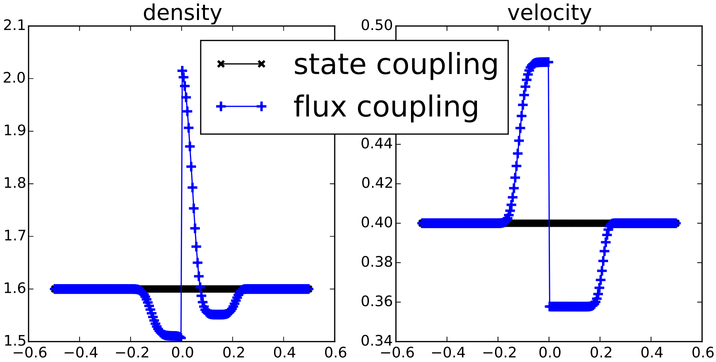
Gas dynamics with different pressure laws

On $x < 0$, the pressure law is $p_L = \rho e(\gamma_L - 1)$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_L) = 0, \\ \partial_t E + \partial_x(u(E + p_L)) = 0, \end{cases}$$

On $x > 0$, the pressure law is $p_R = \rho e(\gamma_R - 1)$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_R) = 0, \\ \partial_t E + \partial_x(u(E + p_R)) = 0. \end{cases}$$



Conclusions and perspectives

Conclusions

- The proposed method does not use any knowledge on the coupled Riemann problem ...
- at the cost of solving a nonlinear system to compensate the waves entering the interface.
- The scheme preserves the equilibrium and conservation type interface conditions.
- It is quite easy to implement on different test cases.

Perspectives

- Extend the method to networks of (system of) conservation laws. We are lacking some conditions at the junction.
- How to deal with moving interfaces?

Thank you for your attention