Non-compactness of initial data sets in high dimensions.
Seminar on Mathematical General Relativity
LJLL, Université Paris 6

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Scalar-field theory in General Relativity

Let \((\mathcal{M}^{n+1}, h), \ n \geq 3\), be a Lorentzian manifold, \(\Psi \in C^\infty(\mathcal{M}^{n+1})\) a scalar-field and \(V \in C^\infty(\mathbb{R})\) a potential.

\((\mathcal{M}^{n+1}, h, \Psi)\) is said to be a space-time if it satisfies the following Einstein equations:

\[
\begin{aligned}
Ric(h)_{ij} - \frac{1}{2} R(h) h_{ij} &= \nabla_i \Psi \nabla_j \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi)\right) h_{ij}, \\
\Box_h \Psi &= \frac{dV}{d\Psi}.
\end{aligned}
\]  

(E)

Relevant physical cases:

- Vacuum case with no cosmological constant: \(\Lambda = 0, \ V = 0\).
- Vacuum case with positive cosmological constant: \(\Lambda > 0, V = \Lambda > 0\).
- Klein-Gordon fields: \(V(\Psi) = \frac{1}{2} m^2 \), \(m > 0\).
Let \((\mathcal{M}^{n+1}, h), n \geq 3\), be a Lorentzian manifold, \(\Psi \in C^\infty(\mathcal{M}^{n+1})\) a scalar-field and \(V \in C^\infty(\mathbb{R})\) a potential.

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Relevant physical cases:

- Vacuum case with no cosmological constant: \(\Psi \equiv 0, V \equiv 0\).
- Vacuum case with positive cosmological constant: \(\Psi \equiv 0, V \equiv \Lambda > 0\).
- Klein-Gordon fields: \(V(\Psi) = \frac{1}{2} m \Psi^2, m > 0\).
The Evolution Problem

Assume globally hyperbolic spacetime: $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$ with $(M^n, h|_{M^n})$ Riemannian.
The Evolution Problem

Assume **globally hyperbolic spacetime:** \( M^{n+1} = M^n \times \mathbb{R} \) with \( (M^n, h_{\mid M^n}) \) Riemannian.

Notion of initial data sets on \( M^n \):

**Theorem (Choquet-Bruhat ’52, Choquet-Bruhat-Geroch ’69)**

\((M^n \times \mathbb{R}, h, \Psi)\) solves \((E)\) if and only if \((\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})\) solves *in* \( M^n \) the *constraint system*:

\[
\begin{aligned}
R(\tilde{g}) + tr_{\tilde{g}} \tilde{K}^2 - ||\tilde{K}||_{\tilde{g}}^2 &= \tilde{\pi}^2 + |\tilde{\nabla} \tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}) , \\
\tilde{\nabla}(tr_{\tilde{g}} \tilde{K}) - div_{\tilde{g}} K &= -\tilde{\pi} \tilde{\nabla} \tilde{\psi} ,
\end{aligned}
\]  

\((C)\)

*In particular:* any solution \((\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})\) of \((C)\) evolves into a solution of the Einstein equations. A solution \((\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})\) is therefore called **an initial data set.**
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Here we have let:

- \(\tilde{g} = h|_{M^n}\) and \(\tilde{\nabla}\) is the Levi-Civita connection for \(\tilde{g}\) in \(M^n\),
- \(\tilde{K}\): second fundamental form of the embedding \(M^n \subset M^n \times \mathbb{R}\),
- \(\tilde{\psi} = \Psi|_{M^n}\) and \(\tilde{\pi} = (N \cdot \Psi)|_{M^n}\). \(N\) is the future-directed unit normal to \(M^n\).
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Assume globally hyperbolic spacetime: $\mathcal{M}^{n+1} = \mathcal{M}^n \times \mathbb{R}$ with $(\mathcal{M}^n, h|_{\mathcal{M}^n})$ Riemannian.

Notion of initial data sets on $\mathcal{M}^n$:

**Theorem (Choquet-Bruhat ’52, Choquet-Bruhat-Geroch ’69)**

$(\mathcal{M}^n \times \mathbb{R}, h, \Psi)$ solves $(E)$ if and only if $(\tilde{\mathcal{g}}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ solves in $\mathcal{M}^n$ the constraint system:

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\begin{aligned}
R(\tilde{\mathcal{g}}) + tr_{\tilde{\mathcal{g}}} \tilde{K}^2 - ||\tilde{K}||_{\tilde{\mathcal{g}}}^2 &= \tilde{\pi}^2 + |\tilde{\nabla}\tilde{\psi}|_{\tilde{\mathcal{g}}}^2 + 2V(\tilde{\psi}), \\
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*In particular: any solution $(\tilde{\mathcal{g}}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ of (C) evolves into a solution of the Einstein equations. A solution $(\tilde{\mathcal{g}}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ is therefore called an initial data set.*

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System (C) has $n(n+1) + 2$ unknowns $(\tilde{\mathcal{g}}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ for $n+1$ equations.
The conformal method (Lichnerowicz, Choquet-Bruhat, York)

Goal: produce solution of the constraint equations:

\[
\begin{align*}
R(\tilde{g}) + \text{tr}_{\tilde{g}} \tilde{K}^2 - ||\tilde{K}||_{\tilde{g}}^2 &= \tilde{\pi}^2 + |\tilde{\nabla}\tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}) , \\
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Idea: look for solutions depending on \( n + 1 \) parameters to overcome the underdetermination.
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\begin{cases}
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Idea: look for solutions depending on \(n + 1\) parameters to overcome the underdetermination.

**Conformal parametrization:** look for the unknown initial data set \((\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})\) as:

\[
\left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right) = \left(u^{\frac{4}{n-2}} g, \frac{T}{n} u^{\frac{4}{n-2}} g + u^{-2} (\sigma + \mathcal{L}_g W), \psi, u^{-\frac{2n}{n-2}} \pi\right),
\]

where \(u \in C^\infty(M), \ u > 0, \ W \in T^*M\) and \(\mathcal{L}_g W\) is the conformal Killing operator.
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Goal: produce solution of the constraint equations:

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\begin{cases}
R(\tilde{g}) + \text{tr}_{\tilde{g}} \tilde{K}^2 - \|\tilde{K}\|_{\tilde{g}}^2 = \tilde{\pi}^2 + |\nabla_{\tilde{g}} \tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}), \\
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\left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right) = \left(u^{n^2}g, \frac{T}{n}u^{n^2}g + u^{-2}(\sigma + \mathcal{L}_{g}W), \psi, u^{-\frac{2n}{n-2}}\pi\right),
\]

where \(u \in C^\infty(M), u > 0, W \in T^*M\) and \(\mathcal{L}_{g}W\) is the conformal Killing operator.

These data depend on \(n + 1\) parameters \((u, W)\) and on given physics data \((\psi, \pi, \tau, \sigma, V)\) where:

- \(V\) is the potential of the scalar-field,
- \(\psi, \pi\) are scalar-field data,
- \(\tau\) is a mean curvature,
- \(\sigma\) is a \((2, 0)\)-symmetric tensor field with \(\text{tr}_{g}\sigma = 0\) and \(\text{div}_{g}\sigma = 0\) ("TT tensor").
The Einstein-Lichnerowicz constraint system

The conformal parametrization solves the original constraint equations if and only if \((u, W)\) solve the Einstein-Lichnerowicz constraint system:

\[
\begin{align*}
\Delta_g u + hu &= fu^2 - 1 + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|^2}{u^2 + 1}, \\
\overrightarrow{\Delta}_g W &= u^2 X + Y.
\end{align*}
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(CC)
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The conformal parametrization solves the original constraint equations if and only if $(u, W)$ solve the Einstein-Lichnerowicz constraint system:

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\begin{align*}
\triangle_g u + hu &= f u^{2* - 1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2* + 1}}, \\
\nabla_g W &= u^{2*} X + Y.
\end{align*}
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(\text{CC})

Here: $2^* = \frac{2n}{n-2}$, $(u, W)$ are smooth, $u > 0$. Also $\triangle_g = -\text{div}_g (\nabla \cdot)$, $\triangle_g = -\text{div}_g (\nabla \cdot)$. $\mathcal{L}_g W$ is the conformal Killing derivative:

\[
\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} \text{div}_g W \cdot g_{ij},
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and $\nabla_g W = -\text{div}_g (\mathcal{L}_g W)$ is the Lamé operator.
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In the physical case, the coefficients \((h, f, \pi, \sigma, X, Y)\) depend on the choice of the given physics data \((\psi, \pi, \tau, \sigma, V)\) of the conformal method.

**Our goal:** understand the blow-up behavior of the solutions of (CC).
Setting of our problem:

In the following: for us, \( M \) will always be compact without boundary. The coefficients \((h, f, \pi, X, Y, \sigma)\) will satisfy the assumptions of the focusing case:

\[
f > 0, \quad \Delta_g + h \quad \text{coercive}, \quad \text{and} \quad \pi \neq 0.
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In the physical case, the coefficients are related to the physics data by:

$$h = \frac{n-2}{4(n-1)} \left( S_g - |\nabla \psi|_g^2 \right),$$

$$f = 2V(\psi) - \frac{n-1}{n} \tau^2,$$

$$X = -\frac{n-1}{n} \nabla \tau, \quad Y = -\pi \nabla \psi.$$
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Solutions of $(CC)$ exist under mild conditions on the coefficients (P., Gicquaud-Nguyen).
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Solutions of $(CC)$ exist under mild conditions on the coefficients (P., Gicquaud-Nguyen). In the following we will investigate the system for general focusing coefficients $(h, f, \pi, X, Y, \sigma)$, not only the physical ones.
Criticality of the system and defects of compactness

\[
\begin{align*}
\Delta_g u + hu &= fu^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\
\overrightarrow{\Delta}_g W &= u^{2^*} X + Y.
\end{align*}
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The critical nonlinearity $u^{2^*-1}$, with the “mean” focusing sign $f > 0$, implies that concentration phenomena (or blow-up) are likely to occur.
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**Model example:** standard bubbles. For \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n, n \geq 3 \):

\[
B_{\lambda,x_0}(x) = \left( \frac{\lambda}{\lambda^2 + \frac{|x-x_0|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}, \quad \nabla_\zeta B_{\lambda,x_0} = B_{\lambda,x_0}^{2^*-1} \quad \text{in } \mathbb{R}^n, \quad \|B_{\lambda,x_0}\|_{L^{2^*}} = K_n.
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Criticality of the system and defects of compactness

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\begin{cases}
\Delta_g u + hu = fu^{2^* - 1} + \frac{\pi^2 + |\sigma + L_g W|^2}{u^{2^* + 1}}, \\
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\]

Similar explosive phenomena in \( C^0(M) \) are obtained for critical nonlinear elliptic equations or systems (Druet-Hebey ’04, Robert-Vétois ’14, Pistoia-Vaira ’15, Vétois-Thizy ’16...).

Perturbations of the coefficients increase the chance of appearance of defects of compactness.

**Example:** the Yamabe equation (Brendle ’08, Esposito-Pistoia-Vétois ’14).
Criticality of the system and defects of compactness

\begin{align*}
\left\{ \begin{array}{l}
\Delta_g u + hu &= fu^{2*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|^2}{u^{2*+1}}, \\
\nabla \Delta_g W &= u^{2*} X + Y.
\end{array} \right.
\end{align*}

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Similar explosive phenomena in $C^0(M)$ are obtained for critical nonlinear elliptic equations or systems (Druet-Hebey ’04, Robert-Vétois ’14, Pistoia-Vaira ’15, Vétois-Thizy ’16...).

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**Example:** the Yamabe equation (Brendle ’08, Esposito-Pistoia-Vétois ’14).

**Natural Question:** when do these blow-up phenomena occur for the EL system?
The notion of Stability for the Einstein-Lichnerowicz constraint system

**Definition**

Let \((h, f, \pi, X, Y, \sigma) \in C^2(M)\). The Einstein-Lichnerowicz system is said to be **stable** if, for any sequence \((h_k, f_k, \pi_k, X_k, Y_k, \sigma_k)_k\) converging to \((h, f, \pi, X, Y, \sigma)\) in \(C^2(M)\) and for any sequence \((u_k, W_k)_k\) of solutions of:

\[
\begin{cases}
\triangle_g u_k + h_k u_k = f_k u_k^{2^* - 1} + \frac{\pi_k^2 + |\sigma_k + L_g W_k|^2}{u_k^{2^* + 1}}, \\
\nabla_g W_k = u_k^{2^*} X_k + Y_k,
\end{cases}
\]

there exists a solution \((u, W)\) of \((CC)\) such that \((u_k, W_k) \to (u, W)\) in \(C^2(M)\) (up to a subsequence and up to elements in the kernel of \(L_g\)).
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there exists a solution $(u, W)$ of $(CC)$ such that $(u_k, W_k) \rightarrow (u, W)$ in $C^2(M)$ (up to a subsequence and up to elements in the kernel of $\mathcal{L}_g$).

The system will be said to be **unstable**... if it is not stable. **(Non)-Compactness** is defined similarly for **constant perturbations** of the coefficients.
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The system will be said to be **unstable**... if it is not stable. (Non)-Compactness is defined similarly for constant perturbations of the coefficients.

The stability of an equation/system yields structural informations. Instability is a failure of uniform (in the choice of the coefficients) a priori bounds for solutions.
Stability results

Stability holds (on a locally conformally flat manifold) under the following conditions:

\[ \text{If} \quad n \geq 5 \quad (\text{Druet-P. '14, } n = 3, \text{ P. '15}) \]

\[ \text{If} \quad n \leq a \quad \text{and} \quad \mathcal{r} \quad \text{and} \quad X \quad \text{have no common zero in} \quad M. \quad \text{Or, if they do, provided at these zeroes there holds:} \]

\[ h < n^2 \quad S_g \left( n^2 \right) \left( n^4 \right) / 8 \quad (n-1)^4 \quad g \quad f \quad f. \quad (0.1) \]

(P. '15)

It is a second-order compatibility condition between the geometric and physics data.

For the physical case of the Einstein-scalar field setting, these conditions ensure that stability holds when the scalar-field \( \mathcal{r} \) and the mean curvature \( \mathcal{r} \) have no common critical point in \( M \).

What about the sharpness of these conditions in high dimensions: can blow-up phenomena happen in high dimensions?
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Stability holds (on a locally conformally flat manifold) under the following conditions:

- If $3 \leq n \leq 5$ (Druet-P. '14, $n = 3$, P. '15)
- If $n \geq 6$ and $\nabla f$ and $X$ have no common zero in $M$. Or, if they do, provided at these zeroes there holds:

$$h < \frac{n - 2}{4(n - 1)} S_g - \frac{(n - 2)(n - 4)}{8(n - 1)} \frac{\Delta_g f}{f}.$$  \hspace{1cm} (0.1)

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It is a second-order compatibility condition between the geometric and physics data. For the physical case of the Einstein-scalar field setting, these conditions ensure that stability holds when the scalar-field $\psi$ and the mean curvature $\tau$ have no common critical point in $M$. 

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What about the sharpness of these conditions in high dimensions: can blow-up phenomena happen in high dimensions?
Main result: instability examples in high dimensions

Theorem (Non-compactness in high dimensions $n \geq 6$, P., ‘16)

Let $(M, g)$ be a closed Riemannian manifold of dimension $n \geq 6$, such that $\nabla_g$ has no kernel. There exist coefficients $(h, f, \pi, \sigma, X, Y)$ of class $C^2$, satisfying the assumptions of the focusing case and $X \neq 0$ such that the Einstein-Lichnerowicz system:

$$
\begin{align*}
\triangle_g u + hu &= fu^{2* - 1} + \frac{|\mathcal{L}_g W + \sigma|^2 + \pi^2}{u^{2*+1}} \\
\nabla_g W &= u^{2*} X + Y
\end{align*}
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possesses a blowing-up sequence of solutions $(u_k, W_k)_k$, that is: $\|u_k\|_{L^\infty(M)} \to +\infty$ and $\|\mathcal{L}_g W_k\|_{L^\infty(M)} \to +\infty$ as $k \to +\infty$. Here the $u_k$ are positive, have one concentration point and have a non-zero limit profile.
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In particular: these coefficients $(h, f, \pi, \sigma, X, Y)$ satisfy

\[
h \geq \frac{n - 2}{4(n - 1)} S_g - \frac{(n - 2)(n - 4)}{8(n - 1)} \frac{\triangle_g f}{f} \text{ somewhere.}
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Main result: instability examples in high dimensions

Theorem (Non-compactness in high dimensions $n \geq 6$, P., ‘16)

Let $(M, g)$ be a closed Riemannian manifold of dimension $n \geq 6$, such that $\hat{\Delta}_g$ has no kernel. There exist coefficients $(h, f, \pi, \sigma, X, Y)$ of class $C^2$, satisfying the assumptions of the focusing case and $X \neq 0$ such that the Einstein-Lichnerowicz system:

$$\begin{cases}
\Delta_g u + hu = fu^{2^*-1} + \frac{|L_g W + \sigma|^2_g + \pi^2}{u^{2^*+1}} \\
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Surprising consequence: the Einstein-Lichnerowicz system has an infinite number of (blowing-up) solutions in high dimensions!
Two dual approaches in the blow-up analysis of critical elliptic equations

1) The *a priori* approach. It is the one used to prove stability results.

Answers the following question: given an (arbitrary) blowing-up sequence of solutions of EL, what can I say about it? It gives informations about: the pointwise blow-up behavior of sequences of solutions, the localisation of concentration points, the mutual interactions between different defects of compactness,...
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In P., '15 it is for instance proven that any blowing-up sequence $(u_k, W_k)$ of the E-L system satisfies:

$$u_k = B_k + o(B_k) \text{ in } C^0$$

in the neighbourhood of a concentration point, where $B_k$ is a given bubbling profile modeled on the standard bubble. And, \textit{as a consequence}, that at a concentration point $x_0$ there holds:

$$\nabla f(x_0) = X(x_0) = 0.$$
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Approach developed by: Li, Zhu, Druet, Schoen, Marques, Zhang, Khuri, Hebey, Robert.
Two dual approaches in the blow-up analysis of critical elliptic equations II

2) The Lyapounov-Schmidt approach, or $H^1$-constructive approach: used to construct blowing-up sequences of solutions under suitable assumptions on the coefficients.

Idea: look for solutions as:

$$u_{t,\xi,k} = B_{t,\xi,k} + u_0 + \varphi_{t,\xi,k},$$

$u_0 > 0$ weak limit, $B_{t,\xi,k}$ is a bubbling profile and $\varphi_{t,\xi,k}$ is small in $H^1(M)$. 
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Example: to solve $\triangle_g u + hu = u^{2^*-1}$, find $u_{t,\xi,k}$ critical point of the energy. Reduces to find $(t, \xi)$ critical point of:

$$(t, \xi) \mapsto \frac{1}{2} \int_M |\nabla B_{t,\xi,k}|^2 + hB_{t,\xi,k}^2 \, dv_g - \frac{1}{2^*} \int_M B_{t,\xi,k}^{2^*} \, dv_g$$
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The method implicitly relies on the informations provided by the a priori techniques.
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The method **implicitly** relies on the informations provided by the a priori techniques.

Developed by Wei, Rey, Del Pino, Pacard (over the last 15 years)
Proof: A constructive approach via a glueing method

We hope to find blowing-up solutions of the Einstein-Lichnerowicz system:

\[
\begin{align*}
\Delta_g u + hu &= fu^{2* - 1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|^2}{u^{2* + 1}}, \\
\nabla_g W &= u^{2*}X + Y
\end{align*}
\]

having the following form:

\[u_{t,\xi,k} = B_{t,\xi,k} + u_0 + \varphi_{t,\xi,k}.\]

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\triangle_g u + h u &= f u^{2^* - 1} + \pi^2 + \frac{|\sigma + \mathcal{L}_g W|^2}{u^{2^* + 1}}, \\
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The system is strongly coupled (\( X \not\equiv 0 \)) and the vector equation is supercritical in the natural energy space \( H^1(M) \): the system is non-variational and ill-posed in \( H^1(M) \). The system therefore exhibits a double (super-)criticality that cannot be handled with standard constructive energy methods.
Solution: a $C^0$ constructive approach that relies on the a priori analysis

For us today: constructive method in strong spaces by combining a priori analysis techniques with the standard $H^1$ reduction method to perform the ping-pong method.
Solution: a $C^0$ constructive approach that relies on the a priori analysis

**For us today:** constructive method *in strong spaces* by combining a priori analysis techniques with the standard $H^1$ reduction method to perform the ping-pong method.

Look again for $u_k$ under the form:

$$u_{t, \xi, k} = B_{t, \xi, k} + u_0 + \varphi_{t, \xi, k},$$
Solution: a $C^0$ constructive approach that relies on the a priori analysis

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Look again for $u_k$ under the form:

$$u_{t,\xi,k} = B_{t,\xi,k} + u_0 + \varphi_{t,\xi,k},$$

with a remainder small in a $C^0(M)$ sense, with explicit pointwise bounds depending on the ansatz of the solution:

$$|\varphi_{t,\xi,k}| \leq \varepsilon_k (B_{t,\xi,k} + u_0), \quad (0.3)$$

where $\varepsilon_k \to 0$ and is independent of $t$ and $\xi$. 
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where $\varepsilon_k \to 0$ and is independent of $t$ and $\xi$.

Our solution depends on $(n+1)$ parameters $(t, \xi)$ – just like the standard bubbling profiles.

**Goal:** find, for every $k$, a value $(t_k, \xi_k)_k$ of the parameters and a suitable remainder $\varphi_{t_k,\xi_k,k}$ (small in $C^0(M)$) for which $u_{t_k,\xi_k,k}$ is indeed a solution!
Sketch of the proof I

The proof is a fixed-point ("ping-pong") method in $1 + 3$ main steps:

1. Choose the following zeroth-order approximation:
   
2. Choose $s$ satisfying
3. By the choice of $s$ and $X$, we now have pointwise estimates on this $W_k$ that blows-up:
4. Problem: it blows up too fast to plug it into the scalar equation and perform a usual ping-pong method!
Sketch of the proof

The proof is a fixed-point ("ping-pong") method in 1 + 3 main steps:

**Step 0:** Choose the following zeroth-order approximation:

\[
B_{t, \xi, k}(x) = \Lambda_\xi(x) \cdot \chi \left( \frac{d_{g_\xi}(\xi, x)}{r_k} \right) \left( (t \mu_k)^{\frac{n-2}{2}} \right) \frac{\left( (t \mu_k)^2 + \frac{f(\xi)}{n(n-2)} d_{g_\xi}(\xi, x)^2 \right)^{\frac{n-2}{2}}}{\text{conformal correction + cutoff}}.
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- **conformal correction + cutoff**
- **standard bubble**

Choose $\varphi$ satisfying $|\varphi| \leq \varepsilon_k (B_{t,\xi,k} + u_0)$ and consider the only solution of:

$$\nabla_g W_{t,\xi,k} = (B_{t,\xi,k} + u_0 + \varphi)^{2^*} X + Y.$$
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By the choice of $\varphi$ and $X$, we now have pointwise estimates on this $W_k$ that blows-up:

$$|\mathcal{L}_g W_{t,\xi,k}| \sim \frac{\frac{n-1}{2}}{\mu_k^{\frac{1}{2}}} \left( \mu_k^2 + d_{g_\xi}(\xi, x)^2 \right)^{\frac{n-1}{2}} \text{ close to } \xi.$$
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By the choice of $\varphi$ and $X$, we now have pointwise estimates on this $W_k$ that blows-up:

$$|\mathcal{L}_g W_{t, \xi, k}| \sim \frac{\mu_k^{n-1}}{\left(\mu_k^2 + d_{g_{\xi}}(\xi, x)^2\right)^{n-2}} \text{ close to } \xi.$$

**Problem:** it blows up too fast to plug it into the scalar equation and perform a usual ping-pong method!
Sketch of the proof II: Semi-decoupling

Step 1: Construct a solution of:

$$\nabla_g u + hu = fu^{2^* - 1} + \frac{|\mathcal{L}_g W_0 + \sigma|_g^2 + \pi^2}{u^{2^* + 1}} + \left( \frac{|\mathcal{L}_g W_{t, \xi, k} + \sigma|_g^2 - |\mathcal{L}_g W_0 + \sigma|_g^2}{(B_{t, \xi, k} + u_0 + \varphi)^{2^* + 1}} \right)$$

$$+ \sum_{j=0}^{n} \chi_j^k(t, \xi, \varphi) (\nabla_g + h) Z_{j,k}.$$ 

Done via a nonlinear fixed-point method in $H^1$ in the orthogonal of the kernel of the linearized equation at $B_{t, \xi, k}$ (spanned by the $Z_{j,k}$). Here $\mathcal{L}_g W_{t, \xi, k}$ is a coefficient.
Sketch of the proof II: Semi-decoupling

**Step 1:** Construct a solution of:

$$
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+ \sum_{j=0}^n \chi_j^j(t, \xi, \varphi) (\triangle_g + h) Z_{j,k}.
$$

Done via a nonlinear fixed-point method in $H^1$ in the orthogonal of the kernel of the linearized equation at $B_{t,\xi,k}$ (spanned by the $Z_{j,k}$). Here $\mathcal{L}_g W_{t,\xi,k}$ is a coefficient.

It works since the red term comes with explicit pointwise estimates on it.
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+ \sum_{j=0}^n \lambda_j(t, \xi, \varphi) (\triangle_g + h) Z_{j,k}.
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The solution is of the form \(B_{t,\xi,k} + u_0 + \psi_{t,\xi,k}\) for a new remainder \(\psi \in H^1(M)\), orthogonal to the \(Z_{j,k}\).
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+ \sum_{j=0}^{n} \chi_j(t, \xi, \varphi)(\nabla g + h) Z_{j,k}.
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Done via a nonlinear fixed-point method in $H^1$ in the orthogonal of the kernel of the linearized equation at $B_{t,\xi,k}$ (spanned by the $Z_{j,k}$). Here $\mathcal{L}_g W_{t,\xi,k}$ is a coefficient.

It works since the red term comes with explicit pointwise estimates on it.

The solution is of the form $B_{t,\xi,k} + u_0 + \psi_{t,\xi,k}$ for a new remainder $\psi \in H^1(M)$, orthogonal to the $Z_{j,k}$.

**Goal:** Get an (almost) solution of the system if $\psi = \varphi$. 
Step 2: The goal is now to fix-point $\varphi \mapsto \psi$, in the set of $C^0$ functions satisfying:

$$|\varphi| \leq \varepsilon_k (B_{t,\xi,k} + u_0).$$

(0.4)
**Step 2:** The goal is now to fix-point \( \varphi \mapsto \psi \), in the set of \( C^0 \) functions satisfying:

\[
|\varphi| \leq \varepsilon_k (B_t, \xi, k + u_0).
\] (0.4)

Since \( \psi \) comes from an \( H^1 \) procedure it is not even clear that \( |\psi| \leq \varepsilon_k (B_t, \xi, k + u_0) \). We prove this by a priori analysis techniques.
Step 2: The goal is now to fix-point $\varphi \mapsto \psi$, in the set of $C^0$ functions satisfying:

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Since $\psi$ comes from an $H^1$ procedure it is not even clear that $|\psi| \leq \varepsilon_k (B_{t,\xi,k} + u_0)$. We prove this by a priori analysis techniques.

This is again done in three steps:

Step a): Extend the a priori asymptotic techniques of the $C^0$-theory of Druet-Hebey-Robert to this scalar equation. Possible here since the red term comes with explicit (and suitable) pointwise bounds.
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Step a): Extend the a priori asymptotic techniques of the $C^0$-theory of Druet-Hebey-Robert to this scalar equation. Possible here since the red term comes with explicit (and suitable) pointwise bounds.

This shows that

$$\psi = o \left( B_{t,\xi, k} + u_0 \right) \quad \text{in } C^0(M).$$
Sketch of the proof III: Fixed-point in $C^0$ 2

**Step b):** Quantify the $o(1)$. This requires to obtain second-order estimates on $\psi$ (again blow-up arguments).
Sketch of the proof III: Fixed-point in $C^0$ 2

**Step b):** Quantify the $o(1)$. This requires to obtain second-order estimates on $\psi$ (again blow-up arguments). They are for instance, at finite distance from $\xi$:

$$|\psi(x)| \lesssim \left[ \mu_k^n + \mu_k \| \nabla f \|_{L^\infty} + \| h - c_n S_g \|_{L^\infty} \mu_k^2 \ln \left( \frac{\mu_k + d_g(\xi, x)}{\mu_k} \right) \right] + \left[ h - c_n S_g \right]_{L^\infty} d_g(\xi, x)^2 + d_g(\xi, x)^4 \chi_{nlcf} B_{t, \xi, k}(x) + \left( \frac{\mu_k}{\mu_k + d_g(\xi, x)} \right)^2.$$
Sketch of the proof III: Fixed-point in $C^0$ 2

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$$
|\psi|(x) \lesssim \left[ \mu_k^{\frac{n}{2}} + \mu_k \|\nabla f\|_{L^\infty} + \|h - c_n S_g\|_{L^\infty} \mu_k^2 \ln \left( \frac{\mu_k + d_g(\xi, x)}{\mu_k} \right) \\
+ \|h - c_n S_g\|_{L^\infty} d_g(\xi, x)^2 + d_g(\xi, x)^4 \mathbb{1}_{nlcf} \right] B_{t, \xi, k}(x) + \left( \frac{\mu_k}{\mu_k + d_g(\xi, x)} \right)^2.
$$

We also prove that these estimates are uniform in $t$, $\xi$ and $\varphi$. 
Step b): Quantify the \( o(1) \). This requires to obtain second-order estimates on \( \psi \) (again blow-up arguments). They are for instance, at finite distance from \( \xi \):

\[
|\psi|(x) \lesssim \left[ \mu_k^2 + \mu_k \|\nabla f\|_{L^\infty} + \|h - c_n S_g\|_{L^\infty} \mu_k^2 \ln \left( \frac{\mu_k + d_g(\xi, x)}{\mu_k} \right) \right]
\]

\[
+ \|h - c_n S_g\|_{L^\infty} d_g(\xi, x)^2 + d_g(\xi, x)^4 1_{nlcf} \left[ B_{t, \xi, k}(x) + \left( \frac{\mu_k}{\mu_k + d_g(\xi, x)} \right)^2 \right].
\]

We also prove that these estimates are uniform in \( t, \xi \) and \( \varphi \).

Step c): Choose a suitable \( \varepsilon_k \) (according to the red term). And then show that \( \varphi \mapsto \psi \) is a contraction. Relies on the second-order estimates.
Step 3: At the end of Step 2, after point-fixing the remainders, we have a solution 
\((u_{t, \xi, k}, W_{t, \xi, k})\) of:

\[
\begin{cases}
\Delta_g u + hu = f u^{2* - 1} + \pi^2 + |\sigma + \mathcal{L}_g W|^2_{\bar{u}} + \sum_{j=0}^{n} \lambda_j^k(t, \xi) (\Delta_g + h) Z_{j,k,t,\xi}, \\
\overrightarrow{\Delta}_g W = u^{2*} X + Y,
\end{cases}
\]

where \(u_{t, \xi, k}\) writes as:

\[
\begin{aligned}
&u_{t, \xi, k} = B_{t, \xi, k} + u_0 + \varphi_{t, \xi, k}, \\
&|\varphi_{t, \xi, k}| \leq \varepsilon_k (B_{t, \xi, k} + u_0),
\end{aligned}
\]

and \(\varepsilon_k\) is known.
Step 3: At the end of Step 2, after point-fixing the remainders, we have a solution $(u_{t,\xi,k}, W_{t,\xi,k})$ of:

\[
\begin{align*}
\Delta_g u + hu &= f u^{2*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|^2}{u^{2*+1}} + \sum_{j=0}^{n} \lambda_j^j(t, \xi) (\Delta_g + h) Z_{j,k,t,\xi}, \\
\overrightarrow{\Delta}_g W &= u^{2*} X + Y,
\end{align*}
\]

where $u_{t,\xi,k}$ writes as:

$$u_{t,\xi,k} = B_{t,\xi,k} + u_0 + \varphi_{t,\xi,k},$$

and $|\varphi_{t,\xi,k}| \leq \varepsilon_k(B_{t,\xi,k} + u_0)$, where $\varepsilon_k$ is known.

To conclude: use the second-order estimates on $\varphi_{t,\xi,k}$ to obtain an asymptotic expansion of the $\lambda_{k,j}(t, \xi)$ in $C^0_{\text{loc}}(\mathbb{R}^{n+1})$ as $k \to +\infty$. And we are left to annihilate $(n + 1)$ functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$. 
Thank you for your attention.
Bonus: Explicit expressions of $h$ and $X$.

The explicit expressions of $h, f$ and $X$ are the following:

$$f(x) \approx f_0$$

$$h(x) = \frac{n-2}{4(n-1)} S_g(x) + \sum_{k \geq 1} \tau_k H \left( \frac{1}{\beta_k} \left( \exp_{\xi_0} \right)^{-1}(x) \right),$$

$$X(x) = X_0(x) + \sum_{k \geq 1} \mu_k \frac{n-1}{2} Z \left( \left( \exp_{\xi_0} \right)^{-1}(x) \right),$$

where $\tau_k$ depends on $\mu_k$ and on the dimension and if $(M, g)$ is locally conformally flat or not. Also, $\mu_k \ll \beta_k \ll 1$ is another scale parameter.

The function $H$ has a strict local maximum at 0 and $|Z(0)|_{\xi} > 0$. 
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where $\tau_k$ depends on $\mu_k$ and on the dimension and if $(M, g)$ is locally conformally flat or not. Also, $\mu_k << \beta_k << 1$ is another scale parameter.

The function $H$ has a strict local maximum at 0 and $|Z(0)|_{\xi} > 0$.

We did not just play around with the values of the parameters so that everything fits well in the end: the relations between the parameters are rigid and are given by the a priori pointwise stability analysis.