An extension procedure for the constraint equations

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Overview of the talk

- The Cauchy problem
    - The constraint equations
    - Initial data
- The extension problem, past results and motivation
- Our extension procedure
- Sketch of the proof
    - Part 1: The prescribed divergence equation for a 2-tensor
    - Part 2: The prescribed scalar curvature equation

The Cauchy problem of general relativity

Spacetime $=$ 4-dim. Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ solving the *Einstein vacuum equations*

$$\mathrm{Ric}(\mathbf{g})_{\mu\nu} = 0$$

Initial data $=$ A triple (Σ, g, k) where (Σ, g) is a Riem. 3-manifold, k a symmetric 2-tensor solving the *constraint equations*

$$\begin{aligned} R(g) &= |k|_g^2 - (\mathrm{tr}_g k)^2 \\ \mathrm{div}_g k &= d(\mathrm{tr}_g k) \end{aligned}$$

In the future development $(\mathcal{M}, \mathbf{g})$, $\Sigma \subset \mathcal{M}$ is a spacelike Cauchy hypersurface with induced metric g and second fundamental form k

The initial data

We consider in the following $\Sigma \subset (\mathcal{M}, \mathbf{g})$ that are *maximal*

$$\mathrm{tr}_g k = 0$$

With this assumption, we arrive at the *maximal constraint equations* for (g, k),

$$\begin{aligned} R(g) &= |k|_g^2 \\ \mathrm{div}_g k &= 0 \\ \mathrm{tr}_g k &= 0 \end{aligned}$$

The initial data

The trivial solution to the Einstein vacuum equations is *Minkowski spacetime*

$$(\mathbb{R}^{1+3}, \mathbf{m})$$

The corresponding initial data is

$$(\Sigma, g, k) = (\mathbb{R}^3, e, 0)$$

Asymptotic flatness

Consider *asymptotically flat* initial data

$$g(x) = e + O\left(\frac{1}{|x|^{1/2}}\right), k(x) = O\left(\frac{1}{|x|^{3/2}}\right)$$

as $|x| \to \infty$

The initial data

Regularity of the data

The critical scaling is at $s_c = 3/2$, that means

$$(g, k) \in \mathcal{H}^{3/2}_{loc} \times \mathcal{H}^{1/2}_{loc}$$

In the following, consider

$$(g, k) \in \mathcal{H}^{2}_{loc} \times \mathcal{H}^{1}_{loc}$$

The extension problem

Extension problem. *Given initial data* (g, k) *on the unit ball* $B_1 \subset \mathbb{R}^3$*, does there exist a regular asymptotically flat initial data set* (g', k') *on* $\mathbb{R}^3$ *that isometrically contains* (g, k) *and continuously depends on it?*

Appears in the context of

- analysing the space of solutions of the constraint equations: Bartnik, Smith-Weinstein, Isenberg, Shi-Tam
- considering the rigidity of solutions of the constraint equations: Corvino-Schoen, Chruściel-Delay, Isenberg, Pollack
- Bartnik's definition of quasi-local mass: Bartnik, Huisken-Ilmanen, Miao, Shi-Tam

Our motivation to study the extension problem

Theorem (Bounded L^2 curvature theorem, Klainerman-Rodnianski-Szeftel)

Let (Σ, g, k) be initial data on a non-compact, maximal Σ. Then there exists a time $T > 0$ depending on

$$\|\mathrm{Ric}\|_{L^2(\Sigma)}, \|k\|_{H^1(\Sigma)}$$

such that the space-time can be continued and controlled up to time T

We want to prove a *localised* version of this theorem!

The main theorem

Theorem (S.C., 2016)

Let $(\bar{g}, \bar{k}) \in \mathcal{H}^2(B_1) \times \mathcal{H}^1(B_1)$ *be a solution to the maximal constraint equations on* $B_1 \subset \mathbb{R}^3$*. There is* $\epsilon > 0$ *small enough such that if*

$$\|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^2(B_1) \times \mathcal{H}^1(B_1)} < \epsilon$$

then there exits a solution (g, k) *on* $\mathbb{R}^3$ *to the maximal constraint equations such that*

- $(g, k)|_{B_1} = (\bar{g}, \bar{k})$
- (g, k) *is asymptotically flat*
- $\|(g - e, k)\|_{\mathcal{H}^2_{-1/2}(\mathbb{R}^3) \times \mathcal{H}^1_{-3/2}(\mathbb{R}^3)} \lesssim \|(\bar{g} - e, \bar{k})\|_{\mathcal{H}^2(B_1) \times \mathcal{H}^1(B_1)}$

Our main theorem

Remarks

- does not need a gluing region
- preserves regularity
- holds also for higher regularity $\mathcal{H}^w_{-1/2} \times \mathcal{H}^w_{-3/2}$ with $w \geq 2$
- is fitted to the assumptions of the bounded L^2 curvature theorem

Sketch of the proof

The idea: Construct a sequence of pairs (g_i, k_i) that extend $(\bar{g}, \bar{k})$ and converge to a solution of the maximal constraints.

The construction: Given (g_i, k_i) on $\mathbb{R}^3$,

1. Let g_{i+i} be an AF metric on $\mathbb{R}^3$ such that

$$\begin{aligned} g_{i+1}|_{B_1} &= \bar{g} \\ R(g_{i+1}) &= |k_i|^2_{g_i} \end{aligned}$$

2. Let k_{i+1} be AF symmetric 2-tensor on $\mathbb{R}^3$ such that

$$\begin{aligned} k_{i+1}|_{B_1} &= \bar{k} \\ \mathrm{div}_{g_{i+1}} k_{i+1} &= 0 \\ \mathrm{tr}_{g_{i+1}} k_{i+1} &= 0 \end{aligned}$$

Sketch of the proof: The divergence equation

Lemma (Extension result for k)

Let g be an AF metric on $\mathbb{R}^3$ and $\bar{k}$ a symmetric 2-tensor on B_1 such that

$$\begin{aligned} \mathrm{div}_g \bar{k} &= 0 \\ \mathrm{tr}_g \bar{k} &= 0 \end{aligned}$$

If $g \approx e$, $\bar{k} \approx 0$, then there exists an AF symmetric 2-tensor k on $\mathbb{R}^3$ such that

$$\begin{aligned} k|_{B_1} &= \bar{k} \\ \mathrm{div}_g k &= 0 \\ \mathrm{tr}_g k &= 0 \end{aligned}$$

Sketch of the proof: The divergence equation

Idea: Extend just via standard Sobolev extension and then correct the error!

Correcting the error: For the error ρ, solve on $\mathbb{R}^3 \setminus \overline{B_1}$ for $\tilde{k}$

$$\mathrm{div}_g \tilde{k} = \rho$$
$$\mathrm{tr}_g \tilde{k} = 0$$

such that

$$\tilde{k} \in \overline{\mathcal{H}}^{w-1}_{-3/2}(\mathbb{R}^3 \setminus \overline{B_1})$$

This means that all derivatives of $\tilde{k}$ must vanish on $\{r = 1\}$

Tools: Use the implicit function theorem + surjectivity at the Euclidean metric

Sketch of the proof: The divergence equation

Surjectivity at the Euclidean metric: Prove that for every ρ, there is an AF $\tilde{k}$ such that on $\mathbb{R}^3$

$$\mathrm{div}_e \tilde{k} = \rho$$
$$\mathrm{tr}_e \tilde{k} = 0$$

Comments:

- This system is under-determined. But the 3-dimensional Hodge system

$$\mathrm{div}_e \tilde{k} = \rho$$
$$\mathrm{curl}_e \tilde{k} = \sigma$$
$$\mathrm{tr}_e \tilde{k} = 0$$

  is determined.
- Energy estimates give regularity on $\mathbb{R}^3 \setminus \overline{B_1}$. But we need that all the derivatives vanish at $r = 1$!
- Carefully pick σ by hand to make sure that all derivatives vanish at $r = 1$

Sketch of the proof: The divergence equation

Analyse the above Hodge system as follows

(1) Decompose $\tilde{k}$ with respect to ∂_r and tensors on 2-sphere S_r

$\rightarrow$ Get equations for scalars, sphere-tangent 1-forms and symmetric tracefree 2-tensors

(2) Expansion of sphere-tangent tensors in spherical harmonics

$\rightarrow$ Leads to: transport equations along r, elliptic equations on spheres S_r, scalar elliptic equations on $\mathbb{R}^3 \setminus \overline{B_1}$

Sketch of the proof: The divergence equation

Sketch: How to control boundary derivatives.

Let $f \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$. Let u solve on $\mathbb{R}^3 \setminus \overline{B}_1$

$$\triangle u = f$$
$$u\big|_{r=1} = 0$$

Observation 1: If in addition

$$\partial_r u\big|_{r=1} = 0$$

then all derivatives of u vanish on the boundary

Observation 2: The above condition can be written as integral conditions on f

Derivation of the integral conditions

Let $f \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B_1})$. Let u solve on $\mathbb{R}^3 \setminus \overline{B_1}$

$$\triangle u = f$$
$$u\big|_{r=1} = 0$$

Rewrite in spherical harmonics modes

$$\begin{aligned} f^{(lm)} = (\triangle u)^{(lm)} &= \left(\partial_r^2 u + \frac{2}{r}\partial_r u + \triangle_{S_r} u\right)^{(lm)} \\ &= \left(\partial_r^2 u + \frac{2}{r}\partial_r u\right)^{(lm)} - \frac{l(l+1)}{r^2} u^{(lm)} \\ &= \frac{1}{r^{l+1}}\partial_r \left(r^{2l+2}\partial_r \left(r^{-l-1} u^{(lm)}\right)\right) \end{aligned}$$

Derivation of the integral conditions

This implies that

$$-\partial_r u^{(lm)}|_{r=1} = \int_1^\infty \partial_r \left(r^{2l+2} \partial_r \left(r^{-l-1} u^{(lm)} \right) \right)$$
$$= \int_1^\infty r^{l+1} f^{(lm)} \stackrel{!}{=} 0$$

for all $l \geq 0, m \in \{-l, \dots, l\}$. Then

$$\partial_r u|_{r=1} = 0$$

Sketch of the proof

The idea: Construct a sequence of pairs (g_i, k_i) that extend $(\bar{g}, \bar{k})$ and converge to a solution of the maximal constraints.

The construction: Given (g_i, k_i) on $\mathbb{R}^3$

1. Let g_{i+i} be such that

$$g_{i+1}|_{B_1} = \bar{g}$$
$$R(g_{i+1}) = |k_i|^2_{g_i}$$

2. Let k_{i+1} be such that

$$k_{i+1}|_{B_1} = \bar{k}$$
$$\mathrm{div}_{g_{i+1}} k_{i+1} = 0$$
$$\mathrm{tr}_{g_{i+1}} k_{i+1} = 0$$

Sketch of the proof: The prescribed scalar curvature equation

The same idea as before: Extend $\bar{g}$ from B_1 to $\mathbb{R}^3$, then perturb its scalar curvature to the prescribed value.

Question: Given a metric g on $\mathbb{R}^3$, how to perturb its scalar curvature on $\mathbb{R}^3 \setminus \overline{B_1}$ without changing g on B_1?

Let

$$g = a^2 dr^2 + \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B)$$

For a scalar function φ and sphere-tangent vectorfield β', let

$$g_{\varphi,\beta'} = a^2 dr^2 + e^{2\varphi}\gamma_{AB}((\beta + \beta')^A dr + d\theta^A)((\beta + \beta')^B dr + d\theta^B)$$

Sketch of the proof: The prescribed scalar curvature equation

Lemma (Surjectivity at the Euclidean metric)

The linearisation of the scalar curvature via the above variations $g \to g_{\varphi,\beta'}$ *is*

$$D_{\varphi,\beta'} R|_e = \partial_r^2 \varphi + \frac{3}{r}\partial_r \varphi + \frac{1}{2}\triangle_{S_r}\varphi - \mathrm{div}_{S_r}\beta'$$

This is a surjective operator

Idea of proof: For a given h, we must show that there exist (φ, β') solving

$$\partial_r^2 \varphi + \frac{3}{r}\partial_r \varphi + \frac{1}{2}\triangle_{S_r}\varphi - \mathrm{div}_{S_r}\beta' = h$$

Rewrite this into

$$\partial_r^2 \varphi + \frac{3}{r}\partial_r \varphi + \frac{1}{2}\triangle_{S_r}\varphi = h + \zeta$$
$$\mathrm{div}_{S_r}\beta' = \zeta$$

Thank you for your attention.