Convection and total variation flow

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simplified model for Bingham fluid + Navier-Stokes equations

$$u \;:\; Q_T \longrightarrow \mathbb{R}$$
$$\partial_t u + \mathrm{div}\, F(x,t,u) - \mathrm{div}\, \frac{\nabla u}{|\nabla u|} = 0 \text{ in } Q_T$$
$$u(x,0) = u_{\mathrm{ini}}(x) \text{ on } \mathbb{R}^d$$

- $Q_T := \mathbb{R}^d \times (0,T)$
- $u_{\mathrm{ini}} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $a_0 \leq u_{\mathrm{ini}} \leq b_0$ a.e.
- $F \in C^1(Q_T \times \mathbb{R}, \mathbb{R}^d)$ locally Lipschitz continuous such that

$$\mathrm{div}_x F(x,t,u) := \sum_{i=1}^{d} \frac{\partial F_i}{\partial x_i}(x,t,u) = 0 \quad \text{Example : } F(x,t,u) = \frac{1}{2} u^2 \mathbf{v}$$

First properties of this model

not regularized if $\frac{\nabla u}{|\nabla u|}$ constant, hence entropy sense needed
$\mathrm{div}\, \frac{\nabla u}{|\nabla u|}$ total variation flow [see Mazón]
TV flow = 1-laplacian = nonlocal problem

$$BV(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d),\ TV(u) < \infty\}$$

with $TV(u) = \sup \left\{ \int_{\mathbb{R}^d} u \operatorname{div}\phi \, \mathrm{d}x; \phi \in C_c^1(\mathbb{R}^d;\mathbb{R}^d) \text{ with } \|\phi\|_{L^\infty(\mathbb{R}^d)} \leq 1 \right\}$

entropy solution : $u \in L^\infty(Q_T) \cap L^1(0,T;BV(\mathbb{R}^d))$ if
there exists $\lambda \in L^\infty(Q_T)^d$, with $|\lambda| \leq 1$ a.e. such that
for all $\varphi \in C_c^\infty(Q_T)$ with $\varphi \geq 0$
for all $\eta \in C^\infty(\mathbb{R})$ convex and $\dfrac{\partial \Phi}{\partial u}(x,t,u) = \eta'(u)\dfrac{\partial F}{\partial u}(x,t,u)$

$$\int_{Q_T} \Big(\eta(u)\partial_t\varphi + \big(\Phi(x,t,u) - \eta'(u)\lambda\big) \cdot \nabla\varphi\Big) \, \mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_T} \varphi \, |D[\eta'(u)]| \mathrm{d}t + \int_{\mathbb{R}^d} \eta(u_{\text{ini}}(x))\varphi(x,0) \, \mathrm{d}x \geq 0$$

obtained by passing to the limit in

$$\partial_t u_\epsilon + \operatorname{div} F(x,t,u_\epsilon) - \operatorname{div} \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|} - \epsilon \Delta u_\epsilon = 0$$

multiplying by $\eta'(u_\epsilon)\varphi$ and using $\dfrac{\nabla u_\epsilon}{|\nabla u_\epsilon|} \cdot \nabla \eta'(u_\epsilon) = |\nabla \eta'(u_\epsilon)|$ and $\dfrac{\nabla u_\epsilon}{|\nabla u_\epsilon|} \rightharpoonup \lambda$

$$\int_{Q_T} \left(\eta(u)\partial_t\varphi - \eta'(u)\lambda \cdot \nabla\varphi\right) \mathrm{d}x\mathrm{d}t - \int_{Q_T} \varphi \left|D[\eta'(u)]\right| \mathrm{d}t + \int_{\mathbb{R}^d} \eta(u_{\mathrm{ini}}(x))\varphi(x,0)\ \mathrm{d}x \geq 0$$

$\eta(s) = s$ and $\eta(s) = -s$ yield

$$\int_{Q_T} \left(u\partial_t\varphi - \lambda \cdot \nabla\varphi\right) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} u_{\mathrm{ini}}(x)\varphi(x,0)\ \mathrm{d}x = 0$$

implies $\mathrm{div}\lambda = \partial_t u$ on Q_T

1D example

$u_{\mathrm{ini}}(x) = 1_{]-1,1[}(x)$

$$\text{solution for } t < 1 : u(x,t) = (1-t)1_{]-1,1[}(x) \text{ and } \lambda(x,t) = \begin{cases} 1 \text{ if } x < -1 \\ -x \text{ if } -1 < x < 1 \\ -1 \text{ if } 1 < x \end{cases}$$

2D example

$u_{\mathrm{ini}}(x) = 1_{B(0,1)}(x)$

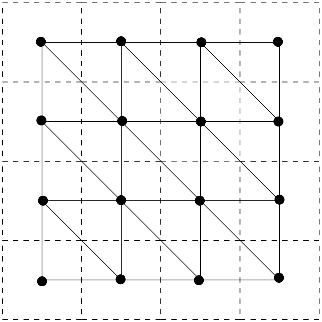
$$\text{solution for } t < \tfrac{1}{2} : u(x,t) = (1-2t)1_{B(0,1)}(x) \text{ and } \lambda(x,t) = \begin{cases} -x \text{ if } |x| < 1 \\ -\frac{x}{|x|^2} \text{ if } |x| > 1 \end{cases}$$

1. Krushkov doubling variable technique
2. start with regular entropy $\eta(\cdot - \kappa)$
3. sign of terms issued from TV flow allows to get rid of them
4. let η tend to Krushkov entropies
5. same conclusion as in the pure convection case

quasi-uniform simplicial mesh of $\mathbb{R}^d$ with nonobtuse angles
(exists in any space dimension [J. Brandts et al, 2009])

$\forall v_h \in \mathbb{R}^{\mathbb{N}}$ $\qquad \widehat{v}_h \in C^0(\mathbb{R}^d)$; $\widehat{v}_{h|K}$ is affine for each $K \in \mathcal{T}_h$, $\widehat{v}_h(x_p) = v_p, \ \forall p \in \mathbb{N}$

$\qquad \overline{v}_h \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$; $\overline{v}_{h|Q_p} = v_p, \ \forall p \in \mathbb{N}$



$$\int_K \nabla \widehat{u}_h(x) \cdot \nabla \widehat{v}_h(x) \mathrm{d}x = \sum_{p \in \mathcal{V}_K} \sum_{q \in \mathcal{V}_K} T^K_{pq} (u_p - u_q)(v_p - v_q)$$

with $T^K_{pq} = -\dfrac{1}{2} \displaystyle\int_K \nabla \phi_p(x) \cdot \nabla \phi_q(x) \mathrm{d}x \geq 0$

and

$$\|\overline{u}_h - \widehat{u}_h\|_{L^1(\mathbb{R}^d)} \leq h \|\nabla \widehat{u}_h\|_{L^1(\mathbb{R}^d)^d}$$

Initialization of $(u_p^0)_{n\in\mathbb{N}}$ such that $\overline{u}_h^0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$:

$$u_p^0 = \frac{1}{m_p}\int_{Q_p} u_{\mathrm{ini}}(x)\mathrm{d}x, \qquad \forall p\in\mathbb{N}$$

Finite volume step Letting $(u_p^n)_{p\in\mathbb{N}}$ such that $\overline{u}_h^n \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ seek $(u_p^{n+\frac{1}{2}})_{p\in\mathbb{N}}$ such that $\overline{u}_h^{n+\frac{1}{2}} \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and

$$m_p\frac{u_p^{n+\frac{1}{2}} - u_p^n}{\delta t} + \sum_{q\in\mathcal{N}_p} F_{p,q}^n(u_p^n, u_q^n) = 0, \qquad \forall p\in\mathbb{N}$$

where $(F_{p,q}^n)_{p,q,n\in\mathbb{N}}$ is admissible and consistent :

- $F_{p,q}^n \in C^0([a_0,b_0]^2,\mathbb{R})$ Lipschitz continuous with constant $m_{p,q}L$
- $F_{p,q}^n(u,v) \nearrow$ with $u \searrow$ with v
- $F_{p,q}^n(u,v) = -F_{q,p}^n(v,u)$
- $F_{p,q}^n(u,u) = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\sigma_{p,q}} F(x,t,u)\cdot\nu_{p,q}\,\mathrm{d}s(x)\mathrm{d}t$

under CFL condition

$$\delta t \le \frac{1}{2L}\inf_{(p,q)\in\mathcal{E}_h}\frac{m_p}{m_{p,q}}$$

Numerical scheme : Finite element step

Seek $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ such that

$$\int_{\mathbb{R}^d} \frac{\overline{u}_h^{n+1} - \overline{u}_h^{n+\frac{1}{2}}}{\delta t} \overline{v}_h \mathrm{d}x + \int_{\mathbb{R}^d} (\lambda_h^{n+1} + \theta(h)\nabla \widehat{u}_h^{n+1}) \cdot \nabla \widehat{v}_h \, \mathrm{d}x = 0, \ \forall v_h \in X_h$$

$$|\lambda_h^{n+1}| \leq 1 \text{ if } \nabla \widehat{u}_h^{n+1} = 0, \ \text{otherwise } \lambda_h^{n+1} = \frac{\nabla \widehat{u}_h^{n+1}}{|\nabla \widehat{u}_h^{n+1}|}$$

with

$$X_h = \{v_h \in \mathbb{R}^{\mathbb{N}}; \ \nabla \widehat{v}_h \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \ \overline{v}_h \in L^2(\mathbb{R}^d)\}$$

$$\Lambda_h := \{\mu_h \in L^\infty(\mathbb{R}^d)^d; \mu_{h|K} \text{ is constant for each } K \in \mathcal{T}_h\}$$

$$\theta(h) > 0 \text{ and } \lim_{h\to 0} \theta(h) = \lim_{h \to 0} \frac{h}{\theta(h)} = 0$$

Remark : approximate finite element step used in practice

Seek $u_h^{n+1} \in X_h$ such that

$$\int_{\mathbb{R}^d} \frac{\overline{u}_h^{n+1} - \overline{u}_h^{n+\frac{1}{2}}}{\delta t} \overline{v}_h \mathrm{d}x + \int_{\mathbb{R}^d} \left(\frac{1}{(\varepsilon(x)^2 + |\nabla \widehat{u}_h^{n+1}|^2)^{1/2}} + \theta(h) \right) \nabla \widehat{u}_h^{n+1} \cdot \nabla \widehat{v}_h \, \mathrm{d}x = 0,$$

$\forall v_h \in X_h$

numerical entropy flux

$$\Phi^n_{p,q}(x,y) := \frac{1}{2}\int_{a_0}^{b_0} \eta''(\kappa)\left(F^n_{p,q}(x\top\kappa, y\top\kappa) - F^n_{p,q}(x\bot\kappa, y\bot\kappa)\right)\, d\kappa + \frac{\eta'(a_0)+\eta'(b_0)}{2} F^n_{p,q}(x,y)$$

entropy inequality satisfied by the finite volume step using CFL condition

$$m_p \frac{\eta(u_p^{n+\frac{1}{2}}) - \eta(u_p^n)}{\delta t} + \sum_{q\in\mathcal{N}_p} \Phi^n_{p,q}(u^n_p, u^n_q) \le 0$$

(proof using Krushkov entropies)
implies for $\eta(0) = 0$, $\eta'(0) = 0$

$$\int_{\mathbb{R}^d} \eta(\overline{u}_h^{n+\frac{1}{2}}(x))\mathrm{d}x \le \int_{\mathbb{R}^d} \eta(\overline{u}_h^n(x))\mathrm{d}x$$

(multiply by $\exp(-\varepsilon|x_p|)$, sum over the mesh and let $\varepsilon \to 0$)

$\eta(s) \to s\top b_0 - b_0$ implies $u_p^{n+\frac{1}{2}} \le b_0$ for all $p \in \mathbb{N}$

$\eta(s) \to a_0 - s\bot a_0$ implies $u_p^{n+\frac{1}{2}} \ge a_0$

$\eta(s) \to |s|$ implies $\overline{u}_h^{n+\frac{1}{2}} \in L^1(\mathbb{R}^d)$

$\eta(s) = s^2$ implies $\|\overline{u}_h^{n+\frac{1}{2}}\|_{L^2(\mathbb{R}^d)} \le \|\overline{u}_h^n\|_{L^2(\mathbb{R}^d)}$

$u_h^{n+1} \in X_h$ minimizer of

$$J_h^{n+1}(v_h) := \frac{1}{2\delta t}\int_{\mathbb{R}^d}\left(\overline{v}_h - \overline{u}_h^{n+\frac{1}{2}}\right)^2 \mathrm{d}x + \int_{\mathbb{R}^d}(|\nabla\widehat{v}_h| + \frac{1}{2}\theta(h)|\nabla\widehat{v}_h|^2)\,\mathrm{d}x$$

thanks to saddle point theorem, deduce that exists $(u_h^{n+1}, \lambda_h^{n+1}) \in X_h \times \Lambda_h$ such that

$$\int_{\mathbb{R}^d}\frac{\overline{u}_h^{n+1} - \overline{u}_h^{n+\frac{1}{2}}}{\delta t}\overline{v}_h \mathrm{d}x + \int_{\mathbb{R}^d}(\lambda_h^{n+1} + \theta(h)\nabla\widehat{u}_h^{n+1})\cdot\nabla\widehat{v}_h\,\mathrm{d}x = 0,\ \forall v_h \in X_h$$

$$|\lambda_h^{n+1}| \leq 1 \text{ if } \nabla\widehat{u}_h^{n+1} = 0,\ \text{otherwise } \lambda_h^{n+1} = \frac{\nabla\widehat{u}_h^{n+1}}{|\nabla\widehat{u}_h^{n+1}|}$$

take $v_p = \eta'(u_p^{n+1})$ with $\eta(0) = 0$, $\eta'(0) = 0$ and η'' bounded gives

$$\int_{\mathbb{R}^d}\eta(\overline{u}_h^{n+1}(x))\mathrm{d}x \leq \int_{\mathbb{R}^d}\eta(\overline{u}_h^{n+\frac{1}{2}}(x))\mathrm{d}x$$

as finite volume step conclusion on L^∞ and L^1 bounds

Remark : approximate finite element step
$u_h^{n+1} \in X_h$ minimizer of

$$\widetilde{J}_h^{n+1}(v_h) := \frac{1}{2\delta t}\int_{\mathbb{R}^d}\left(\overline{v}_h - \overline{u}_h^{n+\frac{1}{2}}\right)^2 \mathrm{d}x + \int_{\mathbb{R}^d}\left((\varepsilon(x)^2 + |\nabla\widehat{v}_h|^2)^{1/2} + \frac{1}{2}\theta(h)|\nabla\widehat{v}_h|^2\right)\mathrm{d}x$$

$L^1(0,T;BV(\mathbb{R}^d))$ *estimate*

$$\frac{1}{2}\|\overline{u}_{h,\delta t}\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))} + \sum_{n=1}^{N} \delta t \int_{\mathbb{R}^d} |\nabla \widehat{u}_h^n| \,\mathrm{d}x \leq \frac{1}{2}\|u_{\mathrm{ini}}\|^2_{L^2(\mathbb{R}^d)}$$

and

$$\sum_{n=0}^{N-1} \delta t \int_{\mathbb{R}^d} \theta(h)|\nabla \widehat{u}_h^{n+1}|^2 \,\mathrm{d}x = \theta(h) \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{T}_h} \sum_{p\in\mathcal{V}_K} \sum_{q\in\mathcal{V}_K} T^K_{pq}(u_p^{n+1} - u_q^{n+1})^2 \leq \frac{1}{2}\|u_{\mathrm{ini}}\|^2_{L^2(\mathbb{R}^d)}$$

proof : use L^2 control on finite volume step and $v_h = u_h^{n+1}$

leads to L^1_{loc} time and space translates estimates for $\overline{u}_{h,\delta t}$ and therefore $\widehat{u}_{h,\delta t}$ converges as well

Used for control of numerical fluxes in the finite volume step only for $n \geq 1$, no need of more regularity for u_{ini}

Entropy estimate

$$\int_{Q_T} \eta(\bar{u}_{h,\delta t})\partial_t\varphi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \Phi(x,t,\bar{u}_{h,\delta t})\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t$$
$$- \int_{Q_T} \eta'(\widehat{u}_{h,\delta t})\lambda_{h,\delta t}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t - \int_{Q_T} \varphi|\nabla\eta'(\widehat{u}_{h,\delta t})|\,\mathrm{d}x\mathrm{d}t + e_{h,\delta t} \geq 0$$

with $\lim_{h\to 0,\delta t\to 0} |e_{h,\delta t}| = 0$

proof : use entropy inequality of finite volume step

$v_p^{n+1} = \eta'(u_p^{n+1})\varphi(x_p,(n+1)\delta t)$ in finite element step

difficulty : error between
$\sum_n \delta t \int_{\mathbb{R}^d} \lambda_h^{n+1}\cdot\nabla v_h^{n+1}\mathrm{d}x\mathrm{d}t$ and $\int_{Q_T} \varphi|\nabla\eta'(\widehat{u}_{h,\delta t})|\mathrm{d}x\mathrm{d}t$

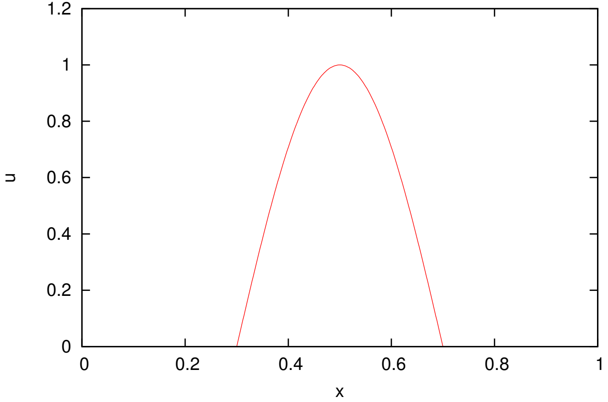
term in $\theta(h)$ with $h/\theta(h)\to 0$ necessary
$\theta(h)\to 0$ necessary for this term to vanish

conclusion by passing to the limit $h\to 0, \delta t\to 0$ using lower semi-continuity of the BV norm

A 1D numerical example

$$\partial_t u + \frac{1}{2}\partial_x(u^2) - g\,\partial_x \frac{\partial_x u}{|\partial_x u|} = 0$$

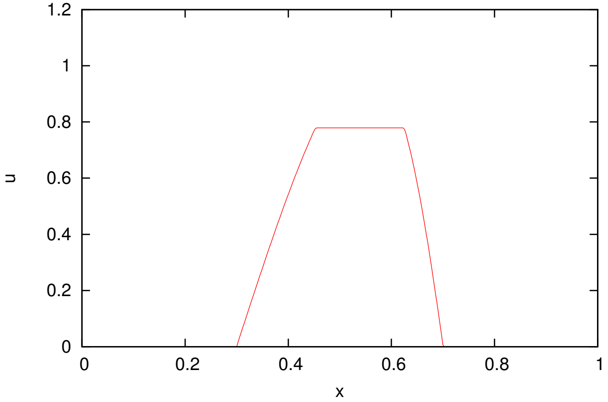
with $g = 5 \cdot 10^{-4}$: $h = 0.002$, $\delta t = 0.001$, initial data



A 1D numerical example

$$\partial_t u + \frac{1}{2}\partial_x(u^2) - g\,\partial_x \frac{\partial_x u}{|\partial_x u|} = 0$$

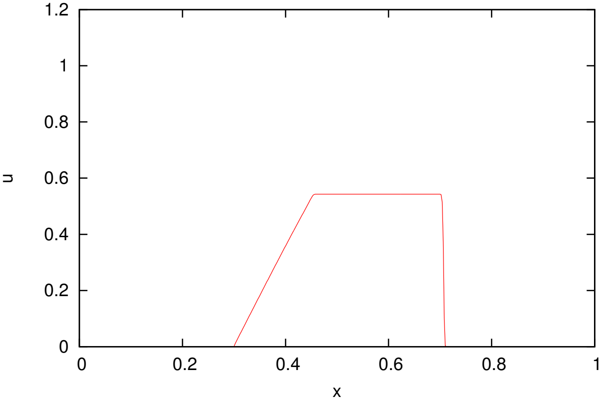
with $g = 5 \cdot 10^{-4}$, $t = 0.05$



A 1D numerical example with convection

$$\partial_t u + \frac{1}{2}\partial_x(u^2) - g\,\partial_x \frac{\partial_x u}{|\partial_x u|} = 0$$

with $g = 5 \cdot 10^{-4}$, $t = 0.15$



A 2D numerical example with convection

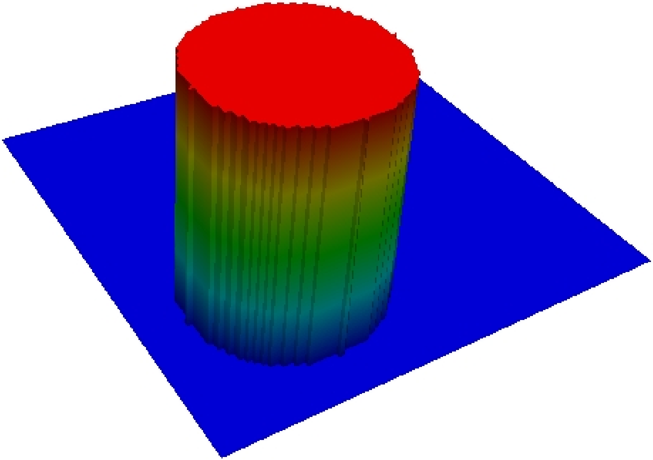
$$\partial_t u + c_x \partial_x u + c_y \partial_y u - g \operatorname{div} \frac{\nabla u}{|\nabla u|} = 0$$

$$u_{\text{ini}}(x, y) = 1_{D_0}(x, y)$$

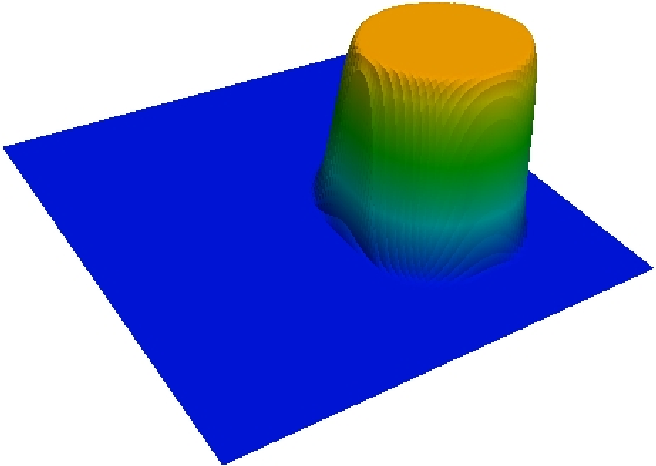
Analytical solution

$$u : (x, y, t) \mapsto \left(1 - \frac{2gt}{r_0}\right)^+ 1_{D_0}(x - c_x t, y - c_y t)$$

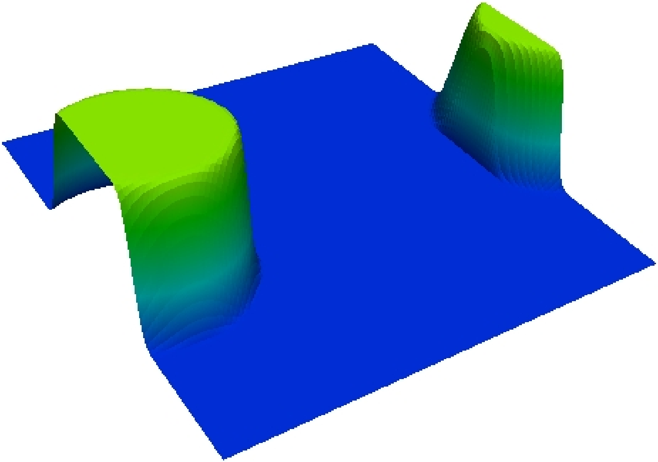
2D initial data



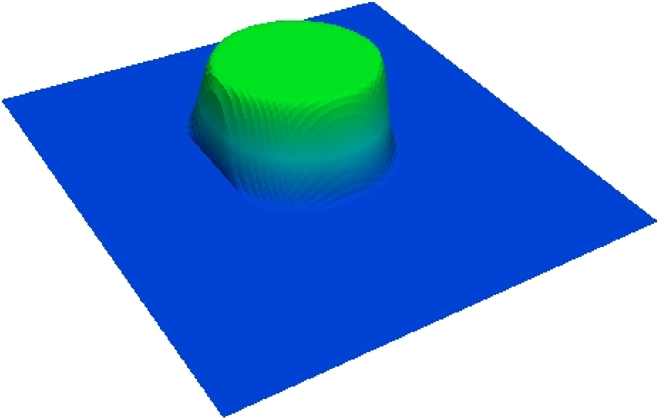
2D solution at $t = 0.5$

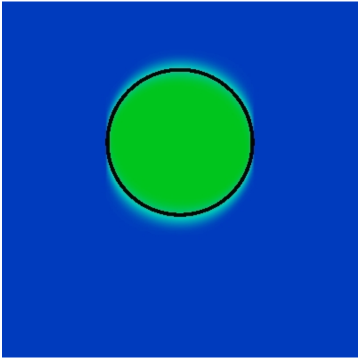


2D solution at $t = 1$

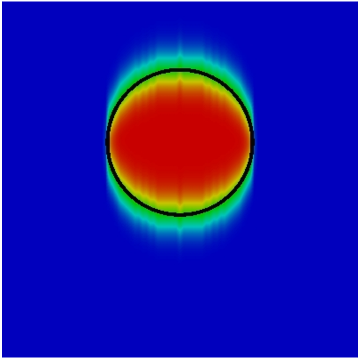


2D solution at $t = 1.5$





convection and TV flow



convection without TV flow

1. a first step to Navier-Stokes equations with solid-fluid transition
2. full finite volume scheme not expected to converge in the TV flow part
3. need of viscous regularization for the analysis of convergence, not needed in the numerical tests