Entropy stable schemes for degenerate convection-diffusion equations

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Outline

Degenerate convection-diffusion systems in one space dimension

- We consider PDE systems of the form:

$$\begin{aligned} \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x &= (\mathbf{k}(\mathbf{u})\mathbf{u}_x)_x, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \end{aligned} \tag{1}$$

  where

  - $\mathbf{u} : \mathbb{R} \times \mathbb{R}_+ \to \Omega \subset \mathbb{R}^N$,
  - $\mathbf{f} = (f_1, \ldots, f_N)^T$ is a nonlinear flux vector,
  - $\mathbf{k}(\mathbf{u}) \in \mathbb{R}^{N \times N}$ is a positive semidefinite diffusion matrix defined in Ω that may vanish in some spatial interval.

- As a consequence of the degeneracy of the viscous terms, the system may have solutions that present shock discontinuities due to the hyperbolic-parabolic dynamics of the problem.
- These systems arise, for instance, in porous media flow, sedimentation, two-phase flow models among others.

Degenerate convection-diffusion systems in one space dimension

- Different consistent numerical methods can provide very different numerical solutions of the shocks:

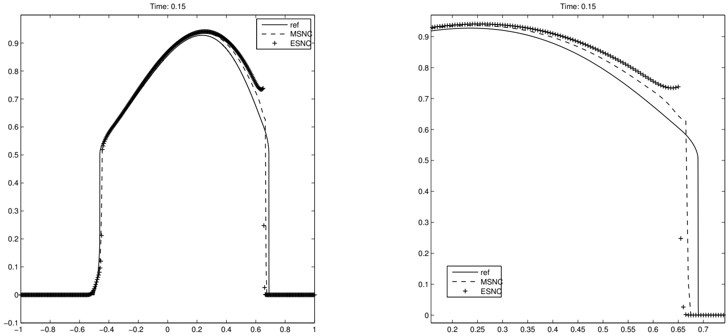

Figure: Numerical solutions given by MS and ES schemes with a centered approximation of the diffusion term in nonconservative for with $\Delta x = 0.005$ (left) and a zoom near the right discontinuity (right).

Degenerate convection-diffusion systems in one space dimension

- If there exists a function $\mathbf{K} : \Omega \to \mathbb{R}^N$ such that

$$\mathbf{K_u} = \mathbf{k},$$

where the subindex $\mathbf{u}$ indicates the Jacobian of the function $\mathbf{K}$, then system (1) can be written in the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{K}(\mathbf{u})_{xx}. \tag{2}$$

- This is always the case for scalar equations by defining

$$\mathbf{K}(\mathbf{u}) = \int_0^{\mathbf{u}} \mathbf{k}(\xi)\, d\xi.$$

Degenerate convection-diffusion systems in one space dimension

- The uniqueness and stability of entropy solutions of degenerate convection-diffusion equations was proved in Karlsen and Risebro (Disc. Cont. Dyn. Sys. 2003) for the multidimensional case.
- The design and the numerical analysis of schemes for the degenerate parabolic problem (1) is difficult because of the complex parabolic-hyperbolic interaction. Some numerical methods have been proposed to approximate degenerate solutions like:
  - monotone/upwind schemes: Evje and Karlsen (SINUM 2000), Karlsen, Risebro, and Towers (IMA J. Num. Anal. 2002), Bürger, Coronel, and Sepúlveda (Appl. Num. Math. 2006), Bessemoulin-Chatard and F. Filbet (SISC 2012), Angelini, Brenner, and Hilhors (Num. Math. 2013)...
  - splitting schemes Evje, Karlsen, Lie, and Risebro (Parallel solution of PDE, Springer 2000), Holden, Karlsen, and Lie (Comput. Geosci. 2000)...
  - ENO/WENO schemes Liu, Shu and Zhang (SISC 2011)...
  - finite element methods Chen, Ewing, Jiang, and Spagnuolo (SINUM 2002), Eymard, Hilhorst, and Vohralík (Num. Math. 2002)...
  - relaxation schemes Cavalli, Naldi, Puppo, and Semplice (SINUM 2007)...
- Our goal: to derive numerical methods that are **stable** for systems for systems in conservative (2) or non-conservative form (1). Jerez and CP (accepted in SINUM).
- In order to do so, we extend the entropy stable methods introduced by Tadmor (Math. Comp. 1987) to degenerate nonlinear convection-diffusion systems.

Entropy conservative methods for systems of conservation laws

- Let us consider a system of conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \tag{3}$$

equipped with an entropy pair (η, g), i.e. a pair of functions from Ω to $\mathbb{R}$ such that η is strictly convex and

$$g_{\mathbf{u}}^T = \eta_{\mathbf{u}}^T \mathbf{f}_{\mathbf{u}},$$

where the superscript T denotes the transpose vector.

- According to Tadmor, a semi-discrete conservative numerical method

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right),$$

is said to be *entropy conservative* (EC) if the equality

$$\frac{d}{dt}\eta(\mathbf{u})_j(t) = -\frac{1}{\Delta x}\left(G_{j+1/2} - G_{j-1/2}\right),$$

is satisfied for some numerical entropy flux $G_{j+1/2}$ consistent with g. T

Entropy conservative methods for systems of conservation laws

- If the numerical flux $\mathbf{F}_{i+1/2}$ satisfies

$$[\mathbf{v}]_{j+1/2}^T \mathbf{F}_{j+1/2} = [\varphi]_{j+1/2},$$

where $\mathbf{v} = \eta_{\mathbf{u}}(\mathbf{u})$ is the so-called *entropy variable* vector and $\varphi = \mathbf{v}^T \mathbf{f} - g$ is known as the *entropy potential* function, then the corresponding conservative method is entropy conservative.

- The following notation is used to denote the jumps and averages of any variable ω:

$$[\omega]_{j+1/2} := \omega_{j+1} - \omega_j, \quad \bar{\omega}_{j+1/2} := (\omega_{j+1} + \omega_j)/2.$$

- To avoid the appearance of strong oscillations near shocks, some numerical viscosity has to be added in such a way that the numerical method becomes *entropy stable* (ES).

Entropy conservative methods for systems of conservation laws

- The accuracy and efficiency of ES methods to approximate classical or non-classical regularization-sensitive shock waves have been shown in several works: Fjordholm, S. Mishra and E. Tadmor (JCP 2011), Fjordholm and S. Mishra (M2AN 2012), Castro, Fjordholm, Mishra, and CP (SINUM 2013)...

- They have also been extended to high-order of accuracy LeFloch and Rhode (SINUM 2000), Chalons and LeFloch (JCP 2001), LeFloch, Mercier, and Rohde (SINUM 2002), U.S. Fjordholm, S. Mishra, and E. Tadmor (SINUM 2012), as well as to discontinuous Galerkin methods Hiltebrand and Mishra (Num. Math. 2014).

Entropy inequality

- The following entropy condition for systems (1) based on a a 3-tuple will be considered here (Karlsen, Risebro, and Towers 2002):
- Let us assume that there exists a 3-tuple (η, g, r) of functions from the set Ω to $\mathbb{R}$, with η strictly convex such that:

$$g_{\mathbf{u}}^T = \eta_{\mathbf{u}}^T \mathbf{f}_{\mathbf{u}}, \tag{4}$$

$$r_{\mathbf{u}}^T = \eta_{\mathbf{u}}^T \mathbf{k}, \tag{5}$$

$$\mathbf{w}^T \eta_{\mathbf{u},\mathbf{u}} \mathbf{k}(\mathbf{u}) \mathbf{w} \geq 0. \quad \forall \mathbf{w} \in \Omega, \tag{6}$$

- A weak solution of the system is said to be an **entropy solution** if the inequality

$$\eta(\mathbf{u})_t + g(\mathbf{u})_x - r(\mathbf{u})_{xx} \leq 0, \tag{7}$$

  is satisfied in the distribution sense.
- For scalar equations, given a strictly convex function η there always exist two functions g and r satisfying (4) and (5). Moreover, inequality (6) is a consequence of the strictly convexity of η and the positivity of $\mathbf{k}$.
- For a scalar equation written in conservative form (2), the choice $\eta = |u - c|$ leads to the the Kruzkov entropy condition.

Entropy inequality

- For systems, inequality (6) is satisfied at least under the following hypothese:
  - (H1) $\mathbf{k}(\mathbf{u}) = k(\mathbf{u})I$ where $k(\mathbf{u})$ is a non-negative function and I is the identity matrix, that is, if the diffusion matrix is scalar.
  - (H2) The diffusion term can be written in terms of the entropy variables:

$$(\widehat{\mathbf{k}}(\mathbf{v})\mathbf{v}_x)_x,$$

where $\widehat{\mathbf{k}}(\mathbf{v})$ is a positive semidefinite matrix.

Theorem

If there exists a function $\mathbf{K}$ such that $\mathbf{K}_{\mathbf{u}} = \mathbf{k}$, then the existence of a 3-tuple (η, g, r) satisfying relations (4) and (5) implies that the system is symmetrizable.

Entropy stable schemes

Definition

A finite difference scheme to solve (1) is said to be entropy stable if there exists two numerical entropy fluxes $G_{j+1/2}$ and $\dot{r}_{j+1/2}$, consistent with g and r_x respectively, such that

$$\frac{d}{dt}\eta(\mathbf{u})_j(t) + \frac{1}{\Delta x}\left(G_{j+1/2} - G_{j-1/2}\right) - \frac{1}{\Delta x}\left(\dot{r}_{j+1/2} - \dot{r}_{j-1/2}\right) \leq 0. \tag{8}$$

Theorem

Consider a system in conservation form (2) *and assume that* $\mathbf{K}$ *satisfies*

$$\left(\eta_{\mathbf{u}}(\mathbf{u}_r) - \eta_{\mathbf{u}}(\mathbf{u}_l)\right)^T \left(\mathbf{K}(\mathbf{u}_r) - \mathbf{K}(\mathbf{u}_l)\right) \geq 0, \quad \forall \mathbf{u}_l, \mathbf{u}_r \in \Omega. \tag{9}$$

Then the finite difference scheme

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) + \frac{1}{\Delta x^2}\left([\mathbf{K}]_{j+1/2} - [\mathbf{K}]_{j-1/2}\right), \tag{10}$$

is entropy stable if the numerical flux, $\mathbf{F}_{j+1/2}$*, is EC.*

Entropy stable schemes

Theorem

The finite difference scheme

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) + \frac{1}{\Delta x^2}\left(\mathbf{k}_{j+1/2}[\mathbf{u}]_{j+1/2} - \mathbf{k}_{j-1/2}[\mathbf{u}]_{j-1/2}\right), \quad (11)$$

is entropy stable if the numerical flux, $\mathbf{F}_{j+1/2}$*, is EC and the numerical viscosity matrix,* $\mathbf{k}_{j+1/2}$*, verifies*

$$\bar{\mathbf{v}}^T_{j+1/2}\mathbf{k}_{j+1/2}[\mathbf{u}]_{j+1/2} = [r]_{j+1/2}, \quad (12)$$

and

$$[\mathbf{v}]^T_{j+1/2}\mathbf{k}_{j+1/2}[\mathbf{u}]_{j+1/2} \geq 0. \quad (13)$$

Entropy stable schemes

Theorem

Let us assume that there is not a pair $(\mathbf{u}_l, \mathbf{u}_r) \in \Omega \times \Omega$ *satisfying*

$$r(\mathbf{u}_l) \neq r(\mathbf{u}_r) \quad \textit{and} \quad \eta_u(\mathbf{u}_l) = -\eta_u(\mathbf{u}_r), \tag{14}$$

at the same time. Then, there exists at least a numerical viscosity matrix, $\mathbf{k}_{j+1/2}$*, verifying identity* (12) *respectively.*

Theorem

The finite difference scheme (10) *and* (11) *are second order accurate.*

Entropy stable schemes

Theorem

If the diffusion term can be written in terms of the entropy variables, he finite difference scheme

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) + \frac{1}{\Delta x^2}\left(\widehat{\mathbf{k}}_{j+1/2}[\mathbf{v}]_{j+1/2} - \widehat{\mathbf{k}}_{j-1/2}[\mathbf{v}]_{j-1/2}\right), \quad (15)$$

is entropy stable if the numerical flux, $\mathbf{F}_{j+1/2}$, is EC and the numerical viscosity matrix, $\widehat{\mathbf{k}}_{j+1/2}$ verifies

$$\bar{\mathbf{v}}^T_{j+1/2}\widehat{\mathbf{k}}_{j+1/2}[\mathbf{v}]_{j+1/2} = [r]_{j+1/2}, \quad (16)$$

and it is positive semidefinite.

Entropy stable schemes

- In order to avoid the appearance of strong oscillation in regions where the diffusion matrix, $\mathbf{k}$, vanishes, some extra numerical diffusion may be necessary:
  - Scheme under a conservative diffusion formulation.

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) + \frac{1}{\Delta x^2}\left([\mathbf{K}]_{j+1/2} - [\mathbf{K}]_{j-1/2}\right) + \frac{\varepsilon}{\Delta x^2}\left([\mathbf{u}]_{j+1/2} - [\mathbf{u}]_{j-1/2}\right), \quad (17)$$

  - Scheme under a non-conservative diffusion formulation.

$$\frac{d}{dt}\mathbf{u}_j(t) = -\frac{1}{\Delta x}\left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right) + \frac{1}{\Delta x^2}\left(\mathbf{k}_{j+1/2}[\mathbf{u}]_{j+1/2} - \mathbf{k}_{j-1/2}[\mathbf{u}]_{j-1/2}\right) + \frac{\varepsilon}{\Delta x^2}\left([\mathbf{u}]_{j+1/2} - [\mathbf{u}]_{j-1/2}\right). \quad (18)$$

- Under the assumptions of the corresponding Theorems, these schemes are entropy-stable. Usually, $\varepsilon = \alpha \Delta x$ for some $\alpha > 0$, and thus these schemes are only first order accurate.

Numerical tests

- The following schemes are compared:
    - MS: monotone flux, conservative discretization of the diffusion term, forward Euler.
    - ESC: entropy conservative flux, conservative discretization of the source term, forward Euler.
    - ESNC: entropy conservative flux, non conservative entropy stable discretization of the source term, forward Euler.
    - ESC-α: ESC with extra viscosity term with coefficient $\varepsilon = \alpha\Delta x$.
    - ESNC-α: ESNC with extra viscosity term with coefficient $\varepsilon = \alpha\Delta x$.
    - ESC2, ESNC2: like ESC, ESNC with TVD-RK2 instead of forward Euler.

Test problem 1

- Let us consider first the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = K(u)_{xx} \qquad (x,t) \in [-2,2] \times [0,T],$$

where $u \in [0,\infty)$ and the diffusion matrix is defined as

$$K(u) = \mu u^2,$$

where μ is a positive constant.

- Entropy 3-tuple:

$$\eta(u) = \frac{u^2}{2}, \quad g(u) = \frac{u^3}{3}, \quad r(u) = \frac{2}{3}\mu u^3.$$

- Entropy conservative numerical flux:

$$F^{EC}_{j+1/2} = \frac{1}{6}\left(u_{j+1}^2 + u_{j+1}u_j + u_j^2\right),$$

- Numerical viscosity matrix satisfying (12):

$$k_{j+1/2} = \begin{cases} 0 & \text{if} \quad u_j, u_{j+1} = 0, \\ \mu \dfrac{4}{3} \dfrac{u_j^2 + u_j u_{j+1} + u_{j+1}^2}{u_j + u_{j+1}} & \text{otherwise.} \end{cases}$$

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} (1-x^2)^2 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

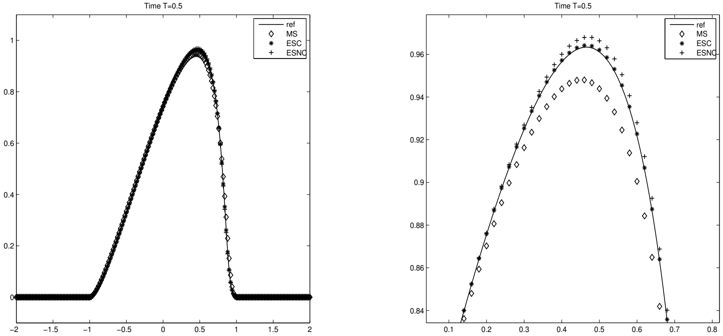

Figure: Numerical solution using MS, ESC ESNC with $\Delta x = 0.01$ (left) and a zoom near the maximum (right).

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} (1-x^2)^2 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

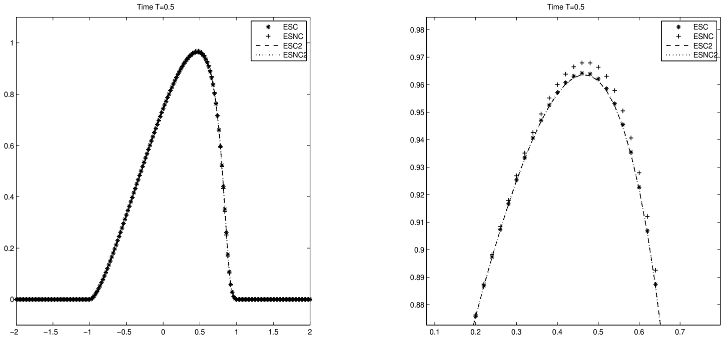

Figure: Numerical solution using ESC, ESNC, ESC2 and ESNC2 schemes with $\Delta x = 0.01$ (left) and a zoom near the maximum (right).

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} (1-x^2)^2 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

	N	200	400	800	1600	3200
Error	MS	1.62406e-2	8.39706e-3	4.27298e-3	2.15598e-3	1.08301e-3
	ESC	9.46103e-4	2.40781e-4	6.08614e-5	1.51969e-5	3.69530e-6
	ESNC	4.13155e-3	1.21311e-3	3.40230e-4	9.06166e-5	2.36939e-5
	ESC2	6.16713e-4	1.55571e-4	3.89052e-5	9.63768e-6	2.31532e-6
	ESNC2	7.02861e-4	1.83268e-4	5.28927e-5	1.42971e-5	4.00493e-6
Order	MS	-	0.9516	0.9746	0.9869	0.9933
	ESC	-	1.9743	1.9841	2.0017	2.0400
	ESNC	-	1.7680	1.8341	1.9087	1.9353
	ESC2	-	1.9870	1.9995	2.0132	2.0575
	ESNC2	-	1.9393	1.7928	1.8873	1.8359

Table: Error and convergence rate at time $T = 0.5$.

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} 1 & \text{if } -0.5 < x < 0.5, \\ 0 & \text{otherwise,} \end{cases}$$

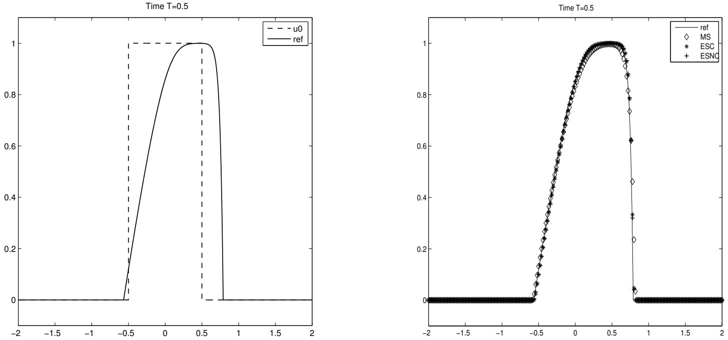

Figure: Initial condition and reference solution (left). Numerical solutions using MS, ESC, ESNC approximations with $\Delta x = 0.01$.

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} 1 & \text{if } -0.5 < x < 0.5, \\ 0 & \text{otherwise}, \end{cases}$$

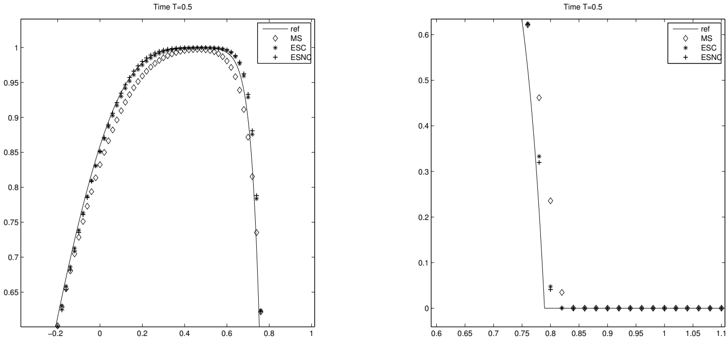

Figure: Numerical solutions using MS, ESC, ESNC with $\Delta x = 0.01$: zoom near the maximum and the discontinuity.

Test problem 1

- Initial condition:

$$u_0(x) = \begin{cases} 1 & \text{if } -0.5 < x < 0.5, \\ 0 & \text{otherwise,} \end{cases}$$

	N	200	400	800	1600	3200
Error	MS	3.04166e-2	1.63092e-2	8.31851e-3	4.17669e-3	2.06881e-3
	ESC	1.93344e-2	9.90348e-3	4.77316e-3	2.25036e-3	9.98193e-4
	ESNC	2.01441e-2	1.00118e-2	4.78085e-3	2.25108e-3	9.98520e-4
	ESC2	1.93294e-2	9.90233e-3	4.77289e-3	2.25030e-3	9.98177e-4
	ESNC2	1.93306e-2	9.90134e-3	4.77283e-3	2.25021e-3	9.98332e-4
Order	MS	-	0.8992	0.9713	0.9940	1.0136
	ESC	-	0.9652	1.0530	1.0848	1.1728
	ESNC	-	1.0086	1.0664	1.0866	1.1728
	ESC2	-	0.9650	1.0529	1.0847	1.1728
	ESNC2	-	0.9652	1.0528	1.0848	1.1725

Table: Error and convergence rate at time $T = 0.5$

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Test problem 2

- Consider first the equation

$$u_t + (u^2)_x = (k(u)u_x)_x, \qquad (x,t) \in [-1,1] \times [0,1],$$

where $u \in [0, \infty)$ and the diffusion matrix is defined as

$$k(u) = \begin{cases} 0 & \text{if} \quad u \le 0.5, \\ 2.5u - 1.25 & \text{if} \quad 0.5 < u < 0.6, \\ 0.25 & \text{if} \quad u \ge 0.6. \end{cases}$$

- Entropy 3-tuple:

$$\eta(u) = \frac{u^2}{2}, \quad g(u) = \frac{u^3}{3}, \quad r(u) = \begin{cases} 0 & \text{if} \quad u \le 0.5, \\ \frac{5}{6}u^3 - \frac{5}{8}u^2 + \frac{5}{96} & \text{if} \quad 0.5 < u < 0.6, \\ \frac{u^2}{8} - \frac{91}{2400} & \text{if} \quad u \ge 0.6. \end{cases}$$

- Entropy conservative flux for Burgers.
- $k_{j+1/2}$ can be explicitly obtained...

Test problem 2

- Initial condition:

$$u_0(x) := u(x,0) = \begin{cases} 0 & \text{if} \quad x \leq -0.5, \\ 5(x+0.5) & \text{if} \quad -0.5 < x < -0.3, \\ 1 & \text{if} \quad -0.3 < x < 0.3, \\ 5(0.5-x) & \text{if} \quad 0.3 < x < 0.5, \\ 0 & \text{if} \quad x \geq 0.5, \end{cases}$$

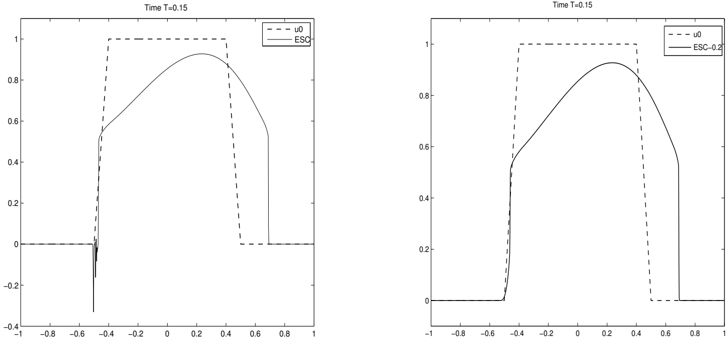

Figure: Initial condition and numerical solutions given by ESC and ESC-0.2 at time $T = 0.15$ with $\Delta x = 0.02$.

Test problem 2

- Initial condition:

$$u_0(x) := u(x,0) = \begin{cases} 0 & \text{if} \quad x \le -0.5, \\ 5(x+0.5) & \text{if} \quad -0.5 < x < -0.3, \\ 1 & \text{if} \quad -0.3 < x < 0.3, \\ 5(0.5-x) & \text{if} \quad 0.3 < x < 0.5, \\ 0 & \text{if} \quad x \ge 0.5, \end{cases}$$

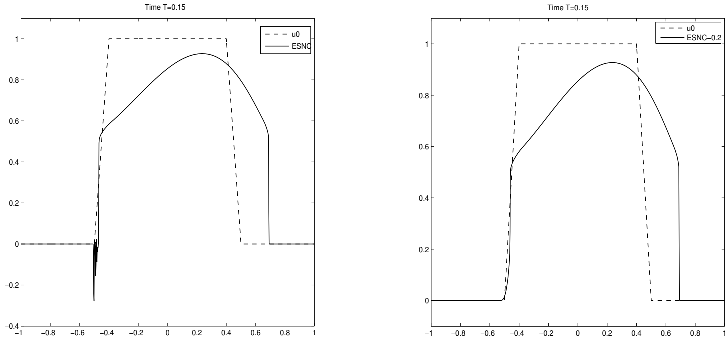

Figure: Initial condition and numerical solutions given by ESNC and ESNC-0.2 at time $T = 0.15$ with $\Delta x = 0.02$.

Test problem 2

- Initial condition:

$$u_0(x) := u(x,0) = \begin{cases} 0 & \text{if} \quad x \leq -0.5, \\ 5(x+0.5) & \text{if} \quad -0.5 < x < -0.3, \\ 1 & \text{if} \quad -0.3 < x < 0.3, \\ 5(0.5-x) & \text{if} \quad 0.3 < x < 0.5, \\ 0 & \text{if} \quad x \geq 0.5, \end{cases}$$

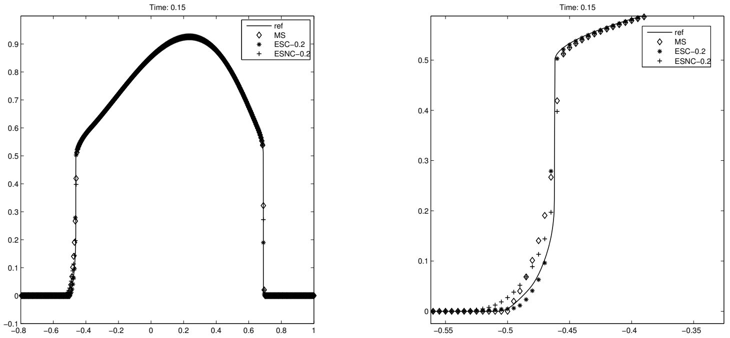

Figure: Numerical solutions given by MS, ESC-0.2 and ESNC-0.2 schemes with $\Delta x = 0.005$ (left) and a zoom near the left discontinuity (right).

Test problem 2

- Initial condition:

$$u_0(x) := u(x,0) = \begin{cases} 0 & \text{if} \quad x \leq -0.5, \\ 5(x+0.5) & \text{if} \quad -0.5 < x < -0.3, \\ 1 & \text{if} \quad -0.3 < x < 0.3, \\ 5(0.5-x) & \text{if} \quad 0.3 < x < 0.5, \\ 0 & \text{if} \quad x \geq 0.5, \end{cases}$$

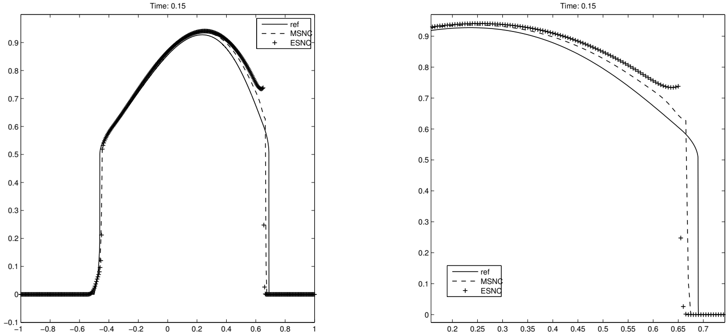

Figure: Numerical solutions given by MS and ES schemes with a centered approximation of the diffusion term in nonconservative for with $\Delta x = 0.005$ (left) and a zoom near the right discontinuity (right).

Test problem 2

- Initial condition:

$$u_0(x) := u(x,0) = \begin{cases} 0 & \text{if} \quad x \leq -0.5, \\ 5(x+0.5) & \text{if} \quad -0.5 < x < -0.3, \\ 1 & \text{if} \quad -0.3 < x < 0.3, \\ 5(0.5-x) & \text{if} \quad 0.3 < x < 0.5, \\ 0 & \text{if} \quad x \geq 0.5, \end{cases}$$

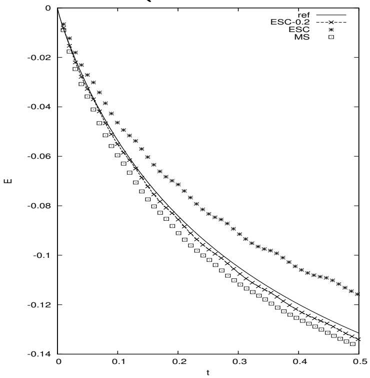

Figure: Total entropy decay.

Test problem 3

- Consider the system

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 - \mu \partial_x((u_1^2 + u_2^2)\partial_x u_1) = 0, \\ \partial_t u_2 + u_2 \partial_x u_2 - \mu \partial_x((u_1^2 + u_2^2)\partial_x u_2) = 0. \end{cases},$$

  This system cannot be written in conservative form.

- Entropy 3-tuple:

$$\eta = \frac{u_1^2 + u_2^2}{2}; \quad g = \frac{u_1^3 + u_2^3}{3}, \quad r = \frac{(u_1^2 + u_2^2)^2}{4}.$$

- Entropy conservative numerical flux for Burgers equation.
- Numerical viscosity matrix satisfying (12):

$$k_{j+1/2} = \mathbf{k}_{j+1/2} = k_{j+1/2} I, \quad k_{j+1/2} = \frac{u_{1,j}^2 + u_{1,j+1}^2 + u_{2,j}^2 + u_{2,j+1}^2}{2}$$

Test problem 3

- Initial condition:

$$u_1(x,0) = \begin{cases} 1 & \text{if} \quad -1.5 < x < -1.3, \\ 1 & \text{if} \quad -0.5 < x < 0.5, \\ 0 & \text{otherwise,} \end{cases} \qquad u_2(x,0) = \begin{cases} 1 & \text{if} \quad -0.5 < x < 0.5, \\ 1 & \text{if} \quad 1.3 < x < 1.5. \\ 0 & \text{otherwise,} \end{cases}$$

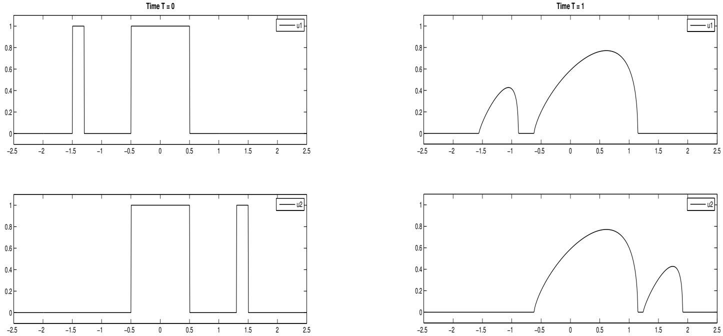

Figure: Numerical solution for at time $T = 0$ (left) and $T = 1$ (right) obtained with ESNC using a mesh of 1000 cells.

Test problem 3

- Initial condition:

$$u_1(x,0) = \begin{cases} 1 & \text{if} \quad -1.5 < x < -1.3, \\ 1 & \text{if} \quad -0.5 < x < 0.5, \\ 0 & \text{otherwise}, \end{cases} \qquad u_2(x,0) = \begin{cases} 1 & \text{if} \quad -0.5 < x < 0.5, \\ 1 & \text{if} \quad 1.3 < x < 1.5. \\ 0 & \text{otherwise}, \end{cases}$$

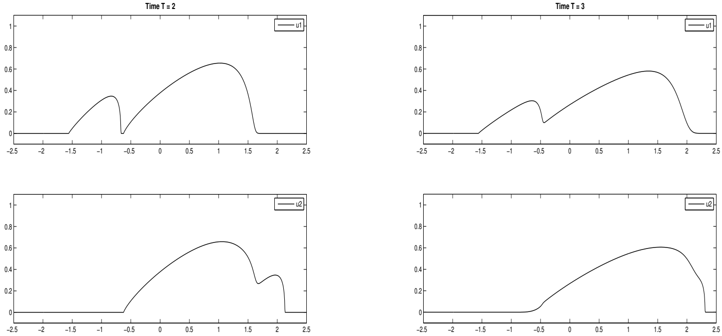

Figure: Numerical solution for at time $T = 2$ (left) and $T = 3$ (right) obtained with ESNC using a mesh of 1000 cells.

Test problem 4

- Consider the hyperbolic shallow water system:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{g_r}{2} h^2 \right) = 0. \end{cases}$$

  Here, h is the height of the fluid column, q is the discharge, $u = q/h$ is the velocity and finally, g_r is the gravity.

- Entropy pair is given by

$$\eta(\mathbf{u}) = \frac{1}{2}\left(hu^2 + g_r h^2 \right), \quad g(\mathbf{u}) = \frac{u^3}{2} + g_r h^2 u,$$

  where

$$\mathbf{u} = \begin{bmatrix} h \\ q \end{bmatrix}.$$

- Entropy variables are

$$\mathbf{v} = \begin{bmatrix} g_r h - \dfrac{u^2}{2} \\ u \end{bmatrix}.$$

Test problem 4

- Entropy conservative flux Fjordholm, Mishra and Tadmor (JCP 2011):

$$\mathbf{F}_{j+1/2} = \begin{bmatrix} \bar{h}_{j+1/2}\bar{u}_{j+1/2} \\ \frac{g_r}{2}\overline{h^2}_{j+1/2} + \bar{h}_{j+1/2}(\bar{u}_{j+1/2})^2 \end{bmatrix}.$$

- The standard form of the eddy viscosity does not fit in the framework of this work.
- We consider a viscous term written in entropy variables as follows:

$$\mu(\widehat{\mathbf{k}}(\mathbf{v})\mathbf{v}_x)_x,$$

where $\mathbf{v} = \mu > 0$ and

$$\widehat{\mathbf{k}}(\mathbf{v}) = \begin{bmatrix} v_1^2 + v_2^2 & 0 \\ 0 & v_1^2 + v_2^2 \end{bmatrix},$$

for which the diffusion entropy flux is

$$r(\mathbf{v}) = \frac{(v_1^2 + v_2^2)^2}{4}.$$

Test problem 4

- Initial condition:

$$h(x,0) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}, \quad q(x,0) \equiv 0.$$

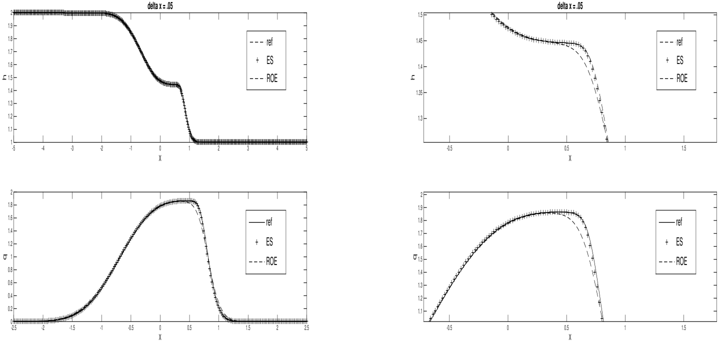

Figure: Numerical solution for the viscous shallow water model with $\mu = 10^{-3}$ at time $T = 0.2$ with the ES and Roe methods using a mesh of 500 cells (left). Zoom (right).

Test problem 4

- Initial condition:

$$h(x,0) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}, \quad q(x,0) \equiv 0.$$

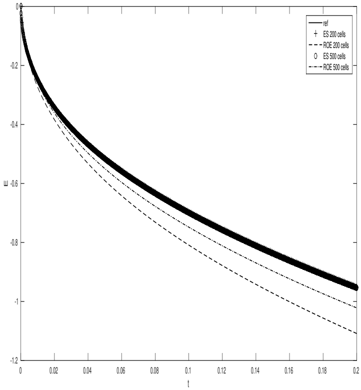

Figure: Total entropy decay for the viscous shallow water model.

Conclusions

- Entropy stable methods can be designed for parabolic degenerate 1d systems provided that the system is equipped with a 3-tuple of entropy.
- Extra numerical viscosity may be needed to avoid the appearance of spurious oscillations near shocks.
- The numerical tests show that the solutions provided by entropy stable schemes converge to the right weak solutions, at least for scalar 1d equations, either if the viscous terms are written in conservative or nonconservative form.
- The convergence rate and the total entropy dispersion are better than the corresponding to numerical methods based on a monotone flux.
- **Further developments:** applications to parabolic degenerate models coming from fluid models; extension to high order and multdimenional problems.