Numerical methods for conservation laws with a stochastically driven flux

Håkon Hoel, Kenneth Karlsen, Nils Henrik Risebro, Erlend Briseid Storrøsten

Department of Mathematics, University of Oslo, Norway

December 8, 2016

Overview

1. Problem description
2. Deterministic conservation laws
   - Characteristics and shocks
   - Well-posedness
3. Stochastic scalar conservation laws
   - Definition and well-posedness
   - Numerical methods
   - Flow map cancellations
4. Conclusion

The model problem

Consider the stochastic conservation law

$$du + \partial_x f(u) \circ dz = 0, \quad \text{in} \quad (0, T] \times \mathbb{R},$$
$$u(0, \cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}).$$

Regularity assumptions

- $f \in C^2(\mathbb{R}; \mathbb{R})$
- $z \in C^{0,\alpha}([0, T]; \mathbb{R})$ for some $\alpha > 0$. That is,

$$\sup_{s \neq t \in [0,T]} \frac{|z(t) - z(s)|}{|t - s|^\alpha} < \infty.$$

Examples $z(t) = t$, Wiener processes, fractional Brownian motions.

The model problem

Consider the stochastic conservation law

$$du + \partial_x f(u) \circ dz = 0, \quad \text{in} \quad (0, T] \times \mathbb{R},$$
$$u(0, \cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}).$$

Regularity assumptions

- $f \in C^2(\mathbb{R}; \mathbb{R})$
- $z \in C^{0,\alpha}([0, T]; \mathbb{R})$ for some $\alpha > 0$. That is,

$$\sup_{s \neq t \in [0,T]} \frac{|z(t) - z(s)|}{|t - s|^\alpha} < \infty.$$

Examples $z(t) = t$, Wiener processes, fractional Brownian motions.

Motivation

For the mean-field SDE

$$dX^i = \sigma\left(X^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X^j}\right)\circ dW, \quad \text{for } i = 1, 2, \ldots, L,$$

with $\sigma : \mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$, one has that

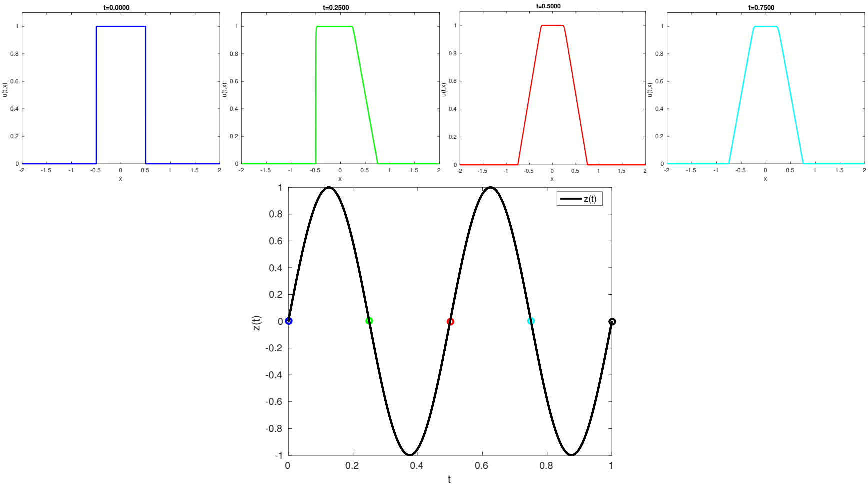
$$\frac{1}{L}\sum_{i=1}^{L}\delta_{X^i(t)} \to \pi(t)\in\mathcal{P}(\mathcal{P}(\mathbb{R})), \text{ as } L\to\infty.$$

The measure's density satisfies the dynamics

$$d\rho_\pi + \partial_x\sigma(x,\rho_\pi)\circ dW = 0.$$

Our contribution

- Develop numerical methods for solving the SSCL
- Show that oscillations in z may lead to cancellations in the flow map.



Overview

1 Problem description

2 Deterministic conservation laws
- Characteristics and shocks
- Well-posedness

3 Stochastic scalar conservation laws
- Definition and well-posedness
- Numerical methods
- Flow map cancellations

4 Conclusion

The deterministic conservation law

The equation

$$u_t + \partial_x f(u) = 0 \quad \text{in } (0,\infty) \times \mathbb{R}$$
$$u(0,\cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$$

takes its name from the property

$$\frac{d}{dt}\int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_t dx = -\int_{\mathbb{R}} f(u)_x dx = 0.$$

Classical notion of weak solutions

$$\int_0^\infty \int_{\mathbb{R}} \phi_t u + f(u)\phi_x dxdt + \int_{\mathbb{R}} \phi(0,x)u_0(x)dx = 0, \quad \forall \phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}),$$

leads to existence, but not uniqueness, due to formation of shocks.

The deterministic conservation law

The equation

$$u_t + \partial_x f(u) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$u(0,\cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$$

takes its name from the property

$$\frac{d}{dt}\int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_t dx = -\int_{\mathbb{R}} f(u)_x dx = 0.$$

Classical notion of weak solutions

$$\int_0^\infty \int_{\mathbb{R}} \phi_t u + f(u)\phi_x dxdt + \int_{\mathbb{R}} \phi(0,x)u_0(x)dx = 0, \quad \forall \phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}),$$

leads to existence, but not uniqueness, due to formation of shocks.

Well-posedness

Definition 1 (Kruzkov's entropy condition)

$$\partial_t \eta(u) + \partial_x q(u) \leq 0, \quad \phi \in \mathcal{D}'_+(\mathbb{R} \times \mathbb{R}),$$

holds for all smooth and convex $\eta : \mathbb{R} \to \mathbb{R}$, and $q'(u) := f'(u)\eta'(u)$.

Theorem 2

Consider

$$u_t + \partial_x f(u) = 0, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}$$
$$u(0,x) = u_0.$$

Assume that $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$ *and* $f \in C^2(\mathbb{R};\mathbb{R})$*. Then there exists a unique solution* $u \in C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R})$ *which satisfies the Kruzkov entropy condition. Moreover, for any* $t \geq 0$*,*

$$\|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1.$$

Well-posedness

Definition 3 (Kruzkov's entropy condition for $z \in C^1$)

$$\partial_t \eta(u) + \dot{z} \partial_x q(u) \leq 0, \quad \phi \in \mathcal{D}'_+(\mathbb{R} \times \mathbb{R}),$$

holds for all smooth and convex $\eta : \mathbb{R} \to \mathbb{R}$, and $q'(u) := f'(u)\eta'(u)$.

Theorem 4

Consider

$$u_t + \dot{z} f(u)_x = 0, \quad \textit{in } \mathbb{R}_+ \times \mathbb{R}$$
$$u(0,x) = u_0.$$

Assume that $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$, z piecewise C^1 and $f \in C^2(\mathbb{R};\mathbb{R})$. Then there exists a unique solution $u \in C(\mathbb{R}_+; L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R})$ which satisfies the Kruzkov entropy condition. Moreover, for any $t \geq 0$,

$$\|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1.$$

Overview

1 Problem description

2 Deterministic conservation laws
- Characteristics and shocks
- Well-posedness

3 Stochastic scalar conservation laws
- Definition and well-posedness
- Numerical methods
- Flow map cancellations

4 Conclusion

Definition

The problem formulation

$$du + \partial_x f(u) \circ dz = 0, \quad \text{in} \quad (0, T] \times \mathbb{R},$$
$$u(0, \cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}).$$

Kruzkov's entropy condition

$$d\eta(u) + \partial_x q(u) \circ dz \leq 0, \quad \text{in } \mathcal{D}'_+(\mathbb{R} \times \mathbb{R})$$

is difficult to work with: If w is a standard Wiener process, then

$$\partial_x q(u) \circ dw = \ldots + \eta''(u)(f'(u))^2 (u_x)^2 dt.$$

Kinetic formulation

Consider the kinetic formulation instead

$$d\chi + f'(\xi)\chi_x \circ dz = \partial_\xi m \,\mathrm{d}t \quad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_\xi)$$

for some non-negative, bounded measure $m(t,x,\xi)$ and the constraint

$$\chi(t,x,\xi) = \chi(\xi; u(t,x)) := \begin{cases} 1 & \text{if } 0 \le \xi \le u(t,x) \\ -1 & \text{if } u(t,x) \le \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Formal motivation for equivalence:

$$\chi_t(\xi; u(t,x)) + f'(\xi)\chi_x(\xi; u(t,x)) \circ dz = \delta(u=\xi)(u_t + f'(u)u_x \circ dz).$$

And L^1 isometry:

$$\int_{\mathbb{R}} \chi(\xi; u(t,x))d\xi = u(t,x) \implies \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \chi(\xi; u) - \chi(\xi; v) d\xi \right| dx = \int_{\mathbb{R}} |u-v| dx.$$

Kinetic formulation

Consider the kinetic formulation instead

$$d\chi + f'(\xi)\chi_x \circ dz = \partial_\xi m \, \mathrm{d}t \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_\xi)$$

for some non-negative, bounded measure $m(t, x, \xi)$ and the constraint

$$\chi(t, x, \xi) = \chi(\xi; u(t, x)) := \begin{cases} 1 & \text{if } 0 \le \xi \le u(t, x) \\ -1 & \text{if } u(t, x) \le \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Formal motivation for equivalence:

$$\chi_t(\xi; u(t, x)) + f'(\xi)\chi_x(\xi; u(t, x)) \circ dz = \delta(u = \xi)(u_t + f'(u)u_x \circ dz).$$

And L^1 isometry:

$$\int_{\mathbb{R}} \chi(\xi; u(t, x)) d\xi = u(t, x) \implies \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \chi(\xi; u) - \chi(\xi; v) d\xi \right| dx = \int_{\mathbb{R}} |u - v| dx.$$

Kinetic formulation

Consider the kinetic formulation instead

$$d\chi + f'(\xi)\chi_x \circ dz = \partial_\xi m \, \mathrm{d}t \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_\xi)$$

for some non-negative, bounded measure $m(t, x, \xi)$ and the constraint

$$\chi(t, x, \xi) = \chi(\xi; u(t, x)) := \begin{cases} 1 & \text{if } 0 \leq \xi \leq u(t, x) \\ -1 & \text{if } u(t, x) \leq \xi < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Formal motivation for equivalence:

$$\chi_t(\xi; u(t, x)) + f'(\xi)\chi_x(\xi; u(t, x)) \circ dz = \delta(u = \xi)(u_t + f'(u)u_x \circ dz).$$

And L^1 isometry:

$$\int_{\mathbb{R}} \chi(\xi; u(t, x)) d\xi = u(t, x) \implies \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \chi(\xi; u) - \chi(\xi; v) d\xi \right| dx = \int_{\mathbb{R}} |u - v| dx.$$

Notion of solution

The term $f'(\xi)\chi_x \circ dz$ is difficult to treat, even as distribution.

Workaround: introduce $\rho^0 \in \mathcal{D}(\mathbb{R})$ and

$$\rho(t, x, \xi; y) := \rho^0(y - x + f'(\xi)z(t)),$$

Then, if $z \in C^1([0, T])$,

$$d\rho + f'(\xi)\rho_x \circ dz = 0, \text{ in } (0, T] \times \mathbb{R}_x \times \mathbb{R}_\xi.$$

Notion of solution

The term $f'(\xi)\chi_x \circ dz$ is difficult to treat, even as distribution.

Workaround: introduce $\rho^0 \in \mathcal{D}(\mathbb{R})$ and

$$\rho(t,x,\xi;y) := \rho^0(y - x + f'(\xi)z(t)),$$

Then, if $z \in C^1([0,T])$,

$$d\rho + f'(\xi)\rho_x \circ dz = 0, \text{ in } (0,T] \times \mathbb{R}_x \times \mathbb{R}_\xi.$$

Consequently,

$$d(\rho\chi) + f'(\xi)(\rho\chi)_x \circ dz = \chi \underbrace{\left(d\rho + f'(\xi)\rho_x \circ dz\right)}_{=0} + \rho \underbrace{\left(d\chi + f'(\xi)\chi_x \circ dz\right)}_{=m_\xi\, dt} = \rho m_\xi dt.$$

Notion of solution

The term $f'(\xi)\chi_x \circ dz$ is difficult to treat, even as distribution.

Workaround: introduce $\rho^0 \in \mathcal{D}(\mathbb{R})$ and

$$\rho(t, x, \xi; y) := \rho^0(y - x + f'(\xi)z(t)),$$

Then, if $z \in C^1([0, T])$,

$$d\rho + f'(\xi)\rho_x \circ dz = 0, \text{ in } (0, T] \times \mathbb{R}_x \times \mathbb{R}_\xi.$$

Consequently,

$$\int_{\mathbb{R}} d(\rho\chi) + f'(\xi)(\rho\chi)_x \circ dz dx = \int_{\mathbb{R}} \rho m_\xi dt dx.$$

Notion of solution

The term $f'(\xi)\chi_x \circ dz$ is difficult to treat, even as distribution.

Workaround: introduce $\rho^0 \in \mathcal{D}(\mathbb{R})$ and

$$\rho(t,x,\xi;y) := \rho^0(y - x + f'(\xi)z(t)),$$

Then, if $z \in C^1([0,T])$,

$$d\rho + f'(\xi)\rho_x \circ dz = 0, \text{ in } (0,T] \times \mathbb{R}_x \times \mathbb{R}_\xi.$$

Leads to condition

$$\frac{d}{dt}\int_{\mathbb{R}} \rho\chi dx = \int_{\mathbb{R}} \rho m_\xi dx, \quad \text{in } \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_\xi).$$

Notion of solution

The term $f'(\xi)\chi_x \circ dz$ is difficult to treat, even as distribution.

Workaround: introduce $\rho^0 \in \mathcal{D}(\mathbb{R})$ and

$$\rho(t,x,\xi;y) := \rho^0(y - x + f'(\xi)z(t)),$$

Then, if $z \in C^1([0,T])$,

$$d\rho + f'(\xi)\rho_x \circ dz = 0, \text{ in } (0,T] \times \mathbb{R}_x \times \mathbb{R}_\xi.$$

Leads to condition

$$\frac{d}{dt}\int_{\mathbb{R}} \rho\chi dx = \int_{\mathbb{R}} \rho m_\xi dx \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}_\xi). \tag{1}$$

Definition 5 (Pathwise entropy solution (PES))

$u \in L^1 \cap L^\infty([0,T] \times \mathbb{R})$ is a PES if there exists a non-negative, bounded measure m such that equation (1) holds for all ρ, as defined above.

Well-posedness

Theorem 6 (Lions, Perthame, Souganidis, 2013)

Assume $f \in C^2(\mathbb{R};\mathbb{R})$, $z \in C([0,T];\mathbb{R})$ and $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$. Then, for all $T > 0$, there exists a unique PES $u \in C([0,T];L^1(\mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R})$. Furthermore, for two solutions u, v generated from the respective driving paths $z, \bar{z}$ and $u_0, v_0 \in BV(\mathbb{R})$,

$$\|u(t,\cdot) - v(t,\cdot)\|_1 \leq \|u_0 - v_0\|_1 + C\sqrt{\sup_{s\in(0,t)} |z(s) - \bar{z}(s)|},$$

for a uniform constant $C(u_0, v_0, f, f', f'') > 0$.

Note that if z^n is a piecewise linear interpolation of $z \in C^{0,\alpha}$ using interpolation points with $z^n(t_k) = z(t_k)$ and $u^n := u(\cdot,\cdot;z^n)$, then

$$\|u(t,\cdot) - u^n(t,\cdot)\|_1 = O(n^{-\alpha/2}).$$

Well-posedness

Theorem 6 (Lions, Perthame, Souganidis, 2013)

Assume $f \in C^2(\mathbb{R};\mathbb{R})$, $z \in C([0,T];\mathbb{R})$ *and* $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$. *Then, for all* $T > 0$, *there exists a unique PES* $u \in C([0,T];L^1(\mathbb{R})) \cap L^\infty([0,T]\times\mathbb{R})$. *Furthermore, for two solutions* u, v *generated from the respective driving paths* $z, \bar{z}$ *and* $u_0, v_0 \in BV(\mathbb{R})$,

$$\|u(t,\cdot) - v(t,\cdot)\|_1 \leq \|u_0 - v_0\|_1 + C\sqrt{\sup_{s\in(0,t)} |z(s) - \bar{z}(s)|},$$

for a uniform constant $C(u_0, v_0, f, f', f'') > 0$.

Note that if z^n is a piecewise linear interpolation of $z \in C^{0,\alpha}$ using interpolation points with $z^n(t_k) = z(t_k)$ and $u^n := u(\cdot,\cdot;z^n)$, then

$$\|u(t,\cdot) - u^n(t,\cdot)\|_1 = O(n^{-\alpha/2}).$$

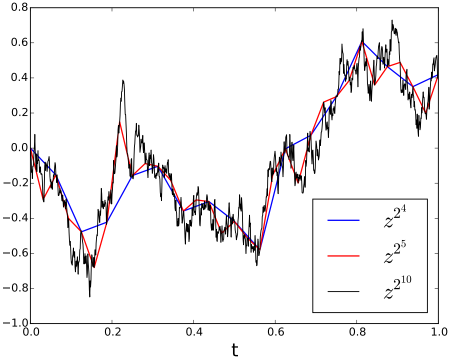
Numerical solution approach

(i) Approximate the rough path z by a piecewise linear interpolation

$$z^n(t) = \left(1 - \frac{t-\tau_k}{\Delta\tau}\right) z(\tau_k) + \frac{t-\tau_k}{\Delta\tau} z(\tau_{k+1}), \quad t \in [\tau_k, \tau_{k+1}],$$

where $\tau_k = k\Delta\tau$ and $\Delta\tau = T/n$

(ii) Solve the conservation law with driving noise z^n using a standard numerical method in classical Kruzkov entropy sense.



Solution with approximated driving noise z^n

The problem to solve:

$$u^n_t + \dot{z}^n \partial_x f(u^n) = 0, \quad \text{in } (0, T] \times \mathbb{R},$$
$$u^n(0, \cdot) = u_0.$$

Let $\mathcal{S}^{(\Delta\tau, \Delta z)} \tilde{u}$ denote the solution of

$$u_t + \frac{\Delta z}{\Delta \tau} \partial_x f(u) = 0, \quad \text{in } (0, \Delta\tau] \times \mathbb{R},$$
$$u(0, \cdot) = \tilde{u}.$$

Then

$$u^n(\tau_k) = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau, \Delta z_j)} u_0, \quad \text{for } k = 0, 1, \ldots, n,$$

where $\Delta z_j := z(\tau_{j+1}) - z(\tau_j)$.

Solution with approximated driving noise z^n

The problem to solve:

$$u^n_t + \dot{z}^n \partial_x f(u^n) = 0, \quad \text{in } (0, T] \times \mathbb{R},$$
$$u^n(0, \cdot) = u_0.$$

Let $\mathcal{S}^{(\Delta\tau, \Delta z)} \tilde{u}$ denote the solution of

$$u_t + \frac{\Delta z}{\Delta \tau} \partial_x f(u) = 0, \quad \text{in } (0, \Delta\tau] \times \mathbb{R},$$
$$u(0, \cdot) = \tilde{u}.$$

Then

$$u^n(\tau_k) = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau, \Delta z_j)} u_0, \quad \text{for } k = 0, 1, \ldots, n,$$

where $\Delta z_j := z(\tau_{j+1}) - z(\tau_j)$.

Numerical schemes

Solve iteratively $k = 0, 1, \ldots, n$

$$u_t^n + \frac{\Delta z_k}{\Delta \tau} \partial_x f(u^n) = 0, \quad \text{in } (\tau_k, \tau_{k+1}] \times \mathbb{R},$$

with $\Delta x = O(N^{-1})$ and time-steps $\Delta t_k = \Delta \tau / N_k$.

Numerical solution $\bar{u}_m^\ell := \bar{u}(t_\ell, x_m; z^n)$.

Solve, for instance, by Lax–Friedrichs (assuming $t_\ell \in (\tau_k, \tau_{k+1})$),

$$\bar{u}_m^{\ell+1} = \frac{\bar{u}_{m+1}^\ell + \bar{u}_{m-1}^\ell}{2} - \Delta t_k \frac{\Delta z_k}{\Delta \tau} \frac{f(\bar{u}_{m+1}^\ell) - f(\bar{u}_{m-1}^\ell)}{2\Delta x}, \text{ over } \ell, m, k. \qquad (2)$$

Solve iteratively $k = 0, 1, \dots, n$

$$u_t^n + \frac{\Delta z_k}{\Delta \tau} \partial_x f(u^n) = 0, \quad \text{in } (\tau_k, \tau_{k+1}] \times \mathbb{R},$$

with $\Delta x = O(N^{-1})$ and time-steps $\Delta t_k = \Delta \tau / N_k$.

Numerical solution $\bar{u}_m^\ell := \bar{u}(t_\ell, x_m; z^n)$.

Solve, for instance, by Lax–Friedrichs

$$\bar{u}_m^{\ell+1} = \frac{\bar{u}_{m+1}^\ell + \bar{u}_{m-1}^\ell}{2} - \frac{\Delta z_k}{N_k} \frac{f(\bar{u}_{m+1}^\ell) - f(\bar{u}_{m-1}^\ell)}{2\Delta x}, \text{ over } \ell, m, k. \tag{2}$$

Numerical schemes

Solve iteratively $k = 0, 1, \ldots, n$

$$u_t^n + \frac{\Delta z_k}{\Delta \tau} \partial_x f(u^n) = 0, \quad \text{in } (\tau_k, \tau_{k+1}] \times \mathbb{R},$$

with $\Delta x = O(N^{-1})$ and time-steps $\Delta t_k = \Delta \tau / N_k$.

Numerical solution $\bar{u}_m^\ell := \bar{u}(t_\ell, x_m; z^n)$.

Solve, for instance, by Lax–Friedrichs

$$\bar{u}_m^{\ell+1} = \frac{\bar{u}_{m+1}^\ell + \bar{u}_{m-1}^\ell}{2} - \frac{\Delta z_k}{N_k} \frac{f(\bar{u}_{m+1}^\ell) - f(\bar{u}_{m-1}^\ell)}{2\Delta x}, \text{ over } \ell, m, k. \tag{2}$$

With initial data

$$\bar{u}_m^0 = \frac{1}{\Delta x} \int_{x_{m-1/2}}^{x_{m+1/2}} u_0(x) dx.$$

Numerical schemes

Solve iteratively $k = 0, 1, \ldots, n$

$$u_t^n + \frac{\Delta z_k}{\Delta \tau} \partial_x f(u^n) = 0, \quad \text{in } (\tau_k, \tau_{k+1}] \times \mathbb{R},$$

with $\Delta x = O(N^{-1})$ and time-steps $\Delta t_k = \Delta \tau / N_k$.

Numerical solution $\bar{u}_m^\ell := \bar{u}(t_\ell, x_m; z^n)$.

Solve, for instance, by Lax–Friedrichs

$$\bar{u}_m^{\ell+1} = \frac{\bar{u}_{m+1}^\ell + \bar{u}_{m-1}^\ell}{2} - \frac{\Delta z_k}{N_k} \frac{f(\bar{u}_{m+1}^\ell) - f(\bar{u}_{m-1}^\ell)}{2\Delta x}, \text{ over } \ell, m, k. \tag{2}$$

$$\text{CFL: } |\dot{z}_k^n| \|f'\|_{L^\infty(-\|u_0\|_\infty, \|u_0\|_\infty)} \frac{\Delta t_k}{\Delta x} \le 1 \implies N_k = O\left(\frac{|\Delta z_k|}{\Delta x}\right) = O(n^{-\alpha} N^{-1})$$

So $\Delta t_k = \Delta \tau / N_k = O(n^{\alpha - 1} N^{-1})$ for all k.

Convergence rates

If $u_0 \in (L^1 \cap BV)(\mathbb{R})$ and $f \in C^2(\mathbb{R};\mathbb{R})$,

$$\begin{aligned} \|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 &\leq \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n}|\Delta z_k| \\ &= O(N^{-1}) + O(N^{-1/2}n^{1-\alpha}). \end{aligned}$$

Recall further

$$\|u(T,\cdot) - u^n(T,\cdot)\|_1 \leq C\sqrt{\sup_{s\in[0,T]}|z(s) - z^n(s)|} = O(n^{-\alpha/2}).$$

Hence,

$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_1 = O(N^{-1/2}n^{1-\alpha} + n^{-\alpha/2}). \tag{3}$$

Balance error contributions:

$$N(n) = O(n^{2-\alpha}).$$

If u_0 has compact support, the cost of achieving $O(\epsilon)$ error in (3) $O(\epsilon^{-(2/\alpha)(5-3\alpha)})$!

Which is $O(\epsilon^{-14})$ for $z \in C^{0,1/2}([0,T])$.

Convergence rates

If $u_0 \in (L^1 \cap BV)(\mathbb{R})$ and $f \in C^2(\mathbb{R};\mathbb{R})$,

$$\begin{aligned}\|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 &\leq \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n}|\Delta z_k| \\ &= O(N^{-1}) + O(N^{-1/2}n^{1-\alpha}).\end{aligned}$$

Recall further

$$\|u(T,\cdot) - u^n(T,\cdot)\|_1 \leq C\sqrt{\sup_{s\in[0,T]}|z(s) - z^n(s)|} = O(n^{-\alpha/2}).$$

Hence,

$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_1 = O(N^{-1/2}n^{1-\alpha} + n^{-\alpha/2}). \tag{3}$$

Balance error contributions:

$$N(n) = O(n^{2-\alpha}).$$

If u_0 has compact support, the cost of achieving $O(\epsilon)$ error in (3)
$O(\epsilon^{-(2/\alpha)(5-3\alpha)})$!
Which is $O(\epsilon^{-14})$ for $z \in C^{0,1/2}([0,T])$.

Convergence rates

If $u_0 \in (L^1 \cap BV)(\mathbb{R})$ and $f \in C^2(\mathbb{R};\mathbb{R})$,

$$\begin{aligned}\|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 &\leq \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n} |\Delta z_k| \\ &= O(N^{-1}) + O(N^{-1/2} n^{1-\alpha}).\end{aligned}$$

Recall further

$$\|u(T,\cdot) - u^n(T,\cdot)\|_1 \leq C\sqrt{\sup_{s\in[0,T]} |z(s) - z^n(s)|} = O(n^{-\alpha/2}).$$

Hence,

$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_1 = O(N^{-1/2} n^{1-\alpha} + n^{-\alpha/2}). \tag{3}$$

Balance error contributions:

$$N(n) = O(n^{2-\alpha}).$$

If u_0 has compact support, the cost of achieving $O(\epsilon)$ error in (3) $O(\epsilon^{-(2/\alpha)(5-3\alpha)})$!

Which is $O(\epsilon^{-14})$ for $z \in C^{0,1/2}([0,T])$.

Convergence rates

If $u_0 \in (L^1 \cap BV)(\mathbb{R})$ and $f \in C^2(\mathbb{R};\mathbb{R})$,

$$\begin{aligned}\|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 &\le \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n} |\Delta z_k| \\ &= O(N^{-1}) + O(N^{-1/2} n^{1-\alpha}).\end{aligned}$$

Recall further

$$\|u(T,\cdot) - u^n(T,\cdot)\|_1 \le C\sqrt{\sup_{s\in[0,T]} |z(s) - z^n(s)|} = O(n^{-\alpha/2}).$$

Hence,

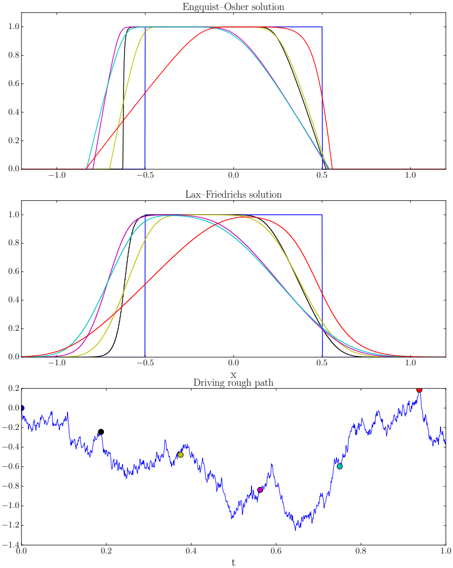
$$\|u(T,\cdot) - \bar{u}(T,\cdot)\|_1 = O(N^{-1/2} n^{1-\alpha} + n^{-\alpha/2}). \tag{3}$$

Balance error contributions:

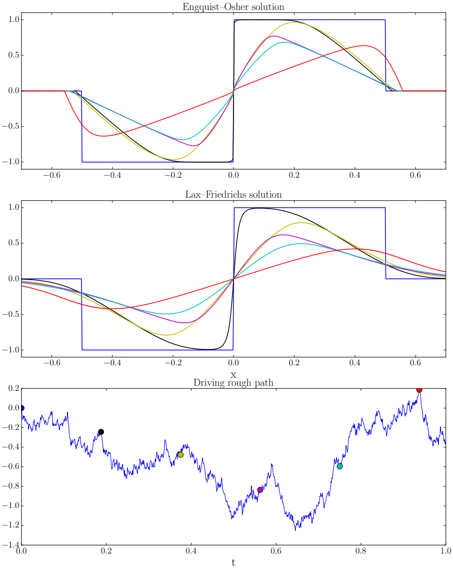
$$N(n) = O(n^{2-\alpha}).$$

If u_0 has compact support, the cost of achieving $O(\epsilon)$ error in (3) $O(\epsilon^{-(2/\alpha)(5-3\alpha)})$!
Which is $O(\epsilon^{-14})$ for $z \in C^{0,1/2}([0,T])$.

Numerical example with $u_0 = \mathbf{1}_{|x|<0.5}$ and $f(u) = u^2/2$.



Numerical example with $u_0 = \text{sign}(x)\mathbf{1}_{|x|<0.5}$ and $f(u) = u^2/2$.



Flow map cancellations

Recall that $\mathcal{S}^{(\Delta\tau,\Delta z)}\tilde{u}$ denotes the solution of

$$u_t + \frac{\Delta z}{\Delta \tau}\partial_x f(u) = 0, \quad \text{in } (0, \Delta\tau] \times \mathbb{R},$$
$$u(0,\cdot) = \tilde{u}.$$

and

$$u^n(\tau_k) = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau,\Delta z_j)} u_0, \quad \text{for } k = 0, 1, \ldots, n,$$

where $\Delta z_j := z(\tau_{j+1}) - z(\tau_j)$.
Provided $u^n(s,\cdot) \in C(\mathbb{R})$ for all $s \in (\tau_\ell, \tau_k)$, then

$$u^n(\tau_k) = \prod_{j=\ell}^{k-1} \mathcal{S}^{(\Delta\tau,\Delta z_j)} u^n(\tau_\ell) = \mathcal{S}^{\left((k-\ell)\Delta\tau, \sum_{j=\ell}^{k-1} \Delta z_j\right)} u^n(\tau_\ell)$$

Benefit $|z(\tau_k) - z(\tau_\ell)|$ replaces $\sum_{j=\ell}^{k-1} |\Delta z_j|$ in the numerical error bound, CFL ...

Flow map cancellations

Recall that $\mathcal{S}^{(\Delta\tau,\Delta z)}\tilde{u}$ denotes the solution of

$$u_t + \frac{\Delta z}{\Delta \tau}\partial_x f(u) = 0, \quad \text{in } (0,\Delta\tau]\times\mathbb{R},$$
$$u(0,\cdot) = \tilde{u}.$$

and

$$u^n(\tau_k) = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau,\Delta z_j)} u_0, \quad \text{for } k = 0,1,\ldots,n,$$

where $\Delta z_j := z(\tau_{j+1}) - z(\tau_j)$.
Provided $u^n(s,\cdot) \in C(\mathbb{R})$ for all $s \in (\tau_\ell, \tau_k)$, then

$$u^n(\tau_k) = \prod_{j=\ell}^{k-1} \mathcal{S}^{(\Delta\tau,\Delta z_j)} u^n(\tau_\ell) = \mathcal{S}^{\left((k-\ell)\Delta\tau, \sum_{j=\ell}^{k-1}\Delta z_j\right)} u^n(\tau_\ell)$$

Benefit $|z(\tau_k) - z(\tau_\ell)|$ replaces $\sum_{j=\ell}^{k-1} |\Delta z_j|$ in the numerical error bound, CFL ...

Local continuity of solutions

One-sided estimates deterministic setting ($z(t) = t$): If $f'' \geq \alpha > 0$ then

$$\frac{u(x+h,t) - u(x,t)}{h} \leq \frac{1}{\alpha t} \quad \forall h > 0, \text{ and } t > 0$$

and if $f'' \leq -\alpha < 0$

$$-\frac{1}{\alpha t} \leq \frac{u(x+h,t) - u(x,t)}{h} \quad \forall h > 0 \text{ and } t > 0.$$

Local continuity of solutions

One-sided estimates deterministic setting ($z(t) = t$): If $f'' \geq \alpha > 0$ then

$$\frac{u(x+h,t) - u(x,t)}{h} \leq \frac{1}{\alpha t} \quad \forall h > 0, \text{ and } t > 0$$

and if $f'' \leq -\alpha < 0$

$$-\frac{1}{\alpha t} \leq \frac{u(x+h,t) - u(x,t)}{h} \quad \forall h > 0 \text{ and } t > 0.$$

One-sided estimates: If $f'' \geq \alpha > 0$ and $\dot{z}^n > 0$ for all $t \in (a,b)$, then

$$\frac{u^n(x+h,t) - u^n(x,t)}{h} \leq \frac{1}{\alpha(z^n(t) - z^n(a))} \quad \forall h > 0 \text{ and } t \in (a,b),$$

and if $\dot{z}^n < 0$ for $t \in (b,c)$

$$\frac{1}{\alpha(z^n(t) - z^n(b))} \leq \frac{u^n(x+h,t) - u^n(x,t)}{h} \quad \forall h > 0 \text{ and } t \in (b,c).$$

Theorem 7 (Flow map product sum property)

Consider Burgers' equation, $f(u) = u^2/2$. Let

$$M^+(t) := \max_{s\in[0,t]} z^n(s), \quad M^-(t) := \min_{s\in[0,t]} z^n(s).$$

For all t and all intervals s.t.: $t \in (a,b) \subseteq [0,T]$ for which $M^-(a) < z^n(t) < M^+(a)$, we have that

$$u^n(s,\cdot) \in C(\mathbb{R}), \quad \forall s \in (a,b),$$

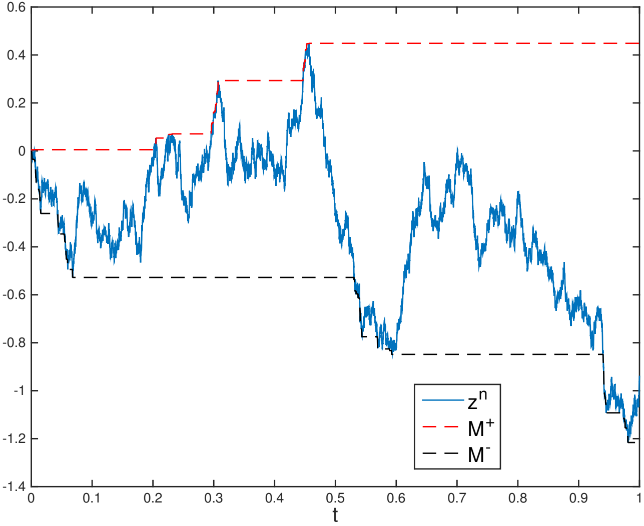
and

$$u^n(b) = \mathcal{S}^{b-a, z^n(b)-z^n(a)} u^n(a).$$

Secondly, whenever $\Delta z_k \Delta z_{k+1} > 0$, then

$$\mathcal{S}^{(\Delta\tau,\Delta z_2)}\mathcal{S}^{(\Delta\tau,\Delta z_1)} = \mathcal{S}^{(2\Delta\tau,\Delta z_1+\Delta z_2)}.$$

Running min and max



An equivalent integration path

Theorem 8 (Oscillating running max/min (ORM) function)

For

$$y^n(t) := \begin{cases} M^+(t)1_{s^+(t)\geq s^-(t)} + M^-(t)1_{s^-(t)\geq s^+(t)} & \text{if } t \in (0,T) \\ z(T) & \text{if } t = T \end{cases}$$

with $s^+(t) = \max\{s \leq t | z^n(s) = M^+(s)\}$ *and*
$s^-(t) = \max\{s \leq t | z^n(s) = M^-(s)\}$.

Then, for Burgers' equation,

$$\prod_{j=0}^{n-1} \mathcal{S}^{(\Delta\tau, \Delta z_j)} = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau, \Delta y_j^n)}.$$

(Note that $\mathcal{S}^{(\Delta\tau, 0)} = I$.*)*

An equivalent integration path

Theorem 8 (Oscillating running max/min (ORM) function)

For

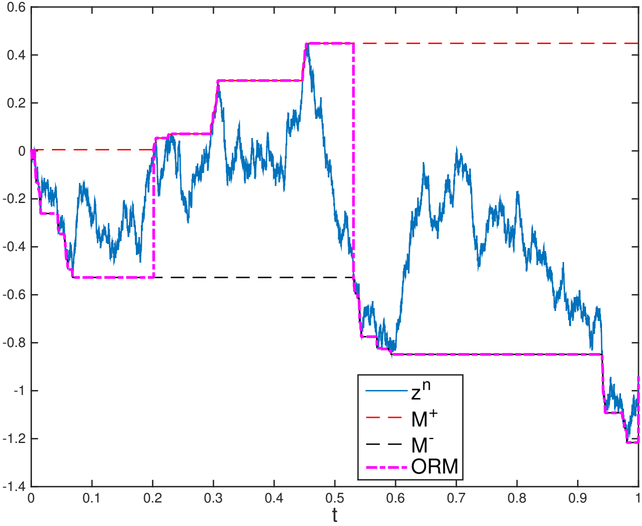
$$y^n(t) := \begin{cases} M^+(t)1_{s^+(t)\geq s^-(t)} + M^-(t)1_{s^-(t)\geq s^+(t)} & \text{if } t \in (0,T) \\ z(T) & \text{if } t = T \end{cases}$$

with $s^+(t) = \max\{s \leq t | z^n(s) = M^+(s)\}$ *and* $s^-(t) = \max\{s \leq t | z^n(s) = M^-(s)\}$.
Then, for Burgers' equation,

$$\prod_{j=0}^{n-1} \mathcal{S}^{(\Delta\tau, \Delta z_j)} = \prod_{j=0}^{k-1} \mathcal{S}^{(\Delta\tau, \Delta y_j^n)}.$$

(Note that $\mathcal{S}^{(\Delta\tau,0)} = I$.*)*

The ORM function



Numerical errors

Numerical integration "along" the ORM yields

$$\|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 \leq \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n} |\Delta y_k^n|$$
$$= O(N^{-1/2}|y^n|_{BV(0,T)}),$$

where $\Delta x = O(N^{-1})$.
Recall that integrating "along" z^n yields $O(N^{-1/2}|z^n|_{BV(0,t)})$ num error bound.
Efficiency to be gained provided

$$\frac{|y^n|_{BV(0,T)}}{|z^n|_{BV(0,T)}} = o(1),$$

since, respectively

$$N(n) = O(|z^n|^2_{BV(0,T)} n^\alpha), O(|y^n|^2_{BV(0,T)} n^\alpha)$$

and

$$\text{Cost}(\bar{u}(T)) = O(N(n)n).$$

Numerical errors

Numerical integration "along" the ORM yields

$$\|\bar{u}(T,\cdot) - u^n(T,\cdot)\|_1 \leq \|\bar{u}_0 - u_0^n\|_1 + C\sqrt{\Delta x}\sum_{k=0}^{n} |\Delta y_k^n|$$
$$= O(N^{-1/2}|y^n|_{BV(0,T)}),$$

where $\Delta x = O(N^{-1})$.
Recall that integrating "along" z^n yields $O(N^{-1/2}|z^n|_{BV(0,t)})$ num error bound.
Efficiency to be gained provided

$$\frac{|y^n|_{BV(0,T)}}{|z^n|_{BV(0,T)}} = o(1),$$

since, respectively

$$N(n) = O(|z^n|^2_{BV(0,T)}n^\alpha), O(|y^n|^2_{BV(0,T)}n^\alpha)$$

and

$$\text{Cost}(\bar{u}(T)) = O(N(n)n).$$

Bounded variation of ORM

Theorem 9 (Bounded variation of Wiener path ORM)

For standard Wiener paths $w : [0, T] \to \mathbb{R}$*, the ORM function* $y^n(\cdot) : [0, T] \to \mathbb{R}$ *associated to* w^n *fulfils*

$$y^n \in BV([0, T]) \quad \forall n > 0 \qquad \textit{almost surely},$$

and

$$\mathrm{E}\left[|y^n|_{BV[0,1]}\right] < \infty, \quad \forall n > 0.$$

The above also holds for the ORM y *of* w*.*

Implication: Cost of achieving $O(\epsilon)$ approximation error is improved by this sharper bound from $O(\epsilon^{-14})$ to $O(\epsilon^{-4})$ for Burgers' equation.

Bounded variation of ORM

Theorem 9 (Bounded variation of Wiener path ORM)

For standard Wiener paths $w : [0, T] \to \mathbb{R}$, *the ORM function* $y^n(\cdot) : [0, T] \to \mathbb{R}$ *associated to* w^n *fulfils*

$$y^n \in BV([0, T]) \quad \forall n > 0 \qquad \textit{almost surely},$$

and

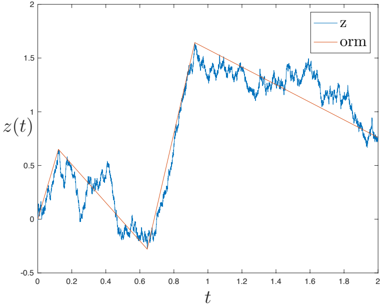
$$\mathrm{E}\left[|y^n|_{BV[0,1]}\right] < \infty, \quad \forall n > 0.$$

The above also holds for the ORM y *of* w.

Implication: Cost of achieving $O(\epsilon)$ approximation error is improved by this sharper bound from $O(\epsilon^{-14})$ to $O(\epsilon^{-4})$ for Burgers' equation.

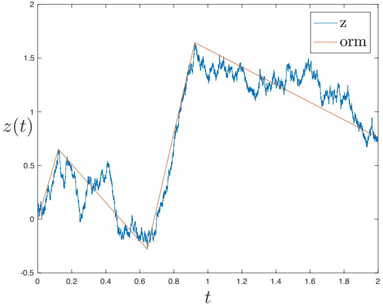
Example

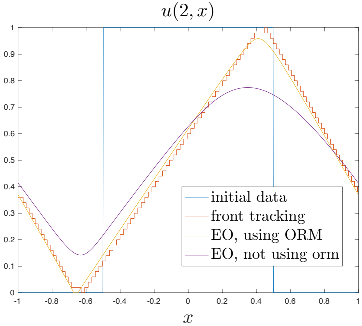
$$u_t + \frac{1}{2}\left(u^2\right)_x \circ dz = 0, \qquad u_0(x) = 1_{|x|<1/2}, \qquad t \in [0, 2]$$



Example

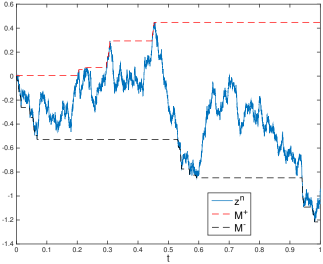
$$u_t + \frac{1}{2}\left(u^2\right)_x \circ dz = 0, \qquad u_0(x) = 1_{|x|<1/2}, \qquad t \in [0,2]$$





Regularity of solutions

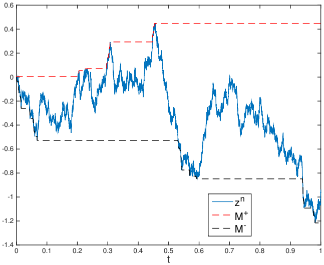
- Does the driving noise z have a regularizing effect on the solution?
- For Burgers', $u^n(t)$ can only be discontinuous at times when $z^n(t) = M^+(t)$ and/or $z^n(t) = M^-(t)$:



- For Wiener processes $\{s \in [0, T] | w(s) = M^+(s) \text{ and/or } w(s) = M^-(s)\}$ has Lebesgue measue 0.
- But, not (presently) clear if regularity behavior of u^n extends to the limit solution.

Regularity of solutions

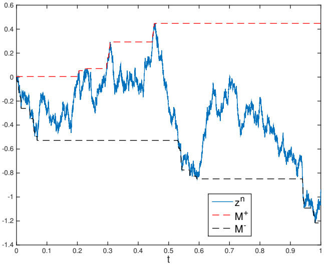
- Does the driving noise z have a regularizing effect on the solution?
- For Burgers', $u^n(t)$ can only be discontinuous at times when $z^n(t) = M^+(t)$ and/or $z^n(t) = M^-(t)$:



- For Wiener processes $\{s \in [0, T] | w(s) = M^+(s) \text{ and/or } w(s) = M^-(s)\}$ has Lebesgue measue 0.
- But, not (presently) clear if regularity behavior of u^n extends to the limit solution.

Regularity of solutions

- Does the driving noise z have a regularizing effect on the solution?
- For Burgers', $u^n(t)$ can only be discontinuous at times when $z^n(t) = M^+(t)$ and/or $z^n(t) = M^-(t)$:



- For Wiener processes $\{s \in [0, T] | w(s) = M^+(s) \text{ and/or } w(s) = M^-(s)\}$ has Lebesgue measue 0.
- But, not (presently) clear if regularity behavior of u^n extends to the limit solution.

Overview

1 Problem description

2 Deterministic conservation laws
- Characteristics and shocks
- Well-posedness

3 Stochastic scalar conservation laws
- Definition and well-posedness
- Numerical methods
- Flow map cancellations

4 Conclusion

Summary

- Developed a numerical method for solving stochastic scalar conservation laws.
- Identified cancellations of oscillations that in some settings lead to sharper error bounds and more efficient numerical algorithms.
- Future challenge: Develop numerics for higher dimensional version

$$du + \sum_{i=1}^{d} \partial_{x_i} f(x, u) \circ dz^i = 0, \quad \text{in} \quad (0, T] \times \mathbb{R}^d,$$
$$u(0, \cdot) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d).$$

References

1. P-L Lions, B Perthame, P E Souganidis *Scalar conservation laws with rough (stochastic) fluxes* .
Stoch. Partial Differ. Equ. Anal. Comput. 1 (2013), no. 4, 664-686.
2. B Gess, P E Souganidis *Long-Time Behavior, invariant measures and regularizing effects for stochastic scalar conservation laws.*
Communications on Pure and Applied Mathematics (2016).
3. P-L Lions, B Perthame, P Souganidis *Scalar conservation laws with rough (stochastic) fluxes: the spatially dependent case.*
Stochastic Partial Differential Equations: Analysis and Computations 2.4 (2014): 517-538.

Thank you for your attention!