The scalar wave equation on general asymptotically flat spacetimes: Stability and instability results

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- Decay in the presence of an evanescent ergosurface.
- Proof of Friedman's instability for spacetimes with an ergoregion and no event horizon.

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We will call $(\mathcal{M}, g)$ asymptotically flat if g approaches the Minkwoski metric η asymptotically, where

$$\eta = -dt^2 + dx^1 + \cdots + (dx^d)^2.$$

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The simplest example of an asymptotically flat spacetime: Minkowski spacetime $(\mathbb{R}^{d+1}, \eta)$. Wave equation:

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- Conservation of energy: For all $t \in \mathbb{R}$,

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- Pointwise decay estimates:

$$|\varphi| \leq C\big(1+|t-r|\big)^{-\frac{1}{2}} (1+t+r)^{-\frac{d-1}{2}} \Big(\sum_{j=1}^{\lceil \frac{d+1}{2} \rceil} \int_{\{t=0\}} r_+^{2j} |\nabla^j \varphi|^2 \, dx \Big)^{\frac{1}{2}}.$$

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- Valid on small radiating perturbations of $(\mathbb{R}^{d+1}, \eta)$

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- Quantitative decay estimates: Trapping enters the picture.

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What can be said for general $\mathcal{O}$ independently of the nature of trapping?

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Theorem (Burq, 1998)

Without any assumptions on the geometry of $\mathcal{O}$, we have:

$$\mathcal{E}_R[\varphi](t) \leq \frac{C}{\big(\log(2+t)\big)^{2m}} \mathcal{E}^{(m)}[\varphi](0).$$

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- C depends on m, R and the size of the initial support of φ.
- The result also holds for the wave equation $\Box_g \varphi = 0$ when $g = -dt^2 + \bar{g}$, with $\bar{g}$ being a compact perturbation of the Euclidean metric on $\mathbb{R}^d$.

Decay on general product spacetimes

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- $\mathcal{E}[\varphi](\tau) = \int_{\overline{\mathcal{M}}} \left(\left| \partial_t \varphi \right|^2 + \left| \bar{\nabla} \varphi \right|_{\bar{g}}^2 \right) d\bar{g}$ is conserved for $\Box_g \varphi = 0$.

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Theorem (Rodnianski–Tao, 2011)

On a general asymptotically conic Riemannian manifold $(\overline{\mathcal{M}}, \bar{g})$, the unique solution $u \in H^2(\mathcal{M})$ of $\Delta_{\bar{g}} u - (\omega + i\varepsilon)^2 u = F$ satisfies:

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- Consequence: Solutions of $\Box_g \varphi = 0$ on the product spacetime $(\mathbb{R} \times \overline{\mathcal{M}}, g = -dt^2 + \bar{g})$ satisfy

$$\mathcal{E}_{\leq R}[\varphi](t) \leq C_{m,R}\left(\log(2+t)\right)^{-2m} \mathcal{E}_w^{(m)}[\varphi](0).$$

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  - $d(g(V,V))|_{\mathcal{H}} = 0$: Degenerate (extremal) horizon, absence of red-shift leads to a mix of stability and instability mechanisms (Aretakis, Aretakis–Angelopoulos–Gajic).

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  - Superradiance for scalar waves acts as an obstacle to stability.

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Question: Do the decay results of Burq and Rodnianski–Tao extend to the case of general stationary and asymptotically flat spacetimes, possibly with a non-degenerate event horizon and a small ergoregion?

A decay result on general spacetimes with small ergoregion

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Theorem (M., 2015)

Let $(\mathcal{M}^{d+1}, g)$, $d \geq 3$, be a stationary and asymptotically flat spacetime, possibly possessing a non-degenerate event horizon $\mathcal{H}$ and a small ergoregion $\mathscr{E}$. Assume that all solutions φ to $\Box_g \varphi = 0$ satisfy

$$\mathcal{E}[\varphi](\tau) \leq C \mathcal{E}[\varphi](0).$$

Then,

$$\mathcal{E}_{\leq R}[\varphi](\tau) \leq C_{Rm\varepsilon} \big(\log(\tau + 2) \big)^{-2m} \mathcal{E}^{(m)}[\varphi](0) + C_{R\varepsilon} \tau^{-\varepsilon} \mathcal{E}_{\varepsilon}[\varphi](0),$$

where

$$\mathcal{E}^{(m)}[\varphi](0) = \sum_{j=0}^{m} \int_{\{t=0\}} \big| \nabla T^j \varphi \big|^2 \, dg_{\Sigma},$$

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- Pointwise estimates can also be obtained.

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- Assume that

$$\mathcal{E}_{\leq R}[\varphi_{\leq \omega_+}](t) \leq C_{R\varepsilon} t^{-\varepsilon}\Big(e^{C_R \omega_+}\mathcal{E}[\varphi](0) + \mathcal{E}_\varepsilon[\varphi](0)\Big).$$

  Then choosing $\omega_+ \sim \varepsilon C_R^{-1} \log t$:

$$\mathcal{E}_{\leq R}[\varphi](t) \leq C_{Rm\varepsilon}\big(\log(t+2)\big)^{-2m}\mathcal{E}^{(m)}[\varphi](0) + C_{R\varepsilon} t^{-\varepsilon}\mathcal{E}_\varepsilon[\varphi](0).$$

Sketch of the proof

The proof is based on seperating φ into frequency decomposed components. The error terms from the cut-off procedure are controlled by the energy boundedness assumption.

Let $\omega_+ \gg 1$. Splitting $\varphi = \varphi_{\leq \omega_+} + \varphi_{\geq \omega_+}$:

$$\mathcal{E}_{\leq R}[\varphi](t) \lesssim \mathcal{E}_{\leq R}[\varphi_{\leq \omega_+}](t) + \mathcal{E}_{\leq R}[\varphi_{\geq \omega_+}](t)$$

- Since $\varphi_{\geq \omega_+}$ has frequency support in $\{\omega \gtrsim \omega_+\}$:

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- Decay on hyperboloids: By using the r^p-weighted energy method of Dafermos–Rodnianski (Dafermos–Rodnianski, 2009; M., 2015).

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In order to obtain a polynomial decay estimate for $\varphi_{\leq \omega_+}$: It suffices to show:

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- For $k \geq 1$: Carleman-type estimates, using the fact that $\partial_t\varphi_k \sim i\omega_k\varphi_k$ (using ideas from Burq and Rodnianski–Tao).

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Remark. The energy boundedness assumption is used in a critical way in the proof of the Carleman estimates.

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$$\bar{\mathcal{E}}_{\leq R}[\varphi](\tau) \geq C_{m,R}\left(\frac{\log\log(\tau+2)}{\log(\tau+2)}\right)^{2m}\bar{\mathcal{E}}_{w}^{(m)}[\varphi](0),$$

for φ depending trivially on the compact directions.

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Question: What happens if $\mathcal{H} = \emptyset$ but $\mathscr{E} \neq \emptyset$?

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For any such solution and any $\tau \geq 0$ (Friedman, 1978):

$$\mathcal{E}_{\mathscr{E}}[\varphi](\tau) = \int_{\{t=\tau\}\cap\mathscr{E}} J^T_\mu(\varphi) n^\mu \leq -1.$$

Friedman's ergoregion instability

Conjecture (Friedman, 1978)

On such a spacetime $(\mathcal{M}, g)$, there exist solutions φ to $\Box_g \varphi = 0$ such that the non-degenerate energy flux of φ through $\{t = \tau\}$ blows up as $\tau \to +\infty$.

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- Rigorous proof?

Friedman's ergoregion instability

Theorem (M., 2016)

Suppose that $(\mathcal{M}^{d+1}, g)$, $d \geq 2$, is as above, satisfying in addition the following unique continuation condition:

UC condition: There exists a point $p \in \partial\mathscr{E}$ and an open neighborhood $\mathcal{U}$ of p in $\mathcal{M}$ such that, for any H^1_{loc} solution $\tilde{\psi}$ to $\square_g \tilde{\psi} = 0$ on $\mathcal{M}$ with $\tilde{\psi} \equiv 0$ on $\mathcal{M} \backslash \mathscr{E}$, we have $\tilde{\psi} = 0$ on $\mathscr{E} \cap \mathcal{U}$.

Then, there exists a smooth φ solving $\square_g \varphi = 0$ with compactly supported initial data , such that

$$\limsup_{\tau \to +\infty} \int_{\{t=\tau\}} |\nabla \varphi|^2 = +\infty.$$

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- Examples of spacetimes where the unique continuation condition holds:
  - Axisymmetric spacetimes with axisymmetric Killing field Φ, such that $[T, \Phi] = 0$ and the span of T, Φ is timelike on $\partial\mathscr{E}$
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- There exist spacetimes violating the unique continuation condition.

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  - Example (Cardoso–Crispino–Oliveira): The hydrodynamic vortex $(\mathbb{R} \times (\mathbb{R}^2 \backslash 0), g_{hyd})$:

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Using the methods of the logarithmic decay result, (1) implies that for any $\varepsilon > 0$, any R, $T, \tau_0 \gg 1$ and any $0 < \delta < 1$, there exists a $\tau_* \geq \tau_0$ such that:

$$\int_{\{\tau_* - T \leq t \leq \tau_* + T\} \cap \{r \leq R\} \setminus \mathscr{E}_\delta} \left(|\nabla \psi|^2 + |\psi|^2 \right) < \varepsilon. \qquad (2)$$

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- $\psi(t + \tau_n, x) \to \tilde{\psi}(t, x)$ and $T\psi(t + \tau_n, x) \to T\tilde{\psi}(t, x)$ weakly in $H^1_{loc}(\mathcal{M})$ and strongly in $L^2_{loc}(\mathcal{M})$, for a sequence $\tau_n \to +\infty$.
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Unique continuation condition $\Longrightarrow$ $\tilde{\psi} \equiv 0$ in $\mathcal{U}$

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So:

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Indefinite inner product associated to the T-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T,\tau} = \int_{\{t=\tau\}} \frac{1}{2} Re\Big\{ T\varphi_1 n\bar{\varphi}_2 + n\varphi_1 T\bar{\varphi}_2 - g(T,n)\partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \Big\}.$$

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(3) & (4): Contradiction!

Thank you for your attention!