The quantum stress-energy tensor
and its intricate relationship
with spacetime geometry

(featuring works w. C. Gérard, O. Oulghazi, A. Vasy)

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INTRODUCTION

At low energies, Quantum Gravity should yield an effective theory (**semi-classical Einstein equations**):

$$(*) \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi\, \omega(:\hat{T}_{\mu\nu}:)$$

where g is classical, and $\omega(:\hat{T}_{\mu\nu}:)$ quantum.

Major problem: If $\hat{\phi}(x)$ is a quantum field (say, scalar, linear):

$$\underbrace{(-\Box_g + m^2)}_{P} \hat{\phi}(x) = 0, \quad [\hat{\phi}(x), \hat{\phi}(y)] = 0 \text{ for spacelike sep. } x, y$$

achieved through $[\hat{\phi}(x), \hat{\phi}(y)] = \mathrm{i}^{-1}(P_+^{-1} - P_-^{-1})(x,y)\mathbf{1}_{\mathcal{H}}$. Hence

$$\hat{\phi}^2(x) = \lim_{x \to y} \hat{\phi}(x)\hat{\phi}(y), \quad (\nabla_\mu \hat{\phi})(x)(\nabla_\nu \hat{\phi})(x), \text{ etc.}$$

need to be **renormalized**. Meaning of $(*)$ unclear.

A bit of history

Formally, $(*)$ obtained from quantum Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_M dx\sqrt{|\det g|}R + S_{\mathrm{matter}},$$

expanding around classical g and keeping only $\hbar^0$ metric terms + $\hbar^1$ quantum field terms ['60s].

Physicists promises:

- Weak energy condition violation (wormholes?)
- Minkowski space still stable
- Black hole evaporation
- Inflation without exotic matter

More rigorous first look: with physicist's prescriptions, $(*)$ ill-posed in Lorentzian signature, inconsistencies.

The program of Kay, Wald et al.

- Given g, construct $\hat{\phi}(x)$. Equivalent to specifying an exp. value

  $$x, y \mapsto \langle \Omega, \hat{\phi}(x)\hat{\phi}(y)\Omega \rangle =: \omega(x, y)$$

  At best we can hope $\lim_{x \to y} \hat{\phi}(x)\hat{\phi}(y) \sim \lim_{x \to y} \omega(x, y)$ is no worse than $m = 0$ **vacuum on Minkowski space:**

  $$\omega_{\text{vac}}(x, y) = \lim_{\epsilon \searrow 0} \frac{1}{4\pi^2} \frac{1}{(x-y)^2 + i\epsilon(x_0 - y_0) + \epsilon^2}$$

  Better characterisation: $(i^{-1}\partial_t - \sqrt{-\Delta})\omega_{\text{vac}} = 0$.

  Hadamard condition: $a(x, D_x)\omega(x, y) \in C^\infty(M \times M)$ for some a with princ. symbol as $(i^{-1}\partial_t - \sqrt{-\Delta})$.

- *Radzikowski's theorem* [Radzikowski '96]: $\omega(x, y)$ is of the form

  $$\lim_{\epsilon \searrow 0} \frac{1}{8\pi^2} \frac{u(x, y)}{\sigma(x-y) + i\epsilon(t(x) - t(y)) + \epsilon^2} + \log \text{term} + C^\infty(M \times M)$$
  $$= H(x, y) + C^\infty(M \times M)$$

The program of Kay, Wald et al.

1. Given g, construct $\hat{\phi}(x)$ or rather $\omega(x, y)$
2. Substract singular part $H(x, y)$ and take $\lim_{x \to y}(\omega(x, y) - H(x, y))$. [Wald]
3. Apply diff. operator $D_{\mu\nu}$ that promotes $:\hat{\phi}^2:$ to $:\hat{T}_{\mu\nu}:$.
    - $\nabla_\mu :\hat{T}^{\mu\nu}: = 0$ with Moretti's redefinition of $D_{\mu\nu}$ [Moretti '01].
    - In $(*)$ the dynamical variables are g and ω.

Consequences of quantum $:\hat{T}_{\mu\nu}:$

- **Violation of weak energy condition:**
  Nevertheless, lower bounds exist [Fewster '00].
- **Chronology protection theorems:**
  By [Kay, Radzikowski, Wald '97], $:\hat{T}_{\mu\nu}:$ diverges at any compactly generated Cauchy horizon.

Problem for $(*)$*:* steps 1. and 2. very indirect.

AN INTRICATE g-DEPENDENCE

But how to construct and control $g \mapsto \omega$? Presently no working example! Present approach via [Hollands & Wald '05]:

1. Axioms on scaling behaviour, local dependence on g, etc. give

$$:T_{\mu\nu}:' - :T_{\mu\nu}: = \alpha I_{\mu\nu} + \beta J_{\mu\nu} + \gamma G_{\mu\nu}$$

   with $I_{\mu\nu} = g_{\mu\nu}(\frac{1}{2}R^2 + 2\Box R) - 2R_{;\mu\nu} - 2RR_{\mu\nu}$, etc.
2. Find constants from experiment!

Gives self-consistent $(*)$. In simplified (low-regularity) FRLW setting, encouraging results [Pinamonti et. al. '10-'14], but does not work in general.

So our primary tasks:

- Need $g \mapsto \hat{\phi}_g$ with special properties (Hartle-Hawking state) on black hole families to describe e.g. evaporation
- Construct $g \mapsto \hat{\phi}_g$ and renormalize $\hat{\phi}_g^2$ *keeping track of g*
  - Original [Fulling, Narcowich, Wald '78] construction not useful here.

I. Pseudodifferential factorization of $-\Box_g + m^2$

Working on $[-\epsilon, \epsilon]_t \times \Sigma$ and coordinates s.t. $P = \partial_t^2 + a(t) + r(t)\partial_t$.

Theorem ([Gérard, Oulghazi, W. '16])

Choice of ω *(pure)* $\Leftrightarrow$ *choice of* $b(t) \in C^\infty(\mathbb{R}; \Psi^1_{\rm ell}(\Sigma))$ *s.t.*

$$(\partial_t + ib(t) + r(t)) \circ (\partial_t - ib(t)) = P + C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma))$$

Similarly $(\partial_t - ib^*(t) + r(t)) \circ (\partial_t + ib^*(t)) = P + C^\infty(\mathbb{R}; \Psi^{-\infty}(\Sigma))$.

- On spacetimes like Kerr, Kerr-de Sitter, pseudodifferential calculus on manifolds of bounded geometry [Kordyukov, Shubin]

Then $:\phi^2(t, {\rm x}):$ amounts to renormalized trace-density $({\rm Tr}_{\rm ren} c(t))({\rm x})$, with $c(t) \in \Psi^1_{\rm ell}(\Sigma)$ closely related to $b(t)$. This has a rich theory [Melrose, Nistor '88] (generalized zeta functions).

Here $b(t)$ determines g!

II. Calderón projector

Vague idea: How to distinguish between $e^{it\sqrt{-\Delta+m^2}}u$ and $e^{-it\sqrt{-\Delta+m^2}}u$? Replace $t = is$ and check L^2 membership.

More precisely: [Gérard '16] Near $\{s = 0\}$, P becomes P_E. Calderón projector $C :=$ proj. on space of $\{s = 0\}$ Cauchy data of L^2 solutions of $P_E u$.

- ✓ Yields the Hartle-Hawking state ω on spacetimes with static bifurcate Killing horizon [Gérard '16]

Work in progress: C is the Cauchy data of $\omega(x, y)$ satisfying Hadamard condition? Gives Hartle-Hawking state on Schwarzschild?

This would give a direct procedure $g \mapsto \omega$, and $:\phi^2:$ amounts to $\mathrm{Tr}_{\mathrm{ren}} P_E^{-1}$.

III. Global analysis

In Vasy's global approach, the pos./neg. frequencies decomposition happens at **radial points** for the bi-characteristic flow.

Example: Extended asymptotically dS spacetimes:

- $g = df^2 - h(f^2, y, dy)$ in $v < 0$ (f^2 times as. dS metric) ,
  $g = df^2 + h_\pm(f^2, y, dy)$ in $v > 0$ (f^2 times as. $\mathbb{H}^d_\pm$ metric)
  (close to conformal horizon $\{v = 0\} = \{f = 0\}$).

The **Vasy operator**

$$P = \begin{cases} f^{\mathrm{i}\nu-(d-1)/2-2}(\Box_{f^2 g} - (\frac{d-1}{2})^2 - \nu^2)f^{-\mathrm{i}\nu+(d-1)/2} \text{ on } \{v < 0\}, \\ f^{\mathrm{i}\nu-(d-1)/2-2}(-\Delta_{f^2 g} + (\frac{d-1}{2})^2 + \nu^2)f^{-\mathrm{i}\nu+(d-1)/2} \text{ on } \{v > 0\}, \end{cases},$$

Solutions of $Pu = 0$, smooth in as. dS region have asymptotics:

$$u = (v + \mathrm{i}0)^{-\mathrm{i}\nu} a^+ + (v - \mathrm{i}0)^{-\mathrm{i}\nu} a^- + a, \quad a^+, a^-, a \in C^\infty(M).$$

Let $C :=$ proj. to a^+ component. Assume no trapping.

Theorem ([Vasy, W. '16])

C is the asymptotic data of $\omega(x, y)$ satisfying Hadamard condition.

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Solutions of $Pu = 0$, smooth in as. dS region have asymptotics:

$$u = (v + i0)^{-i\nu} a^+ + (v - i0)^{-i\nu} a^- + a, \quad a^+, a^-, a \in C^\infty(M).$$

Now $:\phi^2:$ amounts to $\mathrm{Tr}_{\mathrm{ren}} P^{-1}$ with P^{-1} global inverse.

OUTLOOK

Despite its long history, formulation of

$$(*)\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi\, \omega(:\hat{T}_{\mu\nu}:)$$

still very fragile. However progress in understanding $g \mapsto \omega(:\hat{T}_{\mu\nu}:)$

I. $g \mapsto b(t) \mapsto \omega$ gives universal construction but still difficult to apply in this problem.

II. $g \mapsto P_E \mapsto C \mapsto \omega$ seems very robust for **initial value formulation**, but still in progress

III. global approach could be well-suited for clear **action principle**

At the very least this could be used to study questions like:

- When can we treat $:\hat{T}_{\mu\nu}:$ as some perturbation of $T_{\mu\nu}$?

Thank you for your attention!