Conformal Scattering

for Maxwell Fields on the Reissner-Nordstrøm-de Sitter Manifold

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- The general theory of relativity:
  - Spacetime $(\mathcal{M}, g)$.
  - Einstein's field equations: $\mathbf{G}_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}\mathbf{T}_{ab}$.
- Classical electromagnetism:
  - Maxwell's equations: $\nabla^a F_{ab} = 0 \quad ; \quad \nabla_{[a}F_{bc]} = 0$.
  - Maxwell energy-momentum tensor: $\mathbf{T}_{ab} = \frac{1}{4}g_{ab}F^{cd}F_{cd} - F_{ac}F_b{}^c$.
- Einstein-Maxwell coupled system:
  - Coupled equations:

$$\begin{cases} \mathbf{G}_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}\left(\frac{1}{4}g_{ab}F^{cd}F_{cd} - F_{ac}F_b{}^c\right); \\ \nabla^a F_{ab} = 0 \quad ; \quad \nabla_{[a}F_{bc]} = 0 \end{cases}$$

  - A spherically symmetric solution: RNdS black hole spacetime, $\mathcal{M} = \mathbb{R}_t \times]0, +\infty[_r \times \mathcal{S}^2_{\theta,\varphi}$

$$g = f(r)\mathrm{d}t^2 - \frac{1}{f(r)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\varphi^2\right) \ ,$$

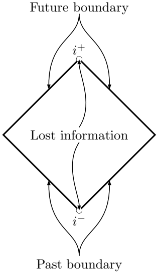
$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2 \ .$$

Decoupled System: Wave as test field on fixed background.

- **Conformal Scattering:** The asymptotic behaviour of the wave in the distant future and past.
  - *Conformal compactification and rescaling*: "make infinity finite", rescale the test field.$\rightarrow$ asymptotic profile
  - *Scattering operator*: Associate the past asymptotic profile to the future asymptotic profile and vice versa.

For conformal scattering:

- **Uniform decay:** make sure that no information is lost at $i^{\pm}$.



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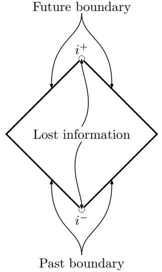
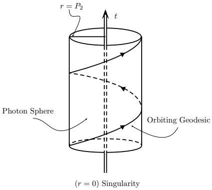


Information is encoded in the energy. We need some decay of energy $i^{\pm}$ (uniform decay).

For decay:

- **Trapping effect:** orbiting null geodesics (*photon sphere*).



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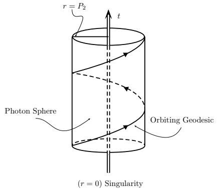
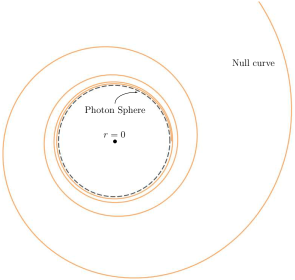


The photon sphere slows down the decay.



Conformal Scattering:

- Scattering:
  - Asymptotic influence of the geometry on fields.
  - Description of phenomena in black holes spacetimes.
- General non-stationary situations.

Conformal Scattering: Main Ingredients

- **Conformal rescaling:** $(\mathscr{M}, g)$ and an equation (E_g) on $\mathscr{M}$.
  - There is $(\hat{\mathscr{M}}, \hat{g})$, such that:
    - $\hat{g} = \Omega^2 g$.
    - $\partial\hat{\mathscr{M}} = \mathscr{I}$ infinity of $(\mathscr{M}, g)$.
    - $\Omega|_{\mathscr{I}} = 0$ and $\mathrm{d}\Omega|_{\mathscr{I}} \neq 0$.
    - $\mathrm{int}\hat{\mathscr{M}} = \mathscr{M}$.
  - (E_g) is conformally invariant: If Φ solution to (E_g) then $\hat{\Phi} = \Omega^s\Phi$ is solution to $(E_{\hat{g}})$.

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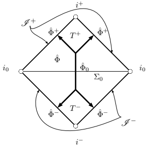
- **Cauchy problem:** Defining the trace operators.



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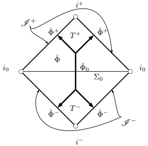
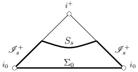


- **Energy estimates:** The trace operators are one-to-one.



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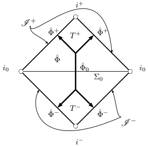

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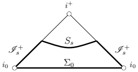

- **Goursat problem:** The trace operators are onto.
- **Scattering operator:** Isometry using the trace operators.

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1. **Photon Sphere:** Finding the necessary and sufficient conditions on the parameters of the RNdS metric to have three horizons, and locating the photon sphere.
2. **Decay:** Proving pointwise decay in time and uniform decay of the energy flux across achronal hypersurfaces for Maxwell fields on the static exterior region of the RNdS black hole.
3. **Conformal Scattering:** Solving the Goursat Problem and constructing a conformal scattering theory for the Maxwell fields on the static exterior region of the RNdS black hole.

The decay and scattering results hold true for a larger class of spherically symmetric spacetimes.

RNdS spacetime: $\mathcal{M} = \mathbb{R}_t \times]0, +\infty[_r \times \mathcal{S}^2_\omega$,

$$g = f(r)\mathrm{d}t^2 - \frac{1}{f(r)}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2 \quad ; \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2.$$

Assuming that f has 3 distinct simple positive zeros, we have three horizons at $r_1 < r_2 < r_3$. $(f(r_i) = 0)$

- $f > 0$ on $]0, r_1[$: static interior region.
- $f < 0$ on $]r_1, r_2[$: dynamic interior region.
- $f > 0$ on $]r_2, r_3[$: static exterior region.
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- $f > 0$ on $]r_2, r_3[$: static exterior region. (Photon sphere)
- $f < 0$ on $]r_3, +\infty[$: dynamic exterior region.

$$\mathcal{N} = \mathbb{R}_t \times]r_2, r_3[_r \times \mathcal{S}^2_\omega$$

One Photon Sphere

Set,

$$R = \frac{1}{\sqrt{6\Lambda}} \quad ; \quad \Delta = 1 - 12Q^2\Lambda$$

$$m_1 = R\sqrt{1 - \sqrt{\Delta}} \quad ; \quad m_2 = R\sqrt{1 + \sqrt{\Delta}}$$

$$M_1 = m_1 - 2\Lambda m_1^3 \quad ; \quad M_2 = m_2 - 2\Lambda m_2^3 \, .$$

Proposition (Three Positive Zeros and One Photon Sphere)

The horizon function f has exactly three positive distinct simple zeros if and only if

$$Q \neq 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{12Q^2} \quad \text{and} \quad M_1 < M < M_2 \, .$$

In this case, there is exactly one photon sphere, and it is located between the two largest zeros of f.

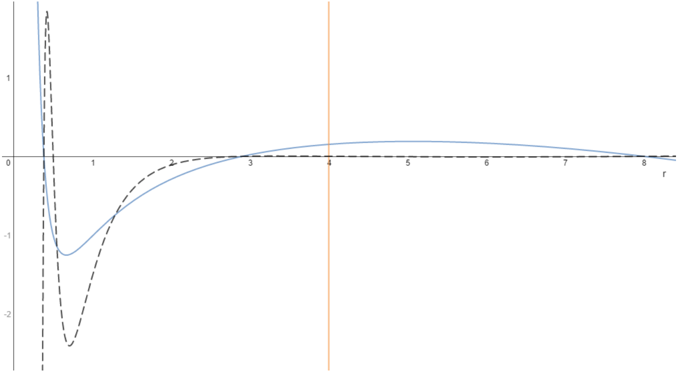
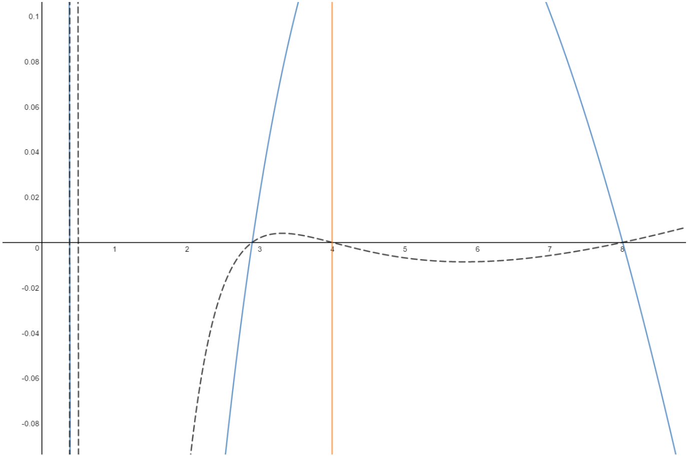


Figure: $Q = 1,\ M = 1.5,\ \Lambda = 0.01$. *The function f is the continuous curve and the radial acceleration $f(2^{-1}f' - r^{-1}f)$ is the doted curve. The vertical line (r=4) is the photon sphere.*



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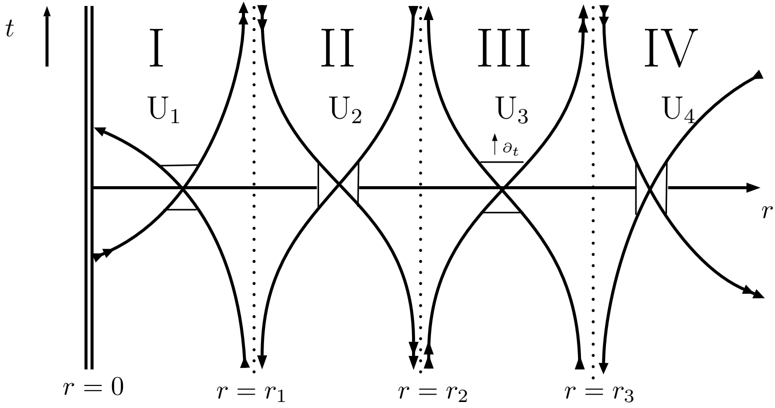


Figure: *The RNdS manifold with the radial null geodesics (integral curves of* $Y^{\mp} = f^{-1}\partial_t \pm \partial_r$*). It admits 16 time-orientations.*

Regge-Wheeler Coordinate: r_*-coordinate

$$\frac{\mathrm{d}r}{\mathrm{d}r_*} = f(r)$$

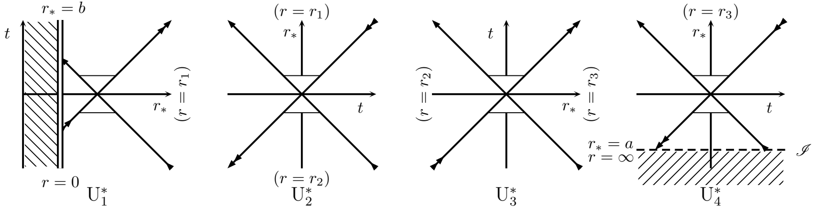


Figure: *The hypersurfaces $r = r_i$ (indicated in parenthesis) are off the chart since they are at infinity in r_*.*

The Eddington-Finkelstein coordinates $u_\pm = t \pm r_*$.
The metric in these coordinates:

$$g = f(r)\mathrm{d}{u_\pm}^2 \mp 2\mathrm{d}u_\pm \mathrm{d}r - r^2\mathrm{d}\omega^2$$

When ∂_r is future-oriented we denote it by $\mathcal{M}_F^\pm$, and $\mathcal{M}_P^\pm$ in the other case.

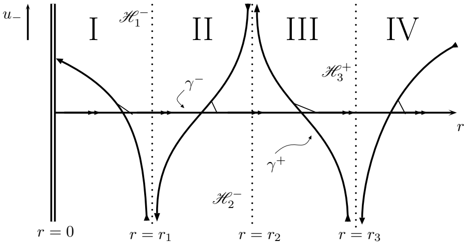


Figure: $\mathcal{M}_F^-$ *and the integral curves of* $Y^\mp$.

Using both $u_\pm$, the metric is: $g = f(r)\mathrm{d}u_-\mathrm{d}u_+ - r^2\mathrm{d}\omega^2$.

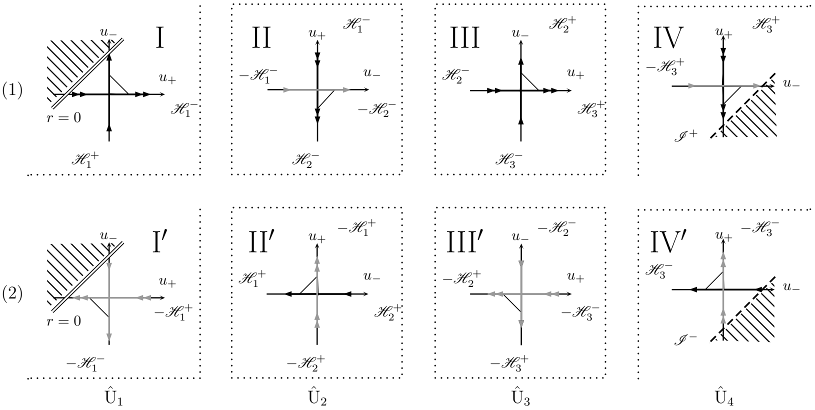
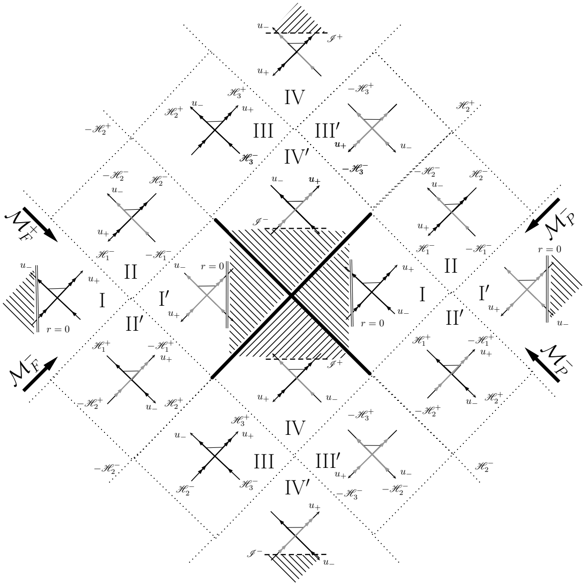


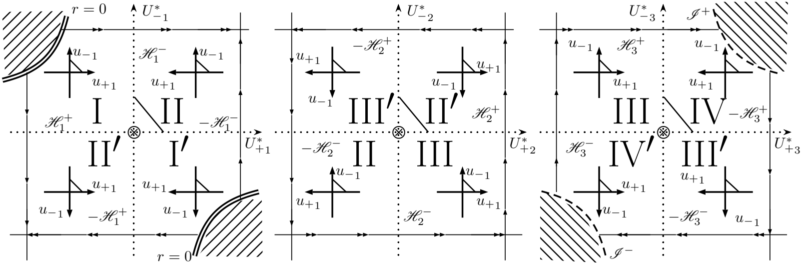
Figure: *In (1), we have time-orientation given by* ∂_t *and* ∂_r*, while in (2) by* $-\partial_t$ *and* $-\partial_r$*. Incoming and outgoing radial null geodesics are integral curves of* $Y^{\mp} = 2f^{-1}\partial_{u_\pm}$ *and of* $-Y^{\mp}$ *(shown in gray). The horizons* $\pm\mathscr{H}_i^{\pm}$ *(dotted lines) are asymptotic to the charts.*



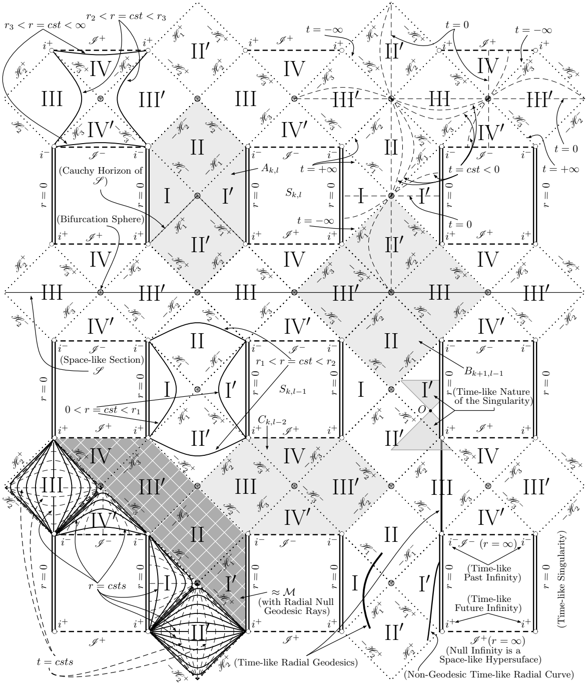
IV
III
III′
IV′
II
I
I′
II′
$\mathcal{M}_F^+$
$\mathcal{M}_F^-$
$\mathcal{M}_P^-$
$r = 0$
$\mathscr{I}^+$
$\mathscr{I}^-$
u_+
u_-
$\mathscr{H}_1^+$
$-\mathscr{H}_1^+$
$\mathscr{H}_1^-$
$-\mathscr{H}_1^-$
$\mathscr{H}_2^+$
$-\mathscr{H}_2^+$
$\mathscr{H}_2^-$
$-\mathscr{H}_2^-$
$\mathscr{H}_3^+$
$-\mathscr{H}_3^+$
$\mathscr{H}_3^-$
$-\mathscr{H}_3^-$

A natural choice of coordinates to glue the charts is the Kruskal-Szekeres coordinates of the form:

$$U_{\pm} = \beta_{\pm} e^{\alpha_{\pm} u_{\pm}} \ .$$



$\mathcal{K}_1$ $\mathcal{K}_2$ $\mathcal{K}_3$



$r_3 < r = cst < \infty$
$r_2 < r = cst < r_3$
$t = -\infty$
$t = 0$
$t = -\infty$
$t = 0$
$t = +\infty$
$t = cst < 0$
$t = +\infty$
$t = 0$
$t = -\infty$
$A_{k,l}$
$S_{k,l}$
(Cauchy Horizon of $\mathscr{S}$)
(Bifurcation Sphere)
(Space-like Section) $\mathscr{S}$
$0 < r = cst < r_1$
$r_1 < r = cst < r_2$
$S_{k,l-1}$
$C_{k,l-2}$
$B_{k+1,l-1}$
(Time-like Nature of the Singularity)
O
$r = csts$
$t = csts$
$\approx \mathcal{M}$ (with Radial Null Geodesic Rays)
(Time-like Radial Geodesics)
(Non-Geodesic Time-like Radial Curve)
$\mathscr{I}^-$ $(r = \infty)$
(Time-like Past Infinity)
(Time-like Future Infinity)
$\mathscr{I}^+(r = \infty)$
(Null Infinity is a Space-like Hypersuface)
(Time-like Singularity)
$r = 0$
I
I′
II
II′
III
III′
IV
IV′

- Null tetrad on $\mathcal{N}$: *"Stationary tetrad"*

$$L = \partial_t + \partial_{r_*}; \quad N = \partial_t - \partial_{r_*}; \quad M = \partial_\theta + \frac{i}{\sin(\theta)}\partial_\varphi; \quad \bar{M}$$

- Spin-components of F on the stationary tetrad:

$$\Phi_1 = F(L, M) \ ;$$
$$\Phi_0 = \frac{1}{2}\left(V^{-1}F(L,N) + F\left(\bar{M}, M\right)\right) \ ; \qquad \left(V = \frac{f}{r^2}\right)$$
$$\Phi_{-1} = F\left(N, \bar{M}\right)$$

Maxwell Compacted Equations

$$N\Phi_1 = VM\Phi_0, \qquad N\Phi_0 = -M_1\Phi_{-1},$$
$$L\Phi_0 = \bar{M}_1\Phi_1, \qquad L\Phi_{-1} = -V\bar{M}\Phi_0,$$

where $M_1 = M + \cot(\theta)$ and $\bar{M}_1$ is its conjugate.

Energies of the Maxwell Field

- Divergence theorem:

$$\int_{\partial\mathcal{U}} (X \lrcorner \mathbf{T})^\sharp \lrcorner \, \mathrm{d}^4x = \frac{1}{2}\int_{\mathcal{U}} (\nabla_a X_b - \nabla_b X_a)\mathbf{T}^{ab}\mathrm{d}^4x \; .$$

- Energy flux across a hypersurface S:

$$E_X[F](S) = \int_S (X \lrcorner \mathbf{T})^\sharp \lrcorner \, \mathrm{d}^4x = \int_S \mathbf{T}_{ab}X^b\eta_S^a(\tau_S \lrcorner \, \mathrm{d}^4x) \; .$$

η_S normal to S, τ_S is transverse to S, such that $g(\eta_S, \tau_S) = 1$.

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- Energy: for $X = T := \partial_t$ and $S = \Sigma_t := \{t\} \times \mathbb{R}_{r_*} \times \mathcal{S}^2_\omega$,

$$E_T[F](t) := E_T[F](\Sigma_t) = \frac{1}{4}\int_{\Sigma_t} |\Phi_1|^2 + \frac{2f}{r^2}|\Phi_0|^2 + |\Phi_{-1}|^2 \mathrm{d}r_*\mathrm{d}^2\omega \; .$$

- Conformal energy: for $X = K := (t^2 + r_*^2)\partial_t + 2tr_*\partial_{r_*}$,

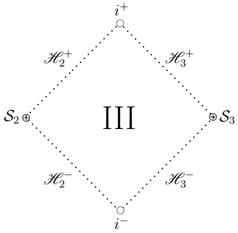
$$E_K[F](t) = \frac{1}{4}\int_{\Sigma_t} u_+^2|\Phi_1|^2 + (u_+^2 + u_-^2)\frac{f}{r^2}|\Phi_0|^2 + u_-^2|\Phi_{-1}|^2\mathrm{d}r_*\mathrm{d}^2\omega \; .$$

Theorem (Uniform Decay)

Let $t_0 \geq 0$ be a real parameter. Let S be any achronal future oriented smooth hypersurface, such that its union with $\Sigma_0 = \{0\} \times \mathbb{R} \times \mathcal{S}^2$ is the boundary of an open submanifold of $\mathcal{N}$, and such that on S, $t \geq |r_*| + t_0$. Then there is a constant $C > 0$ independent of t_0, F, (t, r_*, ω), and S, such that

$$E_T[F](S) \leq {t_0}^{-2} C \left(\sum_{k=0}^{1} E_K[\mathcal{L}_{\mathbb{O}}^k F](0) + \sum_{k=0}^{5} E_T[\mathcal{L}_{\mathbb{O}}^k F](0) \right) .$$

- $\bar{\mathcal{N}}$ the closure of the static exterior region $\mathcal{N}$ in $\mathcal{M}^*$.



- *Stationary tetrad*: $\{L, N, M, \bar{M}\}$ can be extended to $\bar{\mathcal{N}}$ but it will be singular (basis) on the horizons.
- *Outgoing tetrad*: $\{\hat{L} = f^{-1}L = \partial_r, N = 2\partial_{u_-} - f\partial_r, M, \bar{M}\}$ is a regular basis on $\mathcal{M}_F^-$ (but not on $\mathscr{H}_3^-$ and $\mathscr{H}_2^+$).
- *Incoming tetrad*: $\{L, \hat{N} = f^{-1}N, M, \bar{M}\}$ is a basis on $\mathcal{M}_F^+$.

- Spin components of F in the outgoing tetrad:

$$\begin{aligned}\hat{\Phi}_1 &= F\left(\hat{L}, M\right)\\ \Phi_0 &= \frac{1}{2}\left(\hat{V}^{-1}F(\hat{L},N) + F\left(\bar{M},M\right)\right)\\ \Phi_{-1} &= F\left(N,\bar{M}\right)\end{aligned}$$

where $\hat{V} = f^{-1}V = r^{-2}$.

- Maxwell compacted equations in the outgoing tetrad take the following form:

$$\begin{aligned}N\hat{\Phi}_1 &= \hat{V}M\Phi_0 + f'\hat{\Phi}_1,\\ \hat{L}\Phi_0 &= \bar{M}_1\hat{\Phi}_1,\\ N\Phi_0 &= -M_1\Phi_{-1},\\ \hat{L}\Phi_{-1} &= -\hat{V}\bar{M}\Phi_0.\end{aligned}$$

- Similarly for the incoming tetrad.

Function Spaces

- The energy flux across the horizons:

$$E_T[F](\mathscr{H}_3^+) = \frac{1}{4}\int_{\mathscr{H}_3^+} \mathbf{T}_{ab}N^a N^b(\hat{L} \lrcorner \, \mathrm{d}^4 x) = \frac{1}{4}\int_{\mathbb{R}_{u_-}\times\mathcal{S}^2} |\Phi_{-1}|^2 \mathrm{d}u_- \mathrm{d}^2\omega \, .$$

$$E_T[F](\mathscr{H}_2^+) = \frac{1}{4}\int_{\mathscr{H}_2^+} \mathbf{T}_{ab}L^a L^b(\hat{N} \lrcorner \, \mathrm{d}^4 x) = \frac{1}{4}\int_{\mathbb{R}_{u_+}\times\mathcal{S}^2} |\Phi_1|^2 \mathrm{d}u_+ \mathrm{d}^2\omega \, ,$$

- The energy spaces on $\mathscr{H}_i^\pm$: the completions of $\mathcal{C}_0^\infty(\mathscr{H}_i^\pm)$ for

$$\|\phi\|^2_{\mathscr{H}_2^\pm} = \pm\frac{1}{4}\int_{\mathscr{H}_2^\pm} |\phi|^2 \mathrm{d}u_\pm \wedge \mathrm{d}^2\omega \, ; \; \|\phi\|^2_{\mathscr{H}_3^\pm} = \mp\frac{1}{4}\int_{\mathscr{H}_3^\pm} |\phi|^2 \mathrm{d}u_\mp \wedge \mathrm{d}^2\omega$$

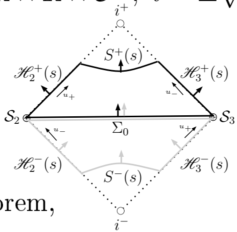
- On $\mathscr{H}^\pm := \mathscr{H}_2^\pm \cup \mathscr{H}_3^\pm$:
  $\mathcal{H}^\pm$ the completions of $\mathcal{C}_0^\infty(\mathscr{H}_2^\pm) \times \mathcal{C}_0^\infty(\mathscr{H}_3^\pm)$ for

$$\|(\phi_\pm, \phi_\mp)\|^2_{\mathcal{H}^\pm} = \|\phi_\pm\|^2_{\mathscr{H}_2^\pm} + \|\phi_\mp\|^2_{\mathscr{H}_3^\pm} .$$

Energy Identity up to $i^{\pm}$

- Consider

$$S^{\pm}(s) = \{(t, r_*, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathcal{S}^2 \,;\; t = \pm\sqrt{1 + r_*^2} + s \;;\; \pm s \geq 0\}.$$



- By the divergence theorem,

$$E_T[F](\Sigma_0) = E_T[F](\mathscr{H}_2^+(s)) + E_T[F](\mathscr{H}_3^+(s)) + E_T[F](S^+(s)).$$

- Thanks to the uniform decay,

$$\lim_{s\to+\infty} E_T[F](S^+(s)) = 0,$$

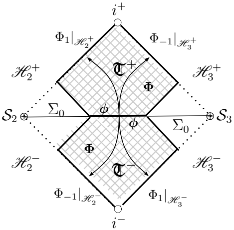
and

$$E_T[F](\Sigma_0) = E_T[F](\mathscr{H}_2^{\pm}) + E_T[F](\mathscr{H}_3^{\pm}).$$

Trace Operators

- The future and past trace operators:

$$\mathfrak{T}^{\pm} : \mathcal{U} \longrightarrow \mathcal{H}^{\pm}$$



$$\mathfrak{T}^{\pm}(\phi) = (\Phi_{\pm 1}|_{\mathscr{H}_2^{\pm}}, \Phi_{\mp 1}|_{\mathscr{H}_3^{\pm}}),$$

- By the energy identity,

$$\|\phi\|_{\mathcal{H}} = \|\mathfrak{T}^{\pm}(\phi)\|_{\mathcal{H}^{\pm}},$$

$\Rightarrow$ $\mathfrak{T}^{\pm}$ injective and have closed ranges.

- The Problem: Prove dense range.
  For $(\phi_+, \phi_-) \in \mathcal{C}_0^\infty\left(\mathscr{H}_2^\pm\right) \times \mathcal{C}_0^\infty\left(\mathscr{H}_3^\pm\right)$ find $\mathbf{\Phi} \in \mathcal{C}(\mathbb{R}_t\,;\ \mathcal{H})$ such that

$$\left(\Phi_{\pm 1}|_{\mathscr{H}_2^\pm}, \Phi_{\mp 1}|_{\mathscr{H}_3^\pm}\right) = (\phi_\pm, \phi_\mp)\ .$$

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  $$\left(\Phi_{\pm 1}|_{\mathscr{H}_2^\pm}, \Phi_{\mp 1}|_{\mathscr{H}_3^\pm}\right) = (\phi_\pm, \phi_\mp)\ .$$
- Simplifications:
  - Future and past are analogous.
  - By linearity and analogy of structure:
    $(0, \phi_-) \in \mathcal{C}_0^\infty\left(\mathscr{H}_2^+\right) \times \mathcal{C}_0^\infty\left(\mathscr{H}_3^+\right)$.

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- The strategy: use Hörmander's results about the characteristic Cauchy problem:
  1. Convert the initial-value problem from Maxwell's equations to wave equations.
  2. Put the problem in a framework for which Hörmander's results apply.
  3. Reinterpret the solution of the wave equations as a Maxwell field.

Solving the Goursat Problem

Theorem (Goursat Problem)

For $\phi_- \in \mathcal{C}_0^\infty\left(\mathscr{H}_3^+\right)$ there is a unique smooth, finite energy, Maxwell field F defined on $\mathcal{N}$, with $\mathbf{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1})$ its spin components in the stationary tetrad, such that

$$\left(\Phi_1|_{\mathscr{H}_2^+}, \Phi_{-1}|_{\mathscr{H}_3^+}\right) = (0, \phi_-).$$

The Scattering Operator

$$\mathfrak{S} = \mathfrak{T}^+ \circ (\mathfrak{T}^-)^{-1} : \mathcal{H}^- \longrightarrow \mathcal{H}^+$$

Wave equations

Let $\hat{\mathbf{\Phi}} = (\hat{\Phi}_1, \Phi_0, \Phi_{-1})$ be the spin components of a smooth Maxwell field in the outgoing tetrad, then

$$\hat{W}\hat{\mathbf{\Phi}} = \begin{pmatrix} \hat{W}_{11} & -\hat{V}'M & 0 \\ 0 & \hat{W}_{00} & 0 \\ 0 & -V'\bar{M} & \hat{W}_{0-1} \end{pmatrix} \begin{pmatrix} \hat{\Phi}_1 \\ \Phi_0 \\ \Phi_{-1} \end{pmatrix} = 0,$$

where the diagonal entries are $\hat{W}_{11} := \hat{L}N_1 - \hat{V}M\bar{M}_1$, $\hat{W}_{00} := \hat{L}N - \hat{V}M_1\bar{M}$, $\hat{W}_{0-1} := \hat{L}N - \hat{V}\bar{M}M_1$, and $N_1 = N - f'$.

The indices of $\hat{W}_{ij}$ indicate their expressions: $\hat{W}_{ij} = \hat{L}\, I - \hat{V}\, J$ with

$$i = \begin{cases} 0 & \text{if } I = N \,; \\ 1 & \text{if } I = N_1 \,, \end{cases} \qquad j = \begin{cases} 1 & \text{if } J = M\bar{M}_1 \,; \\ 0 & \text{if } J = M_1\bar{M} = \bar{M}_1 M \,; \\ -1 & \text{if } J = \bar{M}M_1 \,. \end{cases}$$

Wave equations

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Proof

$$N_1\hat{\Phi}_1 - \hat{V}M\Phi_0 =: E_1 \; ;$$

$$\hat{L}\Phi_0 - \bar{M}_1\hat{\Phi}_1 =: E_2 \; ;$$

$$N\Phi_0 + M_1\Phi_{-1} =: E_3 \; ;$$

$$\hat{L}\Phi_{-1} + \hat{V}\bar{M}\Phi_0 =: E_4 \; ;$$

Wave equations

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Proof

$$\begin{aligned} N_1\hat{\Phi}_1 - \hat{V}M\Phi_0 &=: E_1 \, ; & \hat{L}E_1 + \hat{V}ME_2 &= \hat{W}_{11}\hat{\Phi}_1 - \hat{V}'M\Phi_0 \, ; \\ \hat{L}\Phi_0 - \bar{M}_1\hat{\Phi}_1 &=: E_2 \, ; & N_1E_2 + \bar{M}_1E_1 &= \hat{W}_{00}\Phi_0 \, ; \\ N\Phi_0 + M_1\Phi_{-1} &=: E_3 \, ; & \hat{L}E_3 - M_1E_4 &= \hat{W}_{00}\Phi_0 \, ; \\ \hat{L}\Phi_{-1} + \hat{V}\bar{M}\Phi_0 &=: E_4 \, ; & N_1E_4 - \hat{V}\bar{M}E_3 &= \hat{W}_{0-1}\Phi_{-1} - V'\bar{M}\Phi_0 \, . \end{aligned}$$

Let $\hat{W}\hat{\mathbf{\Phi}} = (\hat{\Omega}_1, \Omega_0, \Omega_{-1})$:

$$\begin{aligned} N_1\hat{\Omega}_1 - \hat{V}M\Omega_0 &= \hat{W}_{01}E_1 + f\hat{V}'ME_2 \ ; \\ \hat{L}\Omega_0 - \hat{V}\bar{M}_1\hat{\Omega}_1 &= \hat{W}_{10}E_2 \ ; \\ N_1\Omega_0 + M_1\Omega_{-1} &= \hat{W}_{00}E_3 \ ; \\ \hat{L}\Omega_{-1} + \hat{V}\bar{M}\Omega_0 &= \hat{W}_{1-1}E_4 - \hat{V}'ME_3 \ . \end{aligned}$$

where

$$\begin{aligned} \hat{W}_{01} &= \hat{L}N - \hat{V}M\bar{M}_1 \ ; \\ \hat{W}_{10} &= \hat{L}N_1 - \hat{V}M_1\bar{M} \ ; \\ \hat{W}_{1-1} &= \hat{L}N_1 - \hat{V}\bar{M}_1 M \ . \end{aligned}$$

Since N is tangent to $\mathscr{H}_3^+$, equations $E_1 = 0$ and $E_2 = 0$ are constraint equations on the null horizon:

$$N_1|_{\mathscr{H}_3^+}\hat{\Phi}_1|_{\mathscr{H}_3^+} - \hat{V}(r_3)M\Phi_0|_{\mathscr{H}_3^+} = 0 \, ,$$
$$N|_{\mathscr{H}_3^+}\Phi_0|_{\mathscr{H}_3^+} + M_1\Phi_{-1}|_{\mathscr{H}_3^+} = 0 \, .$$

Therefore, for $\phi_- \in \mathcal{C}_0^\infty\left(\mathscr{H}_3^+\right)$ we define $\phi_0, \hat{\phi}_+ \in \mathcal{C}^\infty\left(\mathscr{H}_3^+\right)$ consecutively by the constraints initial-value problems in $\mathscr{H}_3^+$:

$$\begin{cases} 2\partial_{u_-}\phi_0 = M_1\phi_- \\ \phi_0|_{\mathcal{S}_p} = 0 \end{cases} \quad ; \quad \begin{cases} (2\partial_{u_-} - f'(r_3))\hat{\phi}_+ = \hat{V}(r_3)M\phi_0 \\ \hat{\phi}_+|_{\mathcal{S}_p} = 0 \end{cases}$$

where $\mathcal{S}_p$ is any sphere of $\mathscr{H}_3^+$ in the future of the support of ϕ_-.

The triplet $\hat{\boldsymbol{\phi}} = (\hat{\phi}_+, \phi_0, \phi_-)$ is the Goursat data for the wave equations. Note that $\hat{\boldsymbol{\phi}}$ vanishes between i^+ and the support of ϕ_-.

Thank you

Transferring to Hörmander's Framework