Splash singularity for a free-boundary incompressible viscoelastic fluid model

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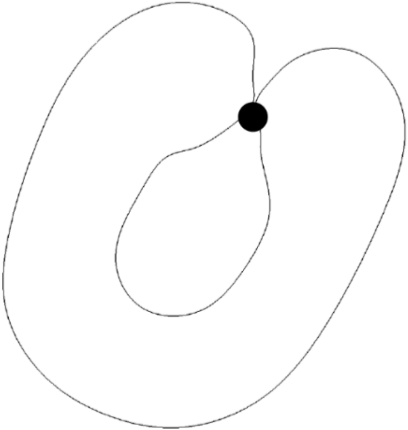
Presentation Outline

Splash Singularity in 2-D



Splash Singularity

This definition appeared in 2011 in a paper on 2-D Water Waves of

Angel Castro, Diego Cordoba, Charles Fefferman, Francisco Gancedo, Javier Gómez-Serrano

They exhibit smooth initial data for the 2D water wave equation for which the smoothness of the interface breaks down in finite time.
Moreover, by a stability result together with numerics they found solutions that starting from a graph, turn over and collapse in a **splash singularity** (**self intersecting curve in one point**) in finite time.

(Highly Incomplete) references

- **Water Waves:** A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, M. Gomez-Serrano (2011-12)
- **Incompressible Euler:** A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, M. Gomez-Serrano (2013 Ann.Math); D. Coutand S. Shkoller (2014); D. Cordoba, A. Enciso, N. Grubic (2014)
- **Incompressible Navier Stokes:** A. Castro, D. Cordoba, C. Fefferman, F. Gancedo,M. Gomez-Serrano (2015)
- **Impossibility for Vortex sheet:** D. Coutand S. Shkoller, (2014) arXiv:1407.1479.
- **No splash singularities for two-fluid interfaces** C. Fefferman, A.D. Ionescu, V. Lie (2013), arXiv:1312.2917

Linear Viscoleastic and Maxwell Material

Viscoelastic behavior has elastic and viscous components modeled as linear combinations of springs and dashpots.

Purely Viscous materials respond to a tangential stress with behavior consistent with Newton's law (tangential force equal to the product of the shear rate and the viscosity)

Purely Elastic materials respond to a normal stress manifesting a coherent behavior with Hooke's law

Let τ, ϵ, η E stress, strain, viscosity and Young modulus

$$\tau_{elastic} = E\epsilon, \quad \tau_{viscous} = \eta \frac{d\epsilon}{dt}$$

Maxwell model for Viscoelastic materials

$$\frac{d\epsilon}{dt} = \frac{d\epsilon_{viscous}}{dt} + \frac{d\epsilon_{elastic}}{dt} = \frac{\tau}{\eta} + \frac{1}{E}\frac{d\tau}{dt}$$

Upper Convective Maxwell and Oldroyd-B model

For large deformations consider the convective derivative and the streching terms
Let $\tau, \mathbf{v}, \lambda, \eta_0, \mathbf{D}$ stress tensor, fluid velocity, relaxation time, the material viscosity, the deformation rate (the rate of strain) tensor.

Upper - Convective Time Derivative

$$\partial_t^{uc}\tau = \partial_t\tau + (\mathbf{v}\cdot\nabla)\tau - (\nabla\mathbf{v})^T\tau + \tau(\nabla\mathbf{v})$$

Upper Convective Maxwell model

$$\tau + \lambda\partial_t^{uc}\tau = 2\eta_0\mathbf{D}, \quad 2\mathbf{D} = (\nabla\mathbf{v}) + (\nabla\mathbf{v})^T$$

Oldroyd-B model

$$\tau + \lambda\partial_t^{uc}\tau = 2\eta_0(\mathbf{D} + \lambda_s\mathbf{D}) \quad \lambda_s = \frac{\eta_{solv}}{\eta_0}\lambda$$

Modelling of polymeric fluids

Viscoelastic fluids, like all fluids, are governed by the momentum equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \operatorname{div} \tau,$$

$$\operatorname{div} u = 0$$

- $u(X,t)$ Eulerian velocity,
- $p(X,t)$ pressure,
- $\tau(X,t)$ stress tensor.
  - Newtonian viscous fluids: $\tau = \nu(\nabla u + \nabla u^T)$,
  - Polymeric fluids: $\tau = \nu(\nabla u + \nabla u^T) + \tau_p$.

Equation for the extra-stress τ_p (Oldroyd-B)

The polymers extra-stress τ_p satisfies

$$\partial_t^{uc}\tau_p = -\frac{1}{\lambda}\tau_p + \frac{\nu_p}{\lambda}(\nabla u + \nabla u^T),$$

- ν_p polymeric viscosity
- λ relaxation time
- $\dot{\gamma}$ is the shear rate $\frac{\text{velocity}}{\text{distance}} \approx t^{-1}$

Weissenberg number $\quad We \approx \frac{\text{viscous forces}}{\text{elastic forces}} = \frac{\nu\lambda}{E\epsilon} = \lambda\dot{\gamma},.$

- $We \ll 1 \Rightarrow \tau_p \approx \nu_p(\nabla u + \nabla u^T)$
- $We > 1 \Rightarrow$ formation of geometrical singularities.

when $We \gg 1$ (we approximate with the limit $We \to \infty$)

$$\partial_t\tau_p + (u\cdot\nabla)\tau_p - (\nabla u)\tau_p - \tau_p(\nabla u)^T = 0.$$

Lemma

Let $F(\alpha, t) = \dfrac{\partial X}{\partial \alpha}$ *the deformation tensor, then we have (in Eulerian coordinates)*

$$\partial_t F + u \cdot \nabla F = \nabla u F.$$

Let the initial condition $\tau(\alpha, 0) = \tau_0(\alpha)$ *be* **positive definite**,

$$\tau(\alpha, t) = F \tau_0 F^T.$$

is **positive definite** *too and* τ *satisfies*

$$\partial_t \tau_p + (u \cdot \nabla)\tau_p - (\nabla u)\tau_p - \tau_p (\nabla u)^T = 0.$$

Viscoelastic High Weissenberg number system (no free boundary)

The viscoelastic fluid system for high Weissenberg number is

$$\begin{cases} \partial_t F + u \cdot \nabla F = \nabla u F \\ \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = \operatorname{div}(F F^T) \text{ in } \mathbb{R}^3 \times (0, T) \\ \operatorname{div} u = 0, \ \operatorname{div} F = 0 \end{cases}$$

- Fang-Hua Lin, Chun Liu & Ping Zhang, *On Hydrodynamics of Viscoelastic Fluids* CPAM 2005.
- Li Xu, Ping Zhang & Zhifei Zhang, *Global solvability of a free boundary three-dimensional incompressible Viscoelastic Fluid System with surface tension* ARMA 2013.

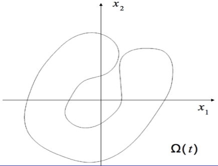
Free boundary problem for the High Weissenberg number system

$$\begin{cases} \partial_t F + u \cdot \nabla F = \nabla u F & \text{in } \Omega(t) \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \operatorname{div}(FF^T) & \\ \operatorname{div} u = 0, \ \operatorname{div} F = 0 & \\ (-p\mathcal{I} + (\nabla u + \nabla u^T) + (FF^T - \mathcal{I}))n = 0 & \text{on } \partial\Omega(t) \\ u(t)_{|t=0} = u_0, \ F(t)_{|t=0} = F_0 & \text{in } \Omega_0. \end{cases}$$

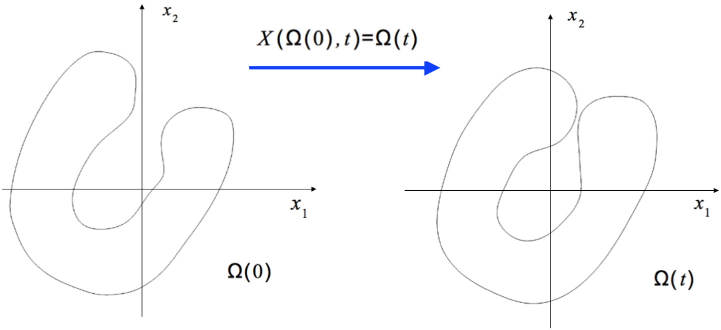
The boundary condition states the equilibrium of the force fields acting on the interface.

Equation for the flux

$$\begin{cases} \dot{X}(\alpha, t) = u(X(\alpha, t), t) \\ X(\alpha, 0) = \alpha, \quad \alpha \in \Omega_0 \end{cases}$$



Evolution of the domain



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Arc-Cord condition and Splash Curves

Definition (Arc-Cord condition)

$z : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ smooth simple closed curve **Arc-Cord condition** if there exists $K > 0$
$|z(\alpha) - z(\alpha')| \geq K\text{dist}(\alpha, \alpha')$ for $\alpha, \alpha' \in \mathbb{R}/2\pi\mathbb{Z}$.

Definition (splash curve)

1. $z_1(\alpha), z_2(\alpha)$ are smooth and 2π-periodic.
2. $z(\alpha)$ satisfies the **Arc-Cord condition** at every point but α_1 and α_2, with $\alpha_1 < \alpha_2$, $z(\alpha_1) = z(\alpha_2)$ and $\frac{dz(\alpha_i)}{d\alpha} \neq 0$.
3. $z(\alpha)$ **separates** the complex plane into a connected fluid region and a vacuum region (not necessarily connected). The normal vector $n = \frac{(-\partial_\alpha z_2(\alpha), \partial_\alpha z_1(\alpha))}{|\partial_\alpha z(\alpha)|}$ points to the vacuum region. The interface is part of the fluid region.

Conformal map

Consider a **conformal map P** from the complex plane to an half plane (e.g. $P(z) = \sqrt{z}$)

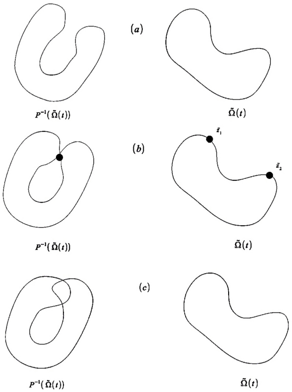
Conformal image of a splash curve

Let z be a splash curve and a branch of the function P on the fluid region, the curve $\tilde{z}(\alpha) = (\tilde{z}_1(\alpha), \tilde{z}_2(\alpha)) = P(z(\alpha))$ satisfies:

1. $\tilde{z}_1(\alpha)$ and $\tilde{z}_2(\alpha)$ are smooth and 2π-periodic.
2. $\tilde{z}$ is a closed contour.
3. $\tilde{z}$ satisfies the Arc-Cord condition.

Idea for proving the existence of splash singularity

Let P a conformal map $P : \Omega \to \tilde{\Omega}$, where $P(z)$, $z \in \mathbb{C}$, is a branch of $\sqrt{z}$



- The initial domain Ω_0 could be nonregular, for instance a splash domain (b), however mapping by P leads to a regular. $\tilde{\Omega}_0$
- Since $\{\tilde{\Omega}_0, \tilde{u}_0, \tilde{F}_0\}$ are regular in the conformal coordinates local existence of smooth solution
- The strategy is take initial domain $P^{-1}(\partial\tilde{\Omega}_0)$ already in splash perturb it and use stability

Choose suitable initial domain and perturb it to get existence

- Choose in a right way the initial velocity, s.t. $\tilde{u}_0(z_{1,s}) \cdot n > 0 \quad \tilde{u}_0(z_{2,s}) \cdot n > 0$, hence there exists $T > 0$, s.t. $P^{-1}(\partial\tilde{\Omega}(T))$ **self-intersects** (c).
- Take a one-parameter family $\{\tilde{\Omega}_\varepsilon(0), \tilde{u}_\varepsilon(0), \tilde{F}_\varepsilon(0)\}$, such that $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$, with $|b| = 1$, s.t. $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ is regular and there exists a local in time smooth solution

Domains local stability in time is needed

- (stability) let $\{\tilde{\Omega}_\varepsilon(t), \tilde{u}_\varepsilon(\cdot, t), \tilde{p}_\varepsilon(\cdot, t), \tilde{F}_\varepsilon(\cdot, t)\}$ the perturbed solution, then
$$\|\partial\tilde{\Omega}_\varepsilon(T) - \partial\tilde{\Omega}(T)\| \sim O(\varepsilon)$$
in a suitable norm, hence
$$P^{-1}(\partial\tilde{\Omega}_\varepsilon(T)) \sim P^{-1}(\partial\tilde{\Omega}(T))$$
and so $P^{-1}(\partial\tilde{\Omega}_\varepsilon(T))$ self-intersects.
- By continuity we have
  - for $t = 0$, $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is regular like (a)
  - for $t = T$, $P^{-1}(\tilde{\Omega}_\varepsilon(T))$ is self-intersecting like (c)

  $\Rightarrow$ define $t^* \in [0, T]$ in the following way
$$t^* = \inf\{t_\epsilon \in [0, T] : P^{-1}(\partial\tilde{\Omega}_\varepsilon(t)) \text{ is splash}\}.$$
- By stability $0 < t^* < T$ hence $(\Omega_\epsilon, u_\epsilon, F_\epsilon, p_\epsilon)$ exists on $[0, t^*]$ and form a splash singulatity in $t = t^*$

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Choice of the initial data $\{u_0, F_0\}$

The initial data $\{u_0, F_0\}$ must satisfy the compatibility condition

$$\mathbf{t}(\nabla u_0 + \nabla u_0^T)n = -\tau(F_0 F_0^T - \mathcal{I})n \qquad (1)$$

- Let $\mathcal{U}$ a neighborhood $\partial\Omega$ given by the parametrization $x(s,\lambda) = z(s) + \lambda z_s^{\perp}(s)$, where $z(s)$ is the parametrization $\partial\Omega$ and $|z_s(s)| = 1$.
- Construct a stream function on $\mathcal{U}$ as follows $\psi(x(s,\lambda)) = \bar{\psi}(s,\lambda) = \psi_0(s) + \lambda\psi_1(s) + \frac{1}{2}\lambda^2\psi_2(s)$.
- Define $u_0 = \nabla^{\perp}\psi$, then $\operatorname{div} u_0 = 0$.

- Extend $\mathbf{t}, n$:

$$\begin{cases} T(s,\lambda) = x_s(s,\lambda) = z_s(s) + \lambda z_{ss}^{\perp}(s) = (1-\lambda k(s)) z_s(s) \\ N(s,\lambda) = x_\lambda = z_s^{\perp}, \end{cases}$$

- By the LHS of (1) in $\lambda = 0$

$$\partial_s^2 \psi_0(s) - \psi_2(s) = T(F_0 F_0^T - \mathcal{I})N \qquad (2)$$

- $u_0 \cdot n = \partial_s \psi_0(s)$ independent from F_0 (depends only on ψ), then for all F_0 we can choose $\psi_0(s)$ and $\psi_2(s)$ by (2).
- Moreover, we can choose $u_0 \cdot n > 0$.

Existence of splash singularity for the free boundary problem for the Navier-Stokes equations

- For the local existence we use a technique introduced by T. Beale in *The initial value problem for the Navier-Stokes equations with a free surface*, CPAM 1981.
- The analysis of the self intersection via conformal maps is a somehow very classical idea resumed recently (for Navier Stokes) by A.Castro, D. Cordoba, C.Fefferman, F.Gancedo J. Gomez-Serrano, arXiv: 1504.02775
- Other Methods on Navier Stokes D. Coutand, S. Shkoller, arXiv: 1505.01929v1.

Conformal Transformation

We define

- A conformal map $P : \Omega \to \tilde{\Omega}$, where $P(z)$, $z \in \mathbb{C}$, as a **branch of** $\sqrt{z}$,
- the conformal velocity $\tilde{u}(\tilde{X}(\alpha, t), t) = u(P^{-1}(\tilde{X}(\alpha, t), t) \Rightarrow u(X(\alpha, t), t) = \tilde{u}(P(X(\alpha, t), t))$,
- the conformal deformation tensor is $\tilde{F}(\tilde{X}(\alpha, t), t) = F(P^{-1}(\tilde{X}(\alpha, t), t) \Rightarrow F(X(\alpha, t), t) = \tilde{F}(P(X(\alpha, t), t))$,
- for the derivatives we will use

$$(\partial_{X_j} u_i) \circ P^{-1} = A_{kj} \partial_{\tilde{X}_k} \tilde{u}_i,$$

where $A_{kj} = \partial_{X_j} P_k \circ P^{-1}$.

Conformally transformed system

The transformed system in $\tilde{\Omega}(t)$

$$\begin{cases} \partial_t \tilde{F}_{ij} + (A_{rk}\tilde{u}_k \partial_r)\tilde{F}_{ij} = \partial_r \tilde{u}_i A_{rk}\tilde{F}_{kj} \\ \partial_t \tilde{u}_i + (A_{rk}\tilde{u}_k \cdot \partial_r)\tilde{u}_i - Q^2 \tilde{\Delta}\tilde{u}_i + A_{ri}\partial_r \tilde{p} = (A_{rk}\tilde{F}_{kl}\partial_r)\tilde{F}_{il} \\ \mathrm{Tr}(\nabla \tilde{u} A) = 0 \\ (-\tilde{p}\mathcal{I} + (\nabla \tilde{u} A + (\nabla \tilde{u} A)^T) + (\tilde{F}\tilde{F}^T - \mathcal{I}))A^{-1}\tilde{n} = 0 \\ \tilde{u}_{|t=0}(t) = \tilde{u}_0, \ \tilde{F}_{|t=0}(t) = \tilde{F}_0. \end{cases}$$

where $Q^2 = \left|\frac{\partial P}{\partial z} \circ P^{-1}\right|^2$.
The transformed flux equation

$$\begin{cases} \dfrac{d}{dt}\tilde{X}(\alpha, t) = (A \circ \tilde{X})(\tilde{u} \circ \tilde{X}) & \text{in} \quad \tilde{\Omega}(t) \\ \tilde{X}(\alpha, 0) = \alpha & \text{in} \quad \tilde{\Omega}(0) \end{cases}$$

Lagrangian Transformation

To have a fixed boundary problem, we transform in Lagrangian coordinates:

$$\begin{cases} \tilde{v}(\alpha, t) = \tilde{u} \circ \tilde{X}(\alpha, t) \\ \tilde{q}(\alpha, t) = \tilde{p} \circ \tilde{X}(\alpha, t) \\ \tilde{G}(\alpha, t) = \tilde{F} \circ \tilde{X}(\alpha, t). \end{cases}$$

and differentiating

$$\partial_{\tilde{X}_j} \tilde{u}_i = \tilde{\zeta}_{lj} \partial_l \tilde{v}_i,$$

where $\tilde{\zeta}_{lj}$ is the lj-th element of $(\nabla_\alpha \tilde{X})^{-1}$ and $\partial_l = \partial_{\alpha_l}$.

Lagrangian system in the conformal domain

The Conformal Lagrangian system is

$$
\begin{cases}
\partial_t \tilde{G}_{ij} = A_{kj} \circ \tilde{X} \tilde{\zeta}_{sr} \partial_s \tilde{v}_i \tilde{G}_{kj} \\
\partial_t \tilde{v}_i - Q^2 \circ \tilde{X} \tilde{\zeta}_{sr} \partial_s (\tilde{\zeta}_{jr} \partial_j \tilde{v}_i) + A_{ri} \circ \tilde{X} \tilde{\zeta}_{sr} \partial_s \tilde{q} = \\
\qquad = A_{rk} \circ \tilde{X} \tilde{G}_{kl} \tilde{\zeta}_{sr} \partial_s \tilde{G}_{il} \\
\mathrm{Tr}(\nabla_{\tilde{\alpha}} \tilde{v} (\nabla_{\tilde{\alpha}} \tilde{X})^{-1} A \circ \tilde{X}) = 0 \\
(-\tilde{q}\mathcal{I} + ((\nabla_{\alpha} \tilde{v} (\nabla_{\alpha} \tilde{X})^{-1} A \circ \tilde{X}) + (\nabla_{\alpha} \tilde{v} (\nabla_{\alpha} \tilde{X})^{-1} A \circ \tilde{X})^T + \\
+(\tilde{G}\tilde{G}^T - \mathcal{I}))A^{-1} \circ \tilde{X} \nabla_J \tilde{X} \tilde{n}_0 = 0 \\
\tilde{v}(\alpha, 0) = \tilde{v}_0(\alpha) = \tilde{u}_0(\alpha), \ \tilde{G}(\alpha, 0) = \tilde{G}_0(\alpha) = \tilde{F}_0(\alpha).
\end{cases}
$$

where $\nabla_J \tilde{X} = -J \nabla \tilde{X} J$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, this is due to the fact that $\tilde{n} = -JA_{|\partial\tilde{\Omega}(t)} Jn$.

Iteration scheme- fluid part

We observed that it is possible to separate the equation for $\tilde{v}$ from the equation for $\tilde{G}$.

$$\begin{cases} \partial_t \tilde{v}^{(n+1)} - Q^2 \Delta \tilde{v}^{(n+1)} + A^T \nabla \tilde{q}^{(n+1)} = \tilde{f}^{(n)} \\ \mathrm{Tr}(\nabla \tilde{v}^{(n+1)} A) = \tilde{g}^{(n)} \\ (-\tilde{q}^{(n+1)} \mathcal{I} + ((\nabla \tilde{v}^{(n+1)} A) + (\nabla \tilde{v}^{(n+1)} A)^T)) A^{-1} \tilde{n}_0) = \tilde{h}^{(n)} \\ \tilde{v}(\alpha, 0) = \tilde{v}_0. \end{cases}$$

where $\tilde{f}^{(n)}, \tilde{g}^{(n)}, \tilde{h}^{(n)}$ contain all the missing terms in the previous time step, for instance in $\tilde{f}^{(n)}$ and in $\tilde{h}^{(n)}$ there are $\tilde{G}^{(n)}$ terms.

Iteration for the Conformal Lagrangian deformation tensor

$$\begin{cases} \partial_t \tilde{G}_{ij}^{(n+1)} = A_{kj} \circ \tilde{X}^{(n)} \tilde{\zeta}_{sr} \partial_s \tilde{v}_i^{(n)} \tilde{G}_{kj}^{(n)} \\ \tilde{G}(\alpha, 0) = \tilde{G}_0. \end{cases}$$

This implies

$$\tilde{G}^{(n+1)}(\alpha, t) = \tilde{G}_0 + \int_0^t (A \circ \tilde{X}^{(n)} \tilde{\zeta}^{(n)} \nabla \tilde{v}^{(n)} \tilde{G}^{(n)})(\alpha, \tau)\, d\tau.$$

For the flux we get

$$\tilde{X}^{(n+1)}(\alpha, t) = \alpha + \int_0^t (A \circ \tilde{X}^{(n)} \tilde{v}^{(n)})(\alpha, \tau)\, d\tau.$$

Assuming the convergence as $n \to \infty$, in the limit we find the solution of the system.

Function spaces

These are the spaces where we will use for the estimates:

$$H^{ht,s}([0,T];\Omega) = L^2([0,T];H^s(\Omega)) \cap H^{\frac{s}{2}}([0,T];L^2(\Omega)),$$

$$H^{ht,s}_{pr}([0,T];\Omega) = \{q \in L^\infty([0,T];\dot{H}^1(\Omega)) :$$
$$\nabla q \in H^{ht,s-1}([0,T];\Omega), q \in H^{ht,s-\frac{1}{2}}([0,T];\partial\Omega)\},$$

$$\bar{H}^{ht,s}([0,T];\Omega) = L^2([0,T];H^s(\Omega)) \cap H^{\frac{s+1}{2}}([0,T];H^{-1}(\Omega)),$$

$$F^{s+1}([0,T];\Omega) = L^\infty_{\frac{1}{4}}([0,T];H^{s+1}(\Omega)) \cap H^2([0,T];H^\gamma(\Omega)),$$
$$\text{for } s-1-\varepsilon < \gamma < s-1,$$

$$F^{s}([0,T];\Omega) = L^\infty_{\frac{1}{4}}([0,T];H^{s}(\Omega)) \cap H^2([0,T];H^{\gamma-1}(\Omega)),$$
$$\text{for } s-2-\varepsilon < \gamma-1 < s-2.$$

$$\|f\|_{L^\infty_{\frac{1}{4}}} = \sup_{t\in[0,T]} t^{-\frac{1}{4}}|f(t)|$$

Idea for solving the linear system for $\tilde{v}$

This theory was developed by T. Beale. The idea is to start with the homogeneous system

$$\begin{cases} \partial_t v - Q^2 \Delta v + A^T \nabla q = f \\ \mathrm{Tr}(\nabla v A) = 0 \\ (-q\mathcal{I} + (\nabla v A) + (\nabla v A)^T) \dfrac{A^T}{Q^2} n = 0 \\ v(\alpha, 0) = v_0. \end{cases}$$

- Weak formulation of the problem
- Projection R on $H^0_\sigma = \{v \in H^1 : \mathrm{Tr}(\nabla v A) = 0\}$

- The main result

Theorem

Let $f \in H^{ht,s-1}$, $2 < s < 3$ such that $Rf(0) = 0$. Then

$$\|v\|_{H^{ht,s+1}} + \|\nabla q\|_{H^{ht,s-1}} + \|q\|_{H^{ht,s-\frac{1}{2}}} \leq C\|f\|_{H^{ht,s-1}},$$

with C indipendent on T.

Write the linear problem as $\partial_t u + S_A u = Rf$
Prove the thm by using the following results:

1. $\|S_A^{-1} f\|_{H^{s+1}} \leq C\|f\|_{H^{s-1}}$,
2. $\|(\lambda + S_A)^{-1} Rf\|_{H^{s+1}} \leq C\left(\|Rf\|_{H^{s-1}} + |\lambda|^{\frac{s-1}{2}}\|Rf\|_{L^2}\right)$
   where $1 \leq s \leq 3$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) \geq 0$.

Inhomogeneous linear problem

For the inhomogeneous problem

$$\begin{cases} \partial_t v - Q^2 \Delta v + A^T \nabla q = f \\ \mathrm{Tr}(\nabla v A) = g \\ (-q\mathcal{I} + (\nabla v A) + (\nabla v A)^T) A^{-1} n = h \\ v(\alpha, 0) = v_0 \end{cases}$$

Let's introduce the compatibility conditions for the initial data:

$$\begin{cases} \mathrm{Tr}(\nabla v_0 A) = g(0) \\ (A^{-1}n)^{\perp} (\nabla v_0 A + (\nabla v_0 A)^T) A^{-1} n = h(0) (A^{-1} n)^{\perp} \end{cases} \quad (3)$$

For this system we define spaces X space of solutions and Y space of data.

$$X := \left\{(v,q) : v \in H^{ht,s+1}, q \in H^{ht,s}_{pr}\right\}$$

$$Y := \{(f,g,h,v_0) : f \in H^{ht,s-1}, g \in \bar{H}^{ht,s},$$
$$h \in H^{ht,s-\frac{1}{2}}(\partial\Omega \times [0,T]), v_0 \in H^s(\Omega) \text{ and } (3) \text{ are satisfied}\}$$

Theorem

Let $2 < s < \frac{5}{2}$. Then $L : X \to Y$ has a bounded inverse:

$$\|(v,q)\|_X \leq C\|(f,g,h,v_0)\|_Y$$

In order to have the constant C independent of the time we define

$$X_0 := \{(v,q) \in X : v(0) = 0, \partial_t v(0) = 0, q(0) = 0\}$$
$$Y_0 := \{(f,g,h,0) \in Y : f(0) = 0, g(0) = 0, \partial_t g(0) = 0, h(0) = 0\}$$

Theorem

$L : X_0 \to Y_0$ is invertible for $2 < s < \frac{5}{2}$. Moreover, $\|L^{-1}\|$ is bounded uniformly if T is bounded above.

Navier Stokes iteration

In order to apply the previous Theorem, we take an approximation $\phi = \tilde{v}_0 + t\exp(-t^2)(Q^2\Delta\tilde{v}_0 - A^T\nabla\tilde{q}_\phi)$, a new function $\tilde{w}^{(n)} = \tilde{v}^{(n)} - \phi$ and the new system is

$$\begin{cases} \partial\tilde{w}^{(n+1)} - Q^2\Delta\tilde{w}^{(n+1)} + A^T\nabla\tilde{q}_w^{(n+1)} = \tilde{f}^{(n)} - \partial_t\phi \\ \qquad +Q^2\Delta\phi - A^T\nabla\tilde{q}_\phi \\ \mathrm{Tr}(\nabla\tilde{w}^{(n+1)}A) = \tilde{g}^{(n)} - \mathrm{Tr}(\nabla\phi A) \\ [-\tilde{q}_w^{(n+1)}\mathcal{I} + ((\nabla\tilde{w}^{(n+1)}A) + (\nabla\tilde{w}^{(n+1)}A)^T)]A^{-1}\tilde{n}_0 = \\ \qquad = \tilde{h}^{(n)} + \tilde{q}_\phi A^{-1}\tilde{n}_0 - ((\nabla\phi A) + (\nabla\phi A)^T)A^{-1}\tilde{n}_0 \\ \tilde{w}^{(n+1)}_{|t=0} = 0 \end{cases}$$

where $\tilde{f}^{(n)}, \tilde{g}^{(n)}, \tilde{h}^{(n)}$ contain all the missing terms.

Flux and deformation gradient iteration

The flux and the deformation gradient

$$\tilde{G}^{(n+1)}(\alpha,t) = \tilde{G}_0 + \int_0^t (A \circ \tilde{X}^{(n)} \tilde{\zeta}^{(n)} \nabla \tilde{w}^{(n)} \tilde{G}^{(n)})(\alpha,\tau)\, d\tau + \int_0^t (A \circ \tilde{X}^{(n)} \tilde{\zeta}^{(n)} \nabla \phi \tilde{G}^{(n)})(\alpha,\tau)\, d\tau,$$

$$\tilde{X}^{(n+1)}(\alpha,t) = \alpha + \int_0^t (A \circ \tilde{X}^{(n)} \tilde{w}^{(n)})(\alpha,\tau)\, d\tau + \int_0^t (A \circ \tilde{X}^{(n)} \phi)(\alpha,\tau)\, d\tau.$$

Results for the Navier-Stokes part

Theorem (Estimate for the flux (Part 1))

Let $\tilde{X}^{(n)} - \alpha \in F^{s+1}$, $\tilde{w}^{(n)} \in H^{ht,s+1}$ *and such that*

- $\tilde{X}^{(n)} - \alpha \in \{\tilde{X} - \alpha \in F^{s+1} : \|\tilde{X} - \alpha - \int_0^t A\phi \, d\tau\|_{F^{s+1}} \leq$
$$\leq \| \int_0^t A\phi \, d\tau\|_{F^{s+1}}\} \equiv B_{A\phi},$$
- $\|\tilde{w}^{(n)}\|_{H^{ht,s+1}} \leq N.$

Then, for small enough $T > 0$, *depending only on* $N, \tilde{v}_0$,
$$\tilde{X}^{(n+1)} - \alpha \in B_{A\phi}.$$

Theorem (Estimate for the flux (Part 2))

Let $\tilde{X}^{(n)} - \alpha, \tilde{X}^{(n-1)} \in F^{s+1}$, with $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in H^{ht,s+1}$ and such that

- $\left\| \tilde{w}^{(n)} \right\|_{H^{ht,s+1}} \leq M, \left\| \tilde{w}^{(n-1)} \right\|_{H^{ht,s+1}} \leq M,$
- $\left\| \tilde{X}^{(n)} - \alpha \right\|_{F^{s+1}} \leq M, \left\| \tilde{X}^{(n-1)} - \alpha \right\|_{F^{s+1}} \leq M$

for some $M > 0$. Then

$$\|\tilde{X}^{(n+1)} - \tilde{X}^{(n)}\|_{F^{s+1}} \leq CT^{\delta}(\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{F^{s+1}} + \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{H^{ht,s+1}})$$

For a small enough δ.

Estimates for $\tilde{G}$

Theorem (Estimates for $\tilde{G}$ (Part1))

Let $\tilde{G}^{(n)} - \tilde{G}_0 \in F^s$, $\tilde{X}^{(n)} - \alpha \in F^{s+1}$, and $\tilde{w}^{(n)} \in H^{ht,s+1}$ and such that

- $\tilde{G}^{(n)} - \tilde{G}_0 \in \{\tilde{G} - \tilde{G}_0 \in F^s : \|\tilde{G} - \tilde{G}_0 - \int_0^t A\nabla\phi\tilde{G}_0\, d\tau\|_{F^s} \leq$
$$\leq \|\int_0^t A\nabla\phi\tilde{G}_0\, d\tau\|_{F^s}\} \equiv B,$$
- $\|\tilde{w}^{(n)}\|_{H^{ht,s+1}} \leq N.$

Then, for $T > 0$ small enough , depending only $N, \tilde{v}_0, \tilde{G}_0$.

$$\tilde{G}^{(n+1)} - \tilde{G}_0 \in B.$$

Estimates for $\tilde{G}$ (Part 2)

Theorem

Let $\tilde{G}^{(n)} - \tilde{G}_0, \tilde{G}^{(n-1)} - \tilde{G}_0 \in F^s$, *with* $\tilde{X}^{(n)} - \alpha, \tilde{X}^{(n-1)} - \alpha \in F^{s+1}$ *and* $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in H^{ht,s+1}$ *and such that*

- $\|\tilde{w}^{(n)}\|_{H^{ht,s+1}} \le M, \|\tilde{w}^{(n-1)}\|_{H^{ht,s+1}} \le M,$
- $\|\tilde{X}^{(n)} - \alpha\|_{F^{s+1}} \le M, \|\tilde{X}^{(n-1)} - \alpha\|_{F^{s+1}} \le M$
- $\|\tilde{G}^{(n)} - \tilde{G}_0\|_{F^s} \le M, \|\tilde{G}^{(n-1)} - \tilde{G}_0\|_{F^s} \le M,$

for some $M > 0$. *Then*

$$\|\tilde{G}^{(n+1)} - \tilde{G}^{(n)}\|_{F^s} \le CT^\delta(\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{F^s} + \\ + \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{H^{ht,s+1}} + \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{F^{s+1}})$$

For a small enough δ.

Theorem (Estimates for $\tilde{v}, \tilde{q}$ (Part 1))

Let $\tilde{X}^{(n)} - \alpha \in F^{s+1}$, $\tilde{q}_w^{(n)} \in H_{pr}^{ht,s}$ *and* $\tilde{w}^{(n)} \in H^{ht,s+1}$, *and such that*

- $\|\tilde{X}^{(n)} - \alpha\|_{F^{s+1}} \leq N, \|\tilde{G}^{(n)} - \tilde{G}_0\|_{F^s} \leq N,$
- $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in \{(\tilde{w}, \tilde{q}) \in H^{ht,s+1} \times H_{pr}^{ht,s} : \tilde{w}_{|t=0} = 0,$
  $\partial_t \tilde{w}_{|t=0} = 0, \|(\tilde{w}, \tilde{q}) - L^{-1}(\tilde{f}_\phi, \tilde{g}_\phi, \tilde{h}_\phi)\|_{H^{ht,s+1} \times H_{pr}^{ht,s}} \leq$
  $\leq \|L^{-1}(\tilde{f}_\phi, \tilde{g}_\phi, \tilde{h}_\phi)\|_{H^{ht,s+1}} \times H_{pr}^{ht,s}\} \equiv B_{L^{-1}(\tilde{f}_\phi, \tilde{g}_\phi, \tilde{h}_\phi)}.$

Then

$$(\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B_{L^{-1}(\tilde{f}_\phi, \tilde{g}_\phi, \tilde{h}_\phi)}.$$

Theorem (Estimates for $\tilde{v}, \tilde{q}$ (Part 2))

Let $\tilde{X}^{(n)} - \alpha, \tilde{X}^{(n-1)} \in F^{s+1}$, $\tilde{G}^{(n)} - \tilde{G}_0, \tilde{G}^{(n-1)} - \tilde{G}_0 \in F^s$, $\tilde{w}^{(n)}, \tilde{w}^{(n-1)} \in H^{ht,s+1}$, *with* $\tilde{w}^{(n)}_{|t=0} = \tilde{w}^{(n-1)}_{|t=0} = 0$, $\partial_t \tilde{w}^{(n)}_{|t=0} = \partial_t \tilde{w}^{(n-1)}_{|t=0} = 0$, $\tilde{q}^{(n)}_w, \tilde{q}^{(n-1)}_w \in H^{ht,s}_{pr}$ *and such that all these function are bounded, in their norm, by a constant* $M > 0$. *Then*

$$\|\tilde{w}^{(n+1)} - \tilde{w}^{(n)}\|_{H^{ht,s+1}} + \|\tilde{q}^{(n+1)}_w - \tilde{q}^{(n)}_w\|_{H^{ht,s}_{pr}}$$
$$\leq CT^{\delta}(\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{F^{s+1}} + \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{H^{ht,s+1}} +$$
$$+ \|\tilde{q}^{(n)}_w - \tilde{q}^{(n-1)}_w\|_{H^{ht,s}_{pr}} + \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{F^s})$$

For a small enough δ.

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Local Existence Theorem

Putting together all the results above and applying the Contraction Mapping Principle we get

Theorem

For $2 < s < 2.5$, $1 < \gamma < s - 1$ and for T sufficiently small we have that $\{\tilde{w}^{(n)}\}_{n=0}^{\infty}$, $\{\tilde{q}_w^{(n)}\}_{n=0}^{\infty}$, $\{\tilde{G}^{(n)}\}_{n=0}^{\infty}$ and $\{\tilde{X}^{(n)}\}_{n=0}^{\infty}$ are Cauchy sequences respectively in $H^{ht,s+1}([0,T],\Omega_0)$, $H_{pr}^{ht,s}([0,T],\Omega_0)$, $F^s([0,T],\Omega_0)$ and $F^{s+1}([0,T],\Omega_0)$.

The limit of the sequence is the desired solution.

Stability estimates

We pick $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$, with $|b| = 1$, for instance $b = -e_2$, such that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is a regular domain. We take the difference between $(\tilde{w}, \tilde{q}, \tilde{X}, \tilde{G})$ and $(\tilde{w}_\varepsilon, \tilde{q}_\varepsilon, \tilde{X}_\varepsilon, \tilde{G}_\varepsilon)$ and we get

$$\begin{cases} \partial_t(\tilde{w} - \tilde{w}_\varepsilon) - Q^2\Delta(\tilde{w} - \tilde{w}_\varepsilon) + A^T\nabla(\tilde{q}_w - \tilde{q}_{w,\varepsilon}) = \tilde{F}_\varepsilon \\ \mathrm{Tr}(\nabla(\tilde{w} - \tilde{w}_\varepsilon)A) = \tilde{K}_\varepsilon \\ [-(\tilde{q}_w - \tilde{q}_{w,\varepsilon})\mathcal{I} + \nabla(\tilde{w} - \tilde{w}_\varepsilon)A + (\nabla(\tilde{w} - \tilde{w}_\varepsilon)A)^T]A^{-1}\tilde{n}_0 = \tilde{H}_\varepsilon \\ \tilde{w}_0 = 0, \end{cases}$$

where $\tilde{F}_\varepsilon, \tilde{K}_\varepsilon, \tilde{H}_\varepsilon$ contain all the other terms.

$$\begin{cases} \tilde{X} - \tilde{X}_\varepsilon = -b\varepsilon + \int_0^t \left(A \circ \tilde{X} \tilde{v} - A \circ \tilde{X}_\varepsilon \tilde{v}_\varepsilon \right) d\tau, \\ \tilde{G} - \tilde{G}_\varepsilon = -b\varepsilon + \int_0^t (A \circ \tilde{X} \tilde{\zeta} \nabla \tilde{v} \tilde{G} - A \circ \tilde{X}_\varepsilon \tilde{\zeta}_\varepsilon \nabla \tilde{v}_\varepsilon \tilde{G}_\varepsilon) \, d\tau. \end{cases}$$

For δ small enough and $2 < s < 2.5$, similar estimates as those for the local existence lead to the following results:

- $\|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}} \leq C\varepsilon +$
$+ CT^\delta (\|\tilde{w} - \tilde{w}_\varepsilon\|_{H^{ht,s+1}} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}}$
$+ \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma})$

$$\bullet \|\tilde w - \tilde w_\varepsilon\|_{H^{ht,s+1}} + \|\tilde q_w - \tilde q_{w,\varepsilon}\|_{H^{ht,s}_{pr}} \le C(\|\tilde F_\varepsilon\|_{H^{ht,s-1}} + \|\tilde K_\varepsilon\|_{\bar H^{ht,s}} + \|\tilde H_\varepsilon\|_{H^{ht,s-\frac{1}{2}}}),$$

$$\bullet \|\tilde F_\varepsilon\|_{H^{ht,s-1}} + \|\tilde K_\varepsilon\|_{\bar H^{ht,s}} + \|\tilde H_\varepsilon\|_{H^{ht,s-\frac{1}{2}}} \le$$
$$\le C\varepsilon + CT^\delta(\|\tilde w - \tilde w_\varepsilon\|_{H^{ht,s+1}} + \|\tilde q_w - \tilde q_{w,\varepsilon}\|_{H^{ht,s}_{pr}} + \|\tilde G - \tilde G_\varepsilon\|_{L^\infty H^s} + \|\tilde G - \tilde G_\varepsilon\|_{H^2 H^{\gamma-1}} + \|\tilde X - \tilde X_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde X - \tilde X_\varepsilon\|_{H^2 H^\gamma}),$$

$$\bullet \|\tilde X - \tilde X_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde X - \tilde X_\varepsilon\|_{H^2 H^\gamma} \le$$
$$\le C\varepsilon + CT^\delta(\|\tilde w - \tilde w_\varepsilon\|_{H^{ht,s+1}} + \|\tilde X - \tilde X_\varepsilon\|_{L^\infty H^{s+1}} + \|\tilde X - \tilde X_\varepsilon\|_{H^2 H^\gamma}).$$

Main result

Putting together these results we get that for $0 < T < \frac{1}{(3C)^{1/\delta}}$

$$\bullet \|\tilde{w} - \tilde{w}_\varepsilon\|_{H^{ht,s+1}} + \|\tilde{q}_w - \tilde{q}_{w,\varepsilon}\|_{H^{ht,s}_{pr}} + \|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}}$$

$$+ \|\tilde{X} - \tilde{X}_\varepsilon\|_{H^2 H^\gamma} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{L^\infty H^s} + \|\tilde{G} - \tilde{G}_\varepsilon\|_{H^2 H^{\gamma-1}} \leq 3C\varepsilon$$

$\Rightarrow$

$$\|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} \leq 3C\varepsilon, \tag{4}$$

this means that the two fluxes are close enough and so that the domains are close:

$$\tilde{X}(\tilde{\Omega}_0, t) \approx \tilde{X}_\varepsilon(\tilde{\Omega}_{0,\varepsilon}, t).$$

$\Rightarrow$ The domains are close enough so we can apply the idea of existence of splash singularity described in the Introduction.

Thank You !!!