The scalar wave equation on general asymptotically flat spacetimes: Stability and instability results

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Structure of the talk

Introduction:
\[ \psi = 0 \text{ on asymptotically flat backgrounds } (M, g) \]
and decay properties on \((\mathbb{R}^d + 1, \eta)\).

Decay in the exterior of a smooth compact obstacle \(O \subset \mathbb{R}^d\): A result of Burq.

Decay on product Lorentzian manifolds: A result of Rodnianski–Tao.

A decay result for general asymptotically flat black hole spacetimes with a small ergoregion.

Decay in the presence of an evanescent ergosurface.

Proof of Friedman's instability for spacetimes with an ergoregion and no event horizon.
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- Introduction: $\Box_g \psi = 0$ on asymptotically flat backgrounds $(\mathcal{M}, g)$ and decay properties on $(\mathbb{R}^{d+1}, \eta)$. 
  - Decay in the exterior of a smooth compact obstacle $O \subset \mathbb{R}^{d+1}$: A result of Burq.
  - Decay on product Lorentzian manifolds: A result of Rodnianski–Tao.
  - A decay result for general asymptotically flat black hole spacetimes with a small ergoregion.
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Introduction: The wave equation on asymptotically flat backgrounds

\[
\Box_g \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial_\nu \phi \right) = 0.
\]

Appears frequently in mathematical physics:

Fluid mechanics:

- \( g \) is the acoustical metric of a fluid in motion

General relativity:

- \( g \) is the spacetime metric of a 3 + 1 dimensional model of our universe.

We will only consider backgrounds \((M, g)\) which are globally hyperbolic.

The initial value problem with initial data on a Cauchy hypersurface \( \Sigma \) is well defined.

We will call \((M, g)\) asymptotically flat if \( g \) approaches the Minkowski metric \( \eta \) asymptotically, where

\[
\eta = -dt^2 + dx_1 + \cdots + (dx_d)^2.
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Scalar wave equation on \((\mathcal{M}^{d+1}, g)\):

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The wave equation on \((\mathbb{R}^{d+1}, \eta)\)

The simplest example of an asymptotically flat spacetime: Minkowski spacetime \((\mathbb{R}^{d+1}, \eta)\).

Wave equation:

\[ \Box \eta \phi = -\partial^2_t \phi + \partial^2 \phi_{x_1} + \ldots + \partial^2 \phi_{x_d} = 0. \]

Conservation of energy: For all \(t \in \mathbb{R}\),

\[ E[\phi](t) = \hat{\mathbb{R}}^{d} \mid \mid \nabla \phi(t, x) \mid \mid^2 \, dx = E[\phi](0). \]

Local energy decay:

\[ E \leq R[\phi](t) \leq C R(1 + t)^{-2} \hat{\{t=0\}} r^2 + \mid \mid \nabla \phi \mid \mid^2 \, dx. \]

Pointwise decay estimates:

\[ |\phi| \leq C \left(1 + \frac{|t - r|}{2} \right)^{-\frac{1}{2}} \left(1 + t + r \right)^{-\frac{d-1}{2}} \left(\left\lceil \frac{d+1}{2} \right\rceil \sum_{j=1}^{\hat{\{t=0\}}} r^2_j + |\nabla_j \phi| \right)^{\frac{1}{2}}. \]

Valid on small radiating perturbations of \((\mathbb{R}^{d+1}, \eta)\).
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\mathcal{E}_{\leq R}[\varphi](t) \leq C_R (1 + t)^{-2} \int_{\{t=0\}} r_+^2 |\nabla \varphi|^2 \, dx.
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- Pointwise decay estimates:

$$|\varphi| \leq C (1 + |t - r|)^{-\frac{1}{2}} (1 + t + r)^{-\frac{d-1}{2}} \left( \sum_{j=1}^{\left\lfloor \frac{d+1}{2} \right\rfloor} \int_{\{t=0\}} r_{+}^{2j} |\nabla^{j} \varphi|^{2} \, dx \right)^{\frac{1}{2}}.$$
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- Valid on small radiating perturbations of \((\mathbb{R}^{d+1}, \eta)\)
The exterior of an obstacle $\mathcal{O}$ in $\mathbb{R}^d$
Let $\mathcal{O}$ be a compact open subset of $\mathbb{R}^d$ with smooth boundary $\partial \mathcal{O}$. Equation $\Box \eta \varphi = 0$ on $\mathcal{M} = \mathbb{R} \times (\mathbb{R}^d \setminus \mathcal{O})$ with Dirichlet or Neumann boundary conditions on $\partial \mathcal{O}$ has been extensively studied in the last 50 years.
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- Conservation of the energy

$$E[\varphi](t) = \int_{\mathbb{R}^d \setminus \mathcal{O}} |\nabla \varphi(t, x)|^2 \, dx,$$

yields boundedness estimates for $\varphi$ and its derivatives, as well as decay without a rate.
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- Quantitative decay estimates: Trapping enters the picture.
The exterior of an obstacle $\mathcal{O}$ in $\mathbb{R}^d$

In the absence of trapping: Morawetz, Ralston and Strauss (1977) showed that

$$\hat{\|E\|} \leq R \left[ \phi(t) \right] \leq C R \left[ \phi(0) \right].$$

In the presence of trapping: no quantitative energy decay estimate possible without loss of derivatives (Ralston, 1969).

Generalisation to trapped null geodesics in Lorentzian manifolds: Sbierski, 2013.

What can be said for general $\mathcal{O}$ independently of the nature of trapping?
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What can be said for general $\mathcal{O}$ independently of the nature of trapping?
A result of Burq for general $\mathcal{O}$. Without any assumptions on the geometry of $\mathcal{O}$, we have:

$$E_{\mathbb{R}}[\phi(t)] \leq C(\log(2 + t))^{2m} E[\phi(0)].$$

$C$ depends on $m$, $R$, and the size of the initial support of $\phi$. The result also holds for the wave equation $\Box g\phi = 0$ when $g = -dt^2 + \bar{g}$, with $\bar{g}$ being a compact perturbation of the Euclidean metric on $\mathbb{R}^d$. 
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Decay on general product spacetimes

Simple non-trivial examples of asymptotically flat spacetimes: Product spacetimes $(\mathbb{R} \times M, -dt^2 + \bar{g})$, where $(M, \bar{g})$ is a Riemannian manifold.

$E[\phi](\tau) = \int_M (|\partial_t \phi|^2 + |\bar{\nabla} \phi|^2 \bar{g}) \, d\bar{g}$ is conserved for $\Box_g \phi = 0$.

Trapped null geodesics act as an obstruction to decay.

Can the result of Burq be generalised in this setting?

Theorem (Rodnianski–Tao, 2011)

On a general asymptotically conic Riemannian manifold $(M, \bar{g})$, the unique solution $u \in H^2(M)$ of $\Delta \bar{g} u - (\omega + i \epsilon)^2 u = F$ satisfies:

$$\hat{M} \frac{-1 - \eta}{m} + \left( |\nabla u|^2 + \omega^2 |u|^2 \right) \, d\bar{g} \leq C \epsilon \frac{1 + \eta}{m} |\omega| \hat{M} \frac{1}{m} |F|^2 \, d\bar{g}.$$
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\int_{\mathcal{M}} r_+^{-1-\eta} (|\nabla u|^2 + \omega^2 |u|^2) \, d\bar{g} \leq C e^{C|\omega|} \int_{\mathcal{M}} r_+^{1+\eta} |F|^2 \, d\bar{g}.
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\]

- Consequence: Solutions of \(\Box_g \varphi = 0\) on the product spacetime \((\mathbb{R} \times \overline{M}, g = -dt^2 + \bar{g})\) satisfy

\[
\mathcal{E}_{\leq R}[\varphi](t) \leq C_{m,R} \left( \log(2 + t) \right)^{-2m} \mathcal{E}_w^{(m)}[\varphi](0).
\]
Going beyond product spacetimes

In the general class of stationary and asymptotically flat spacetimes \((\mathcal{M}, g)\), one encounters geometric features which are absent in the case of product spacetimes.

Event horizon \(\mathcal{H}\) (black hole exterior spacetime). In many interesting cases, \(\mathcal{H}\) is also a Killing horizon, with Killing generator \(V\).

\[
\left. \frac{\partial}{\partial t} \right|_{\mathcal{H}} \neq 0: \text{Non-degenerate horizon, red-shift effect acts as a decay mechanism for scalar waves (Dafermos–Rodnianski)}.
\]

\[
\left. \frac{\partial}{\partial t} \right|_{\mathcal{H}} = 0: \text{Degenerate (extremal) horizon, absence of red-shift leads to a mix of stability and instability mechanisms (Aretakis, Aretakis–Angelopoulos–Gajic)}.
\]

Ergoregion: \(\mathcal{E} = \{ p \in \mathcal{M} : \frac{(T_p T_p)}{g(T_p T_p)} > 0 \} \neq \emptyset\), where \(T\) is the stationary Killing field.

Superradiance for scalar waves acts as an obstacle to stability.
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- **Event horizon** \( \mathcal{H} \) (black hole exterior spacetime). In many interesting cases, \( \mathcal{H} \) is also a **Killing** horizon, with Killing generator \( V \).
  - \( d(g(V, V))|_{\mathcal{H}} \neq 0 \): Non-degenerate horizon, red-shift effect acts as a decay mechanism for scalar waves (Dafermos–Rodnianski).
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- **Ergoregion**:

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- **Event horizon** \(\mathcal{H}\) (black hole exterior spacetime). In many interesting cases, \(\mathcal{H}\) is also a *Killing* horizon, with Killing generator \(V\).
  - \(d(g(V, V))|_{\mathcal{H}} \neq 0\): Non-degenerate horizon, red-shift effect acts as a decay mechanism for scalar waves (Dafermos–Rodnianski).
  - \(d(g(V, V))|_{\mathcal{H}} = 0\): Degenerate (extremal) horizon, absence of red-shift leads to a mix of stability and instability mechanisms (Aretakis, Aretakis–Angelopoulos–Gajic).

- **Ergoregion**:

  \[
  \mathcal{E} \doteq \left\{ p \in \mathcal{M} : g(T_p, T_p) > 0 \right\} \neq \emptyset.
  \]

  where \(T\) is the stationary Killing field.

  - Superradiance for scalar waves acts as an obstacle to stability.
Going beyond product spacetimes

Question:
Do the decay results of Burq and Rodnianski–Tao extend to the case of general stationary and asymptotically flat spacetimes, possibly with a non-degenerate event horizon and a small ergoregion?
Going beyond product spacetimes

**Question:** Do the decay results of Burq and Rodnianski–Tao extend to the case of general stationary and asymptotically flat spacetimes, possibly with a non-degenerate event horizon and a small ergoregion?
A decay result on general spacetimes with small ergoregion

Theorem (M., 2015)

Let \((M, g)\), \(d \geq 3\), be a stationary and asymptotically flat spacetime, possibly possessing a non-degenerate event horizon \(H\) and a small ergoregion \(E\). Assume that all solutions \(\phi\) to \(\Box g \phi = 0\) satisfy

\[E \left[ \phi \right](\tau) \leq C E \left[ \phi \right](0)\]

Then,

\[E \leq \mathcal{R} \left[ \phi \right](\tau) \leq C \mathcal{R} m \varepsilon \left( \log(\tau + 2) \right) - 2 m E(m) \left[ \phi \right](0) + C \mathcal{R} \varepsilon \tau^{-\varepsilon} E\left[ \phi \right](0),\]

where

\[E(m) \left[ \phi \right](0) = \sum_{j=0}^{\hat{t}} \left| \nabla_{\Sigma} T_j \phi \right|^2 dg,\]

\[E\varepsilon \left[ \phi \right](0) = \hat{t} r \varepsilon + \left| \nabla_{\Sigma} T_j \phi \right|^2 dg.\]
A decay result on general spacetimes with small ergoregion

**Theorem (M., 2015)**

Let \((M^{d+1}, g), d \geq 3\), be a stationary and asymptotically flat spacetime, possibly possessing a non-degenerate event horizon \(\mathcal{H}\) and a small ergoregion \(\mathcal{E}\). Assume that all solutions \(\varphi\) to \(\Box_g \varphi = 0\) satisfy

\[
\mathcal{E}[\varphi](\tau) \leq C \mathcal{E}[\varphi](0).
\]

Then,

\[
\mathcal{E}_{\leq R}[\varphi](\tau) \leq C R m \left( \log(\tau + 2) \right)^{-2m} \mathcal{E}^{(m)}[\varphi](0) + C R \varepsilon \tau^{-\varepsilon} \mathcal{E}_\varepsilon[\varphi](0),
\]

where

\[
\mathcal{E}^{(m)}[\varphi](0) = \sum_{j=0}^{m} \int_{\{t=0\}} \left| \nabla T^j \varphi \right|^2 \, dg_{\Sigma},
\]

\[
\mathcal{E}_\varepsilon[\varphi](0) = \int_{\{t=0\}} r_+^\varepsilon \left| \nabla T^j \varphi \right|^2 \, dg_{\Sigma}.
\]
A decay result on general spacetimes with small ergoregion

Remarks:
No assumption is imposed on the trapped set or the topology of the near region.
In the case $H = \emptyset$, the condition on the smallness of $E$ implies that $E = \emptyset$ and $T$ is everywhere timelike.
In the case $E \neq \emptyset$, the energy boundedness assumption cannot be inferred from the rest of the assumptions: Counterexamples can be constructed by suitable deformations of the subextremal Kerr metric (M., 2016).
The local energy $E \leq R[\phi](\tau)$ can be replaced by the energy flux of $\phi$ through a hyperboloidal foliation terminating at $I^+$. Pointwise estimates can also be obtained.
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- Pointwise estimates can also be obtained.
Sketch of the proof

The proof is based on separating $\phi$ into frequency decomposed components. The error terms from the cut-off procedure are controlled by the energy boundedness assumption.

Let $\omega \gg 1$. Splitting $\phi = \phi_{\leq \omega} + \phi_{\geq \omega}$:

$$E \leq R[\phi](t) \lesssim E \leq R[\phi_{\leq \omega}](t) + E \leq R[\phi_{\geq \omega}](t).$$

Since $\phi_{\geq \omega}$ has frequency support in $\{\omega \gtrsim \omega + \}:

$$E \leq R[\phi_{\geq \omega}](t) \leq C R \omega^{m} - 2 m \sum_{j=0}^{m} E[T_{j}\phi](0).$$

Assume that $E \leq R[\phi_{\leq \omega}](t) \leq C R \varepsilon t^{-\varepsilon} (e C R \omega + E[\phi](0) + E[\phi](0))$.

Then choosing $\omega \sim \varepsilon C - 1 R \log t$:

$$E \leq R[\phi](t) \leq C R \varepsilon (\log(t+2)) - 2 m E^{(m)}[\phi](0) + C R \varepsilon t^{-\varepsilon} E[\phi](0).$$

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Let $\omega + \gg 1$. Splitting $\varphi = \varphi \leq \omega + \varphi \geq \omega +$:

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Since $\varphi \geq \omega +$ has frequency support in $\{\omega \gg \omega +\}$:

$$E \leq R[\varphi \geq \omega +](t) \leq C R \varepsilon t^{-\varepsilon} (e C R \omega + E[\varphi](0) + E[\varphi](0))$$

Then choosing $\omega + \sim \varepsilon C^{-1} R \log t$:

$$E \leq R[\varphi](t) \leq C R \varepsilon (\log(t + 2))^{-2} E[m][\varphi](0) + C R \varepsilon t^{-\varepsilon} E[\varphi](0)$$

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- Since \( \varphi_{\geq \omega_+} \) has frequency support in \( \{ \omega \gtrsim \omega_+ \} \):

\[
\mathcal{E}_{\leq R}[\varphi_{\geq \omega_+}](t) \leq C_R m \omega_+^{-2m} \sum_{j=0}^{m} \mathcal{E}[T^j \varphi](0).
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Then choosing $\omega_+ \sim \epsilon C_R^{-1} \log t$:

$$\mathcal{E}_{\leq R}[\varphi](t) \leq C_R m \epsilon \left( \log(t + 2) \right)^{-2m} \mathcal{E}^{(m)}[\varphi](0) + C_R \epsilon t^{-\epsilon} \mathcal{E}_\epsilon[\varphi](0).$$
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$$E_{\leq R}[\varphi_{\leq \omega_+}](t) \leq C_{R\varepsilon} t^{-\varepsilon} \left( e^{C_{R\omega_+} E[\varphi](0)} \right) + E_{\varepsilon}[\varphi](0).$$

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$$E_{\leq R}[\varphi](t) \leq C_{Rm\varepsilon} (\log(t + 2))^{-2m} E^{(m)}[\varphi](0) + C_{R\varepsilon} t^{-\varepsilon} E_{\varepsilon}[\varphi](0).$$

Sketch the proof

In order to obtain a polynomial decay estimate for $\phi \leq \omega + \epsilon$: It suffices to show:

$$\hat{\phi} \leq R \left[ \phi \leq \omega + \epsilon \right] (t) dt \leq C R e^{C R \omega + E [\phi](0)}.$$ 

Decompose $\phi \leq \omega$ into components $\phi_k, 0 \leq k \leq \lceil \log_2 (\omega - \omega_0) \rceil$ with frequency support around $\omega_k \sim 2^k \omega_0$.

For $k \geq 1$: Carleman-type estimates, using the fact that $\partial_t \phi_k \sim i \omega_k \phi_k$ (using ideas from Burq and Rodnianski–Tao).

For $k = 0$: Separate argument.

Remark. The energy boundedness assumption is used in a critical way in the proof of the Carleman estimates.
Sketch the proof

In order to obtain a polynomial decay estimate for \( \varphi_{\leq \omega^+} \): It suffices to show:

\[
\int_0^{+\infty} \mathcal{E}_{\leq R}[\varphi_{\leq \omega^+}](t) \, dt \leq C R e^{C R \omega^+} \mathcal{E}[\varphi](0).
\]
Sketch the proof

In order to obtain a polynomial decay estimate for $\varphi_{\leq \omega_+}$: It suffices to show:

$$\int_0^{+\infty} E_{\leq R}[(\varphi_{\leq \omega_+})(t)] dt \leq C_R e^{C_R \omega_+} E[\varphi](0).$$

- Decompose $\varphi_{\leq \omega_+}$ into components $\varphi_k$, $0 \leq k \leq \lceil \log_2(\omega_0^{-1} \omega_) \rceil$ with frequency support around $\omega_k \sim 2^k \omega_0$. 

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Remark. The energy boundedness assumption is used in a critical way in the proof of the Carleman estimates.
Spacetimes with an evanescent ergosurface

The inverse logaritmic decay rate does not persist for spacetimes with $H = \emptyset$, $E = \emptyset$ possessing an evanescent ergosurface.

Two charge supersymmetric geometries:

$\overline{E} \leq R[\phi](\tau) \geq Cm, R(\log \log (\tau + 2))^{2m} \overline{E}(m)[w][\phi](0)$,

for $\phi$ depending trivially on the compact directions.


Question: What happens if $H = \emptyset$ but $E \neq \emptyset$?
Spacetimes with an evanescent ergosurface

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Spacetimes with $\mathcal{H} = \emptyset$, $\mathcal{E} \neq \emptyset$

Assume that $(M, g)$ is asymptotically flat, is stationary, with stationary Killing field $T$ has a non-empty ergoregion. Every point of $M$ communicates causally with the asymptotically flat region. Then there exist solutions $\phi$ to $\Box_g \phi = 0$ such that $E[\phi](0) = \hat{\{t = 0\}} J T^\mu(\phi) n_\mu = -1$.

For any such solution and any $\tau \geq 0$ (Friedman, 1978): $E[\phi](\tau) = \hat{\{t = \tau\}} \cap E J T^\mu(\phi) n_\mu \leq -1.$
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Assume that $(\mathcal{M}, g)$:

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\mathcal{E}_{\mathcal{E}}[\varphi](\tau) = \int_{\{t=\tau\} \cap \mathcal{E}} J^T_\mu(\varphi) n^\mu \leq -1.
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Friedman’s ergoregion instability

Conjecture (Friedman, 1978)

On such a spacetime \((\mathcal{M}, g)\), there exist solutions \(\varphi\) to \(\Box_g \varphi = 0\) such that the non-degenerate energy flux of \(\varphi\) through \(\{t = \tau\}\) blows up as \(\tau \to +\infty\).
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- Heuristic justification: Friedman (assuming that \((\mathcal{M}, g)\) is globally real analytic)
- Numerical investigation: Comins–Schutz, Yoshida–Eriguchi,...
- Rigorous proof?
Theorem (M., 2016)

Suppose that \( (\mathcal{M}^{d+1}, g), d \geq 2 \), is as above, satisfying in addition the following unique continuation condition:

**UC condition:** There exists a point \( p \in \partial E \) and an open neighborhood \( \mathcal{U} \) of \( p \) in \( \mathcal{M} \) such that, for any \( H^1_{loc} \) solution \( \tilde{\psi} \) to \( \square_g \tilde{\psi} = 0 \) on \( \mathcal{M} \) with \( \tilde{\psi} \equiv 0 \) on \( \mathcal{M} \setminus E \), we have \( \tilde{\psi} = 0 \) on \( E \cap \mathcal{U} \).

Then, there exists a smooth \( \varphi \) solving \( \square_g \varphi = 0 \) with compactly supported initial data, such that

\[
\limsup_{\tau \to +\infty} \int_{\{t=\tau\}} |\nabla \varphi|^2 = +\infty.
\]
Friedman’s ergoregion instability

Remarks:

No assumption on \((M, g)\) being real analytic is necessary. The proof also applies in the case when \((M, g)\) has a non-empty future event horizon \(H^+\) with positive surface gravity, such that \(H^+ \cap E = \emptyset\).

Examples of spacetimes where the unique continuation condition holds:

- Axisymmetric spacetimes with axisymmetric Killing field \(\Phi\), such that \([T, \Phi] = 0\) and the span of \(T, \Phi\) is timelike on \(\partial E\).
- Spacetimes which are real analytic in a neighborhood of \(\partial E\).

There exist spacetimes violating the unique continuation condition.
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Examples of spacetimes where the unique continuation condition holds:

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Applications

General relativity: Scalar wave equation on rapidly rotating self-gravitating dense fluids (Butterworth–Ipser).

Fluid mechanics: Acoustic wave equation on a steady irrotational flow with a supersonic region and no acoustic horizon.

Example (Cardoso–Crispino–Oliveira): The hydrodynamic vortex \((\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}), g_{\text{hyd}})\):

\[
g_{\text{hyd}} = -(1 - C_2 r^2) dt^2 + dr^2 - 2 C_1 dt d\theta + r^2 d\theta^2.
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  $$g_{\text{hyd}} = -\left(1 - C^2 r^2\right) dt^2 + dr^2 - 2Cdtd\theta + r^2 d\theta^2.$$
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g_{\text{hyd}} = -(1 - \frac{C^2}{r^2}) dt^2 + dr^2 - 2 C dt d\theta + r^2 d\theta^2.
  \]
Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions $\varphi$ to $\Box g \varphi = 0$ satisfy

$$\limsup_{\tau \to +\infty} \{ t = \tau \} |\nabla \varphi|^2 < +\infty.$$  \hfill (1)

Let $\psi = T \varphi$, for a solution $\varphi$ of $\Box g \varphi = 0$ with compactly supported initial data to be chosen later.

Using the methods of the logarithmic decay result, (1) implies that for any $\varepsilon > 0$, any $R, T, \tau_0 \gg 1$ and any $0 < \delta < 1$, there exists a $\tau^* \geq \tau_0$ such that:

$$\hat{\{ \tau^* - T \leq t \leq \tau^* + T \} \cap \{ r \leq R \} \setminus E_{\delta}(|\nabla \psi|^2 + |\psi|^2) < \varepsilon.$$  \hfill (2)

(1), (2) $\Rightarrow$ There exists a function $\tilde{\psi} \in H^1_{\text{loc}}(M)$ such that:

$$\psi(t + \tau_n, x) \to \tilde{\psi}(t, x) \text{ and } T \psi(t + \tau_n, x) \to T \tilde{\psi}(t, x) \text{ weakly in } H^1_{\text{loc}}(M) \text{ and strongly in } L^2_{\text{loc}}(M),$$

for a sequence $\tau_n \to +\infty$.

$\tilde{\psi} \equiv 0$ on $M \setminus E$

$\Box g \tilde{\psi} = 0$

Unique continuation condition $\Rightarrow \tilde{\psi} \equiv 0$ in $U$. 

Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions \( \varphi \) to \( \Box_g \varphi = 0 \) satisfy

\[
\limsup_{\tau \to +\infty} \int_{\{t = \tau\}} |\nabla \varphi|^2 < +\infty. \tag{1}
\]

Let \( \psi = T \varphi \), for a solution \( \varphi \) of \( \Box_g \varphi = 0 \) with compactly supported initial data to be chosen later. Using the methods of the logarithmic decay result, (1) implies that for any \( \epsilon > 0 \), any \( R, T, \tau_0 \gg 1 \) and any \( 0 < \delta < 1 \), there exists a \( \tau^* \geq \tau_0 \) such that:

\[
\hat{\{ \tau^* - T \leq t \leq \tau^* + T \}} \cap \{ r \leq R \} \setminus E_\delta (|\nabla \psi|^2 + |\psi|^2) < \epsilon. \tag{2}
\]

(1), (2) \Rightarrow There exists a function \( \tilde{\psi} \in H^1_{\text{loc}}(M) \) such that:

\( \psi(t + \tau_n, x) \to \tilde{\psi}(t, x) \) and \( T \psi(t + \tau_n, x) \to T \tilde{\psi}(t, x) \) weakly in \( H^1_{\text{loc}}(M) \) and strongly in \( L^2_{\text{loc}}(M) \), for a sequence \( \tau_n \to +\infty \).

\( \tilde{\psi} \equiv 0 \) on \( M \setminus E \), \( \Box_g \tilde{\psi} = 0 \)

Unique continuation condition = \Rightarrow \( \tilde{\psi} \equiv 0 \) in \( U \).
Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions \( \varphi \) to \( \square_g \varphi = 0 \) satisfy

\[
\limsup_{\tau \to +\infty} \int_{\{t = \tau\}} |\nabla \varphi|^2 < +\infty. \tag{1}
\]

Let \( \psi = T\varphi \), for a solution \( \varphi \) of \( \square_g \varphi = 0 \) with compactly supported initial data to be chosen later.
Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions \( \varphi \) to \( \Box_g \varphi = 0 \) satisfy

\[
\limsup_{\tau \to +\infty} \int_{\{t=\tau\}} |\nabla \varphi|^2 < +\infty. \tag{1}
\]

Let \( \psi = T \varphi \), for a solution \( \varphi \) of \( \Box_g \varphi = 0 \) with compactly supported initial data to be chosen later.

Using the methods of the logarithmic decay result, (1) implies that for any \( \varepsilon > 0 \), any \( R, T, \tau_0 \gg 1 \) and any \( 0 < \delta < 1 \), there exists a \( \tau_* \geq \tau_0 \) such that:

\[
\int_{\{\tau_* - T \leq t \leq \tau_* + T\} \cap \{r \leq R\} \setminus E_{\delta}} (|\nabla \psi|^2 + |\psi|^2) < \varepsilon. \tag{2}
\]
Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions $\phi$ to $\Box_g \phi = 0$ satisfy

$$\limsup_{\tau \to +\infty} \int_{\{t=\tau\}} |\nabla \phi|^2 < +\infty.$$  \hspace{1cm} (1)

Let $\psi = T\phi$, for a solution $\phi$ of $\Box_g \phi = 0$ with compactly supported initial data to be chosen later.

Using the methods of the logarithmic decay result, (1) implies that for any $\epsilon > 0$, any $R, T, \tau_0 \gg 1$ and any $0 < \delta < 1$, there exists a $\tau_* \geq \tau_0$ such that:

$$\int_{\{\tau_* - T \leq t \leq \tau_* + T\} \cap \{r \leq R\} \setminus E_\delta} (|\nabla \psi|^2 + |\psi|^2) < \epsilon.$$  \hspace{1cm} (2)

(1), (2) $\implies$ There exists a function $\tilde{\psi} \in H^1_{loc}(\mathcal{M})$ such that:

- $\psi(t + \tau_n, x) \rightarrow \tilde{\psi}(t, x)$ and $T\psi(t + \tau_n, x) \rightarrow T\tilde{\psi}(t, x)$ weakly in $H^1_{loc}(\mathcal{M})$ and strongly in $L^2_{loc}(\mathcal{M})$, for a sequence $\tau_n \rightarrow +\infty$.
- $\tilde{\psi} \equiv 0$ on $\mathcal{M} \setminus E$
- $\Box_g \tilde{\psi} = 0$
Sketch of the proof

The proof proceeds by contradiction, assuming that all smooth solutions \( \varphi \) to \( \Box_g \varphi = 0 \) satisfy

\[
\limsup_{\tau \to +\infty} \int_{\{t=\tau\}} |\nabla \varphi|^2 < +\infty. \tag{1}
\]

Let \( \psi = T\varphi \), for a solution \( \varphi \) of \( \Box_g \varphi = 0 \) with compactly supported initial data to be chosen later.

Using the methods of the logarithmic decay result, (1) implies that for any \( \varepsilon > 0 \), any \( R, T, \tau_0 \gg 1 \) and any \( 0 < \delta < 1 \), there exists a \( \tau^* \geq \tau_0 \) such that:

\[
\int_{\{\tau^* - T \leq t \leq \tau^* + T\} \cap \{r \leq R\} \setminus \mathcal{E}_\delta} \left( |\nabla \psi|^2 + |\psi|^2 \right) < \varepsilon. \tag{2}
\]

(1), (2) \implies There exists a function \( \tilde{\psi} \in H^1_{loc}(\mathcal{M}) \) such that:

\begin{itemize}
  \item \( \psi(t + \tau_n, x) \to \tilde{\psi}(t, x) \) and \( T\psi(t + \tau_n, x) \to T\tilde{\psi}(t, x) \) weakly in \( H^1_{loc}(\mathcal{M}) \) and strongly in \( L^2_{loc}(\mathcal{M}) \), for a sequence \( \tau_n \to +\infty \).
  \item \( \tilde{\psi} \equiv 0 \) on \( \mathcal{M}\setminus \mathcal{E} \)
  \item \( \Box_g \tilde{\psi} = 0 \)
\end{itemize}

Unique continuation condition \implies \( \tilde{\psi} \equiv 0 \) in \( \mathcal{U} \)
Sketch of the proof

It is possible to choose the initial data for \( \phi \) (and thus for \( \psi = T \phi \)) on \( \{ t = 0 \} \) so that:

\[
(\psi, T \psi) \big|_{t=0} \text{ is supported in } U \cap \{ t = 0 \}
\]

Conservation of the \( T \)-energy flux: For all \( \tau \geq 0 \)

\[
\hat{\{ t = \tau \}} \cap E \quad J_{T \mu}(\psi_n \mu) \leq -1.
\]

Alternative formula for energy:

\[
\hat{\{ t = \tau \}} J_{T \mu}(\psi_n) = \hat{\{ t = \tau \}} \Re \{ T \psi \cdot n \bar{\psi} - \psi \cdot n (T \bar{\psi}) \}.
\]

So:

\[
\hat{\{ t = 0 \}} J_{T \mu}(\tilde{\psi}) n \mu \leq -1.
\]
Sketch of the proof

It is possible to choose the initial data for $\varphi$ (and thus for $\psi = T\varphi$) on $\{t = 0\}$ so that:

- $(\psi, T\psi)|_{t=0}$ is supported in $U \cap E$
- $\int_{\{t=0\}} J_T^\mu (\psi) n^\mu = -1$
It is possible to choose the initial data for $\varphi$ (and thus for $\psi = T\varphi$) on $\{t = 0\}$ so that:

- $(\psi, T\psi)|_{t=0}$ is supported in $U \cap \mathcal{E}$
- $\int_{\{t=0\}} J^T_\mu (\psi) n^\mu = -1$

Conservation of the $T$-energy flux: For all $\tau \geq 0$

$$\int_{\{t=\tau\} \cap \mathcal{E}} J^T_\mu (\psi) n^\mu \leq -1.$$
Sketch of the proof

It is possible to choose the initial data for $\phi$ (and thus for $\psi = T\phi$) on $\{t = 0\}$ so that:

- $(\psi, T\psi)|_{t=0}$ is supported in $U \cap E$
- $\int_{\{t=0\}} J_\mu^T(\psi)n^\mu = -1$

Conservation of the $T$-energy flux: For all $\tau \geq 0$

$$\int_{\{t=\tau\} \cap E} J_\mu^T(\psi)n^\mu \leq -1.$$ 

Alternative formula for energy:

$$\int_{\{t=\tau\}} J_\mu^T(\psi)n^\mu = \int_{\{t=\tau\}} \text{Re}\left\{ T\psi \cdot n\bar{\psi} - \psi \cdot n(T\bar{\psi}) \right\}.$$
Sketch of the proof

It is possible to choose the initial data for $\varphi$ (and thus for $\psi = T\varphi$) on $\{t = 0\}$ so that:

- $(\psi, T\psi)|_{t=0}$ is supported in $U \cap \mathcal{E}$
- $\int_{\{t=0\}} J^T_\mu (\psi) n^\mu = -1$

Conservation of the $T$-energy flux: For all $\tau \geq 0$

$$\int_{\{t=\tau\} \cap \mathcal{E}} J^T_\mu (\psi) n^\mu \leq -1.$$  

Alternative formula for energy:

$$\int_{\{t=\tau\} \cap \mathcal{E}} J^T_\mu (\psi) n^\mu = \int_{\{t=\tau\}} \text{Re}\left\{ T\psi \cdot n\bar{\psi} - \psi \cdot n(T\bar{\psi}) \right\}.$$  

So:

$$\int_{\{t=0\}} J^T_\mu (\tilde{\psi}) n^\mu \leq -1. \quad (3)$$
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \phi_1, \phi_2 \rangle_T, \tau = \hat{\{t = \tau}\}}^{1,2} \text{Re} \{T \phi_1 \bar{\phi_2} + n \phi_1 T \bar{\phi_2} - g(T, n) \partial_\alpha \phi_1 \partial_\alpha \bar{\phi_2}\}.$$

For all $\tau \geq 0$:

$$\langle \psi, F - \tau \tilde{\psi} \rangle_T, 0 = 0,$$

where $F - \tau \tilde{\psi}(t, x) = \tilde{\psi}(t - \tau, x)$.

Conservation of the inner product:

$$\langle \psi, F - \tau \tilde{\psi} \rangle_T, \tau = 0.$$

Equivalently:

$$\langle F \tau \psi, \tilde{\psi} \rangle_T, 0 = 0.$$

Therefore, for $\tau = \tau_n \to +\infty$:

$$\hat{\{t = 0\}} J T \mu(\tilde{\psi}) n \mu = \langle \tilde{\psi}, \tilde{\psi} \rangle_T, 0 = 0.$$

(4) (3) & (4): Contradiction!
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \int_{\{t = \tau\}} \frac{1}{2} \text{Re} \left\{ T \varphi_1 n \bar{\varphi}_2 + n \varphi_1 T \bar{\varphi}_2 - g(T, n) \partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \right\}.$$
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \int_{\{t=\tau\}} \frac{1}{2} \text{Re}\left\{ T \varphi_1 \bar{n}\varphi_2 + n\varphi_1 T\bar{\varphi}_2 - g(T, n)\partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \right\}. $$

- For all $\tau \geq 0$: $\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \rangle_{T, 0} = 0$, where $\mathcal{F}_{-\tau} \tilde{\psi}(t, x) = \tilde{\psi}(t - \tau, x)$. (4)

(3) & (4): Contradiction!
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \int_{\{t=\tau\}} \frac{1}{2} \text{Re} \left\{ T \varphi_1 \n \overline{\varphi}_2 + n \varphi_1 T \overline{\varphi}_2 - g(T, n) \partial^\alpha \varphi_1 \partial_\alpha \overline{\varphi}_2 \right\}.$$

- For all $\tau \geq 0$: $\left\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \right\rangle_{T, 0} = 0$, where $\mathcal{F}_{-\tau} \tilde{\psi}(t, x) = \tilde{\psi}(t - \tau, x)$.

- Conservation of the inner product: $\left\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \right\rangle_{T, \tau} = 0$. 

Equivalently:

$\left\langle \mathcal{F}_{-\tau} \psi, \mathcal{F}_{-\tau} \tilde{\psi} \right\rangle_{T, 0} = 0.$
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \int_{\{t=\tau\}} \frac{1}{2} \text{Re}\left\{ T \varphi_1 n \bar{\varphi}_2 + n \varphi_1 \tau \bar{\varphi}_2 - g(T, n) \partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \right\}.$$

- For all $\tau \geq 0$: $\left\langle \psi, F_{-\tau} \bar{\psi} \right\rangle_{T,0} = 0$, where $F_{-\tau} \bar{\psi}(t, x) = \bar{\psi}(t - \tau, x)$.

- Conservation of the inner product: $\left\langle \psi, F_{-\tau} \bar{\psi} \right\rangle_{T, \tau} = 0$.

- Equivalently: $\left\langle F_{\tau} \psi, \bar{\psi} \right\rangle_{T,0} = 0$. 

(4) & (3): Contradiction!
Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \int_{\{t=\tau\}} \frac{1}{2} \operatorname{Re} \left\{ T \varphi_1 n \bar{\varphi}_2 + n \varphi_1 T \bar{\varphi}_2 - g(T, n) \partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \right\}.$$

- For all $\tau \geq 0$: $\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \rangle_{T, 0} = 0$, where $\mathcal{F}_{-\tau} \tilde{\psi}(t, x) = \tilde{\psi}(t - \tau, x)$.

- Conservation of the inner product: $\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \rangle_{T, \tau} = 0$.

- Equivalently: $\langle \mathcal{F}_{\tau} \psi, \tilde{\psi} \rangle_{T, 0} = 0$.

- Therefore, for $\tau = \tau_n \to +\infty$:

$$\int_{\{t=0\}} J^T_{\bar{\mu}} (\tilde{\psi}) n^\mu = \langle \tilde{\psi}, \tilde{\psi} \rangle_{T, 0} = 0. \quad (4)$$
Sketch of the proof

Indefinite inner product associated to the $T$-energy:

$$\langle \varphi_1, \varphi_2 \rangle_{T,\tau} = \int_{\{t=\tau\}} \frac{1}{2} \Re \left\{ T \varphi_1 n\bar{\varphi}_2 + n\varphi_1 T\bar{\varphi}_2 - g(T,n) \partial^\alpha \varphi_1 \partial_\alpha \bar{\varphi}_2 \right\}.$$

- For all $\tau \geq 0$: $\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \rangle_{T,0} = 0$, where $\mathcal{F}_{-\tau} \tilde{\psi}(t,x) = \tilde{\psi}(t-\tau, x)$.

- Conservation of the inner product: $\langle \psi, \mathcal{F}_{-\tau} \tilde{\psi} \rangle_{T,\tau} = 0$.

- Equivalently: $\langle \mathcal{F}_{\tau} \psi, \tilde{\psi} \rangle_{T,0} = 0$.

- Therefore, for $\tau = \tau_n \to +\infty$:

$$\int_{\{t=0\}} J^T_\mu (\tilde{\psi}) n^\mu = \langle \tilde{\psi}, \tilde{\psi} \rangle_{T,0} = 0. \quad (4)$$

(3) & (4): Contradiction!
Thank you for your attention!