General Covariant Modulated procedure

Dawei Shen

Laboratoire Jacques-Louis Lions,
Sorbonne Université

Plan of the talk

1. Introduction: Kerr stability for small angular momentum

2. General Covariant Modulated (GCM) spheres

3. General Covariant Modulated (GCM) hypersurfaces

1. Introduction

The Kerr and the Schwarzschild solutions

Einstein vacuum equations (EVE): $\mathbf{Ric}_{\alpha\beta} = 0$ ($\mathbf{Ric}$ Ricci tensor of $\mathbf{g}$)

Kerr metric given in Boyer-Lindquist (t, r, θ, φ) coordinates by

$$\mathbf{g}_{a,m} = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2 \sin^2\theta}{\rho^2}\Big(d\varphi - \frac{2amr}{\Sigma^2}dt\Big)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\Delta = r^2 + a^2 - 2mr, \; \rho^2 = r^2 + a^2\cos^2\theta, \; \Sigma^2 = (r^2+a^2)^2 - a^2\sin^2\theta\Delta$$

The Schwarzschild metric is spherically symmetric and corresponds to the particular case $a = 0, m > 0$

$$\mathbf{g}_m = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2\Big(d\theta^2 + (\sin\theta)^2 d\varphi^2\Big)$$

Stability conjecture for the Kerr family

Schwarzchild spacetimes correspond to non rotating black holes while for $|a| < m$, Kerr spacetimes correspond to rotating black holes

Stability problem: Are these black holes stable?

In the context of asymptotically flat solutions to the Einstein vacuum equation, we have the following conjecture:

Conjecture (Stability of the exterior region of Kerr). Small perturbations of given initial conditions of an exterior Kerr $\mathbf{g}_{a,m}$ with $|a| < m$ have maximal future developments converging to another exterior Kerr solution $\mathbf{g}_{a_f,m_f}$ with $|a_f| < m_f$

Nonlinear stability of Kerr for $|a| \ll m$

Theorem: The stability conjecture holds true for $|a| \ll m$.

- Modulation: Klainerman-Szeftel 19' (arXiv:1911.00697, arXiv:1912.12195), and S. 22' (arXiv:2205.12336)
- Decay estimates, as well as statement of the result and strategy: Klainerman-Szeftel 21' (arXiv:2104.11857)
- Hyperbolic estimates: Giorgi-Klainerman-Szeftel 22' (arXiv:2205.14808)

In this talk, we focus on the modulation procedure that extends [Klainerman-Szeftel 18'] in axial polarized symmetry to general perturbations of Kerr spacetimes

Modulation

$\mathbf{Ric}[\phi^* \mathbf{g}_{a,m}] = 0$ for all $|a| < m$ and diffeomorphism ϕ and hence:

$$\delta\mathbf{Ric}\left[\frac{\partial \mathbf{g}_{a,m}}{\partial m}\right] = \delta\mathbf{Ric}\left[\frac{\partial \mathbf{g}_{m,a}}{\partial a}\right] = \delta\mathbf{Ric}\left[\mathcal{L}_X \mathbf{g}_{m,a}\right] = 0$$

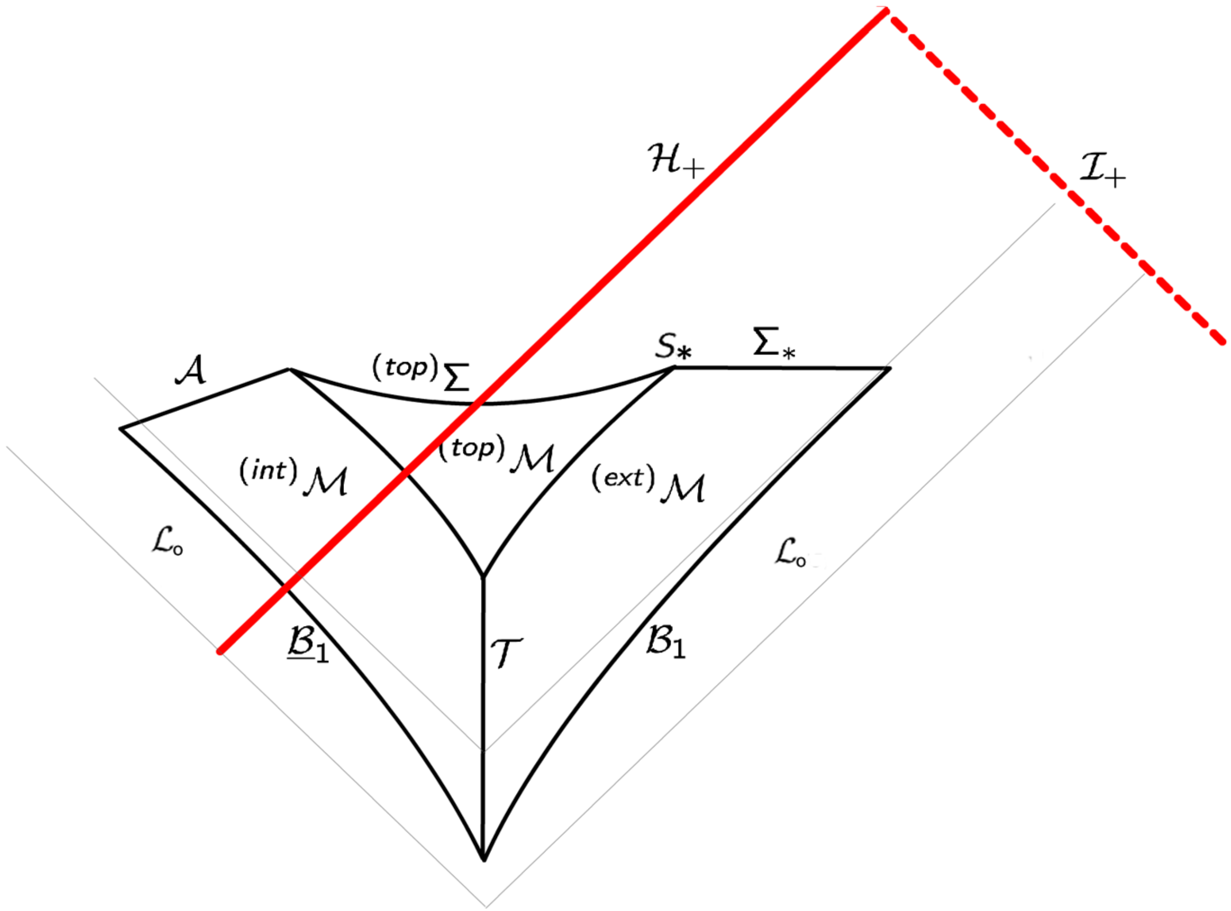
Thus, $\partial_m \mathbf{g}_{m,a}$, $\partial_a \mathbf{g}_{m,a}$ and $\mathcal{L}_X \mathbf{g}_{m,a}$ belong to the kernel of Linearized Gravity System (LGS), which corresponds at the nonlinear level to the tracking of (m_f, a_f).

When dealing with a linearized operator possessing a non trivial kernel, one uses modulation theory

General covariance of Einstein equations generates a kernel of LGS which has infinite dimensions and hence requires to find a strategy to implement modulation in infinite dimensions

The continuity argument and the last slice

The last slice is chosen as a GCM hypersurface, which is the topic of this talk.



2. GCM spheres

Principal quantities

S–adapted null frame (e_1, e_2, e_3, e_4): (e_1, e_2) are tangent to S

Ricci coefficients:

$$\chi_{ab} := g(D_{e_a} e_4, e_b), \qquad \underline{\chi}_{ab} := g(D_{e_a} e_3, e_b), \qquad \xi_a := \frac{1}{2} g(D_{e_4} e_4, e_a),$$

$$\underline{\xi}_a := \frac{1}{2} g(D_{e_3} e_3, e_a), \qquad \omega := \frac{1}{4} g(D_{e_4} e_4, e_3), \qquad \underline{\omega} := \frac{1}{4} g(D_{e_3} e_3, e_4),$$

$$\eta_a := \frac{1}{2} g(D_{e_3} e_4, e_a), \qquad \underline{\eta}_a := \frac{1}{2} g(D_{e_4} e_3, e_a), \qquad \zeta_a := \frac{1}{2} g(D_{e_a} e_4, e_3)$$

Expansion, shear and twist:

$$\mathrm{tr}\,\chi := g^{ab}\chi_{ab}, \qquad \widehat{\chi}_{ab} := \chi_{ab} - \frac{1}{2}\mathrm{tr}\,\chi\, g_{ab}, \qquad {}^{(a)}\mathrm{tr}\,\chi := \in^{ab} \chi_{ab},$$

$$\mathrm{tr}\,\underline{\chi} := g^{ab}\underline{\chi}_{ab}, \qquad \widehat{\underline{\chi}}_{ab} := \underline{\chi}_{ab} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\, g_{ab}, \qquad {}^{(a)}\mathrm{tr}\,\underline{\chi} := \in^{ab} \underline{\chi}_{ab}$$

Principal quantities

Curvature components:

$$\alpha_{ab} := R(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} := R(e_a, e_3, e_b, e_3),$$

$$\beta_a := \frac{1}{2} R(e_a, e_4, e_3, e_4), \qquad \underline{\beta}_a := \frac{1}{2} R(e_a, e_3, e_3, e_4),$$

$$\rho := \frac{1}{4} R(e_3, e_4, e_3, e_4), \qquad \sigma := \frac{1}{4} {}^*R(e_3, e_4, e_3, e_4)$$

Mass aspect function:

$$\mu := -\mathrm{div}\zeta - \rho + \frac{1}{2} \widehat{\chi} \cdot \underline{\widehat{\chi}}$$

Conditions of geodesic foliation on e_4:

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta$$

Choice of last slice Σ_*

Before estimating the Exterior region ${}^{(ext)}\mathcal{M}$, we need to estimate Σ_*

- We only have Hodge-Elliptic equations on Σ_* (e.g. Codazzi equations)
- $\operatorname{tr}\chi$, $\operatorname{tr}\underline{\chi}$ and μ do not verify Hodge-Elliptic equations, (only transport equations)

Idea: Construct Σ_* by combination of well chosen spheres such that $\operatorname{tr}\chi$, $\operatorname{tr}\underline{\chi}$ and μ take Schwarzchild values (Close enough to Kerr values for $r \gg m$):

$$\operatorname{tr}\chi = \frac{2}{r}, \qquad \operatorname{tr}\underline{\chi} = -\frac{2\Upsilon}{r}, \qquad \mu = \frac{2m}{r^3},$$

where m denotes the Hawking mass. These conditions called GCM conditions and these spheres called GCM spheres.

Deformations of spheres

Idea: Start from $\overset{\circ}{S}(u,s)$ having small GCM conditions. Then contruct GCM spheres by the deformation of spheres

$$\Phi : \overset{\circ}{S}(u,s) \to \mathbf{S}'$$
$$(u, s, y^1, y^2) \to (u + U(y^1, y^2), s + S(y^1, y^2), y^1, y^2)$$

Goal: Find $\mathbf{S}'$–adapted frame (e_1', e_2', e_3', e_4') verifying the following GCM conditions:

$$\mathrm{tr}\,\chi' = \frac{2}{r'}, \qquad \mathrm{tr}\,\underline{\chi}' = -\frac{2\Upsilon'}{r'}, \qquad \mu' = \frac{2m'}{r'^3}$$

Frame transformations

Transition functions $F := (f, \underline{f}, \lambda - 1)$ describe $SO(1,3)/SO(2)$:

$$e_4' = \lambda \left(e_4 + f^b e_b \right) + O(|F|^2),$$

$$e_a' = e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2} f_a e_3 + O(|F|^2),$$

$$e_3' = \lambda^{-1} \left(e_3 + \underline{f}^b e_b \right) + O(|F|^2)$$

The condition that (e_1', e_2') is tangent to $\mathbf{S}'$ leads to:

$$\partial_{y^a} U = \Phi^{\#}(\mathcal{U}(f, \underline{f}, \Gamma))_a,$$

$$\partial_{y^a} S = \Phi^{\#}(\mathcal{S}(f, \underline{f}, \Gamma))_a,$$

where $\mathcal{U}, \mathcal{S} = (f, \underline{f}) + O(|f, \underline{f}|^2)$ are 1–forms on $\mathbf{S}'$

Null transformation formulae

Transformation formulae for Ricci coefficients:

$$\mathrm{tr}\,\chi' = \mathrm{tr}\,\chi + \mathrm{div}' f + F \cdot \Gamma + \ldots$$

$$\mathrm{tr}\,\underline{\chi}' = \mathrm{tr}\,\underline{\chi} + \mathrm{div}' \underline{f} + F \cdot \Gamma + \ldots$$

$${}^{(a)}\mathrm{tr}\,\chi' = {}^{(a)}\mathrm{tr}\,\chi + \mathrm{curl}' f + F \cdot \Gamma + \ldots$$

$${}^{(a)}\mathrm{tr}\,\underline{\chi}' = {}^{(a)}\mathrm{tr}\,\underline{\chi} + \mathrm{curl}' \underline{f} + F \cdot \Gamma + \ldots$$

$$\mu' = \mu + \Delta' \lambda + F \cdot \Gamma + F \cdot R + \ldots$$

Elliptic systems

Fixing following conditions:

$$\mathrm{tr}\,\chi' = \frac{2}{r'}, \qquad \mathrm{tr}\,\underline{\chi}' = -\frac{2\Upsilon'}{r'}, \qquad \mu' = \frac{2m'}{(r')^3},$$

$$^{(a)}\mathrm{tr}\,\chi' = 0, \qquad ^{(a)}\mathrm{tr}\,\underline{\chi}' = 0$$

We obtain elliptic systems: $(\overset{\circ}{\lambda} := \lambda - 1)$

$$\mathrm{div}' f = \ldots \qquad \mathrm{div}' \underline{f} = \ldots$$

$$\mathrm{curl}' f = \ldots \qquad \mathrm{curl}' \underline{f} = \ldots$$

$$(\Delta' + V)\,\overset{\circ}{\lambda} = \ldots \qquad V := \frac{2}{(r')^2}$$

Presence of an asymptotic kernel on $\ell = 1$ modes as $r \to +\infty$

GCM systems

We relax the GCM conditions as follows:

$$\operatorname{tr}\chi' = \frac{2}{r'}, \qquad \left(\operatorname{tr}\underline{\chi}' + \frac{2\Upsilon'}{r'}\right)_{\ell\geq 2} = 0, \qquad \left(\mu' - \frac{2m'}{(r')^3}\right)_{\ell\geq 2} = 0,$$

$$^{(a)}\operatorname{tr}\chi' = 0, \qquad ^{(a)}\operatorname{tr}\underline{\chi}' = 0$$

We solve the following GCM systems by iteration:

$$\begin{cases} \operatorname{div}' f = ... \\ \operatorname{curl}' f = ... \\ \operatorname{div}' \underline{f} = ... + \text{Extra terms} \\ \operatorname{curl}' \underline{f} = ... \\ (\Delta' + V)\overset{\circ}{\lambda} = ... + \text{Extra terms} \end{cases} \quad \text{and} \quad \begin{cases} (\operatorname{div}' f)_{\ell=1} = \Lambda, \\ (\operatorname{div}' \underline{f})_{\ell=1} = \underline{\Lambda} \end{cases}$$

6 parameters of $(\Lambda, \underline{\Lambda})$: 3 translations and 3 boosts.

Construction of GCM spheres

Theorem [Klainerman-Szeftel, 19']. Let $\mathcal{R}$ be a fixed spacetime region endowed with an outgoing geodesic foliation $\overset{\circ}{S}(u,s)$, verifying

$$\operatorname{tr}\chi - \frac{2}{r}, \quad \left(\operatorname{tr}\underline{\chi} + \frac{2\Upsilon}{r}\right)_{\ell\geq 2}, \quad \left(\mu - \frac{2m}{r^3}\right)_{\ell\geq 2} = O(\overset{\circ}{\delta}).$$

Then, for any (u,s) and $(\Lambda, \underline{\Lambda}) = O(\overset{\circ}{\delta})$, there exists a unique GCM sphere $\mathbf{S}' = \mathbf{S}'(u,s,\Lambda,\underline{\Lambda})$, which is a deformation of $\overset{\circ}{S}(u,s)$ s.t.

$$\operatorname{tr}\chi' - \frac{2}{r'} = 0, \quad \left(\operatorname{tr}\underline{\chi}' + \frac{2\Upsilon'}{r'}\right)_{\ell\geq 2} = 0, \quad \left(\mu' - \frac{2m'}{(r')^3}\right)_{\ell\geq 2} = 0,$$

and

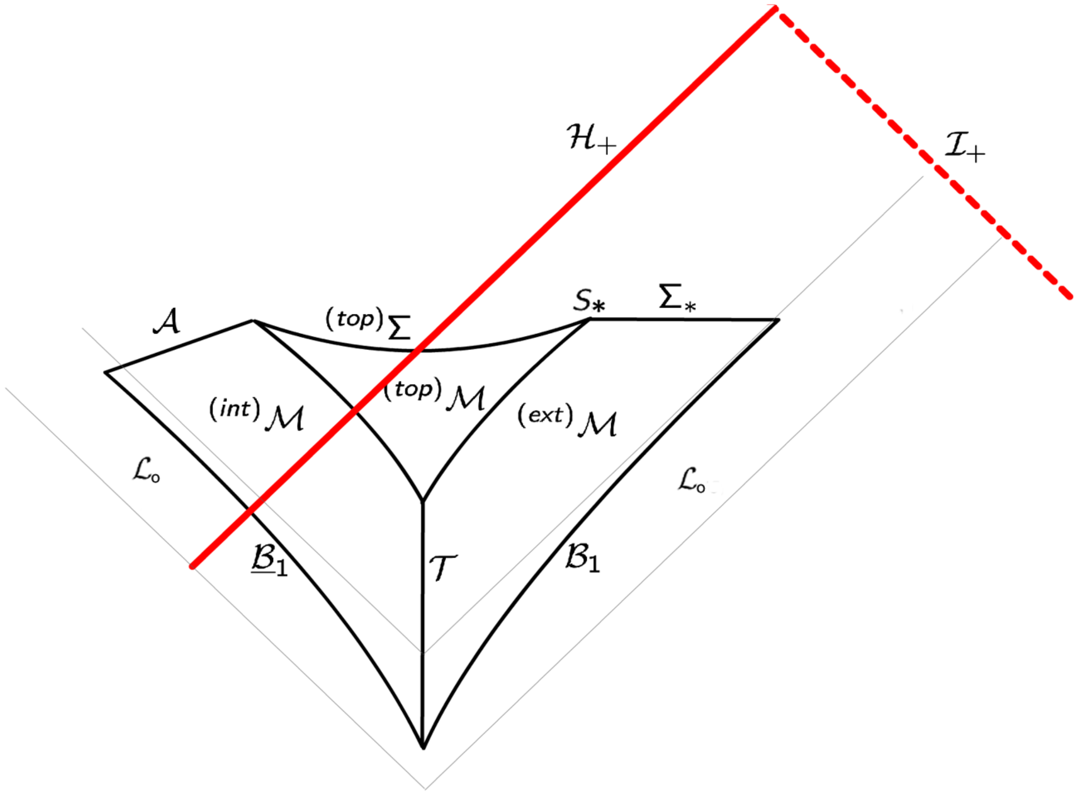
$$(\operatorname{div}' f)_{\ell=1} = \Lambda, \qquad (\operatorname{div}' \underline{f})_{\ell=1} = \underline{\Lambda}.$$

3. GCM hypersurfaces

GCM hypersurfaces

The last slice is foliated by these GCM spheres:

$$\Sigma_* = \bigcup \mathbf{S}'(\Psi(s), s, \Lambda(s), \underline{\Lambda}(s))$$



Choice of Λ, $\underline{\Lambda}$ and Ψ

Denoting $\nu := e_3 + b e_4$ which is tangent to Σ_*

- We only have elliptic equations for ${\not{d}}_2^* \eta$ and ${\not{d}}_2^* \underline{\xi}$ instead of equations of η and $\underline{\xi}$
- The kernel of ${\not{d}}_2^*$ consist of the basis of $\ell = 1$ modes
- Freedom to choose Ψ corresponds to freedom to fix $\overline{b}$

Idea: Choose Λ, $\underline{\Lambda}$ and Ψ s.t. $(\mathrm{div}\eta)_{\ell=1}$, $(\mathrm{div}\underline{\xi})_{\ell=1}$ and $\overline{b}$ take Schwarzchild values:

$$(\mathrm{div}\eta)_{\ell=1} = 0, \qquad (\mathrm{div}\underline{\xi})_{\ell=1} = 0, \qquad \overline{b} = -1 - \frac{2m}{r}$$

Transport systems

Recall the transformation formulae:

$$\eta' = \eta + \frac{1}{2}\nabla_3' f + F \cdot \Gamma + \ldots$$

$$\underline{\xi}' = \underline{\xi} + \frac{1}{2}\nabla_3' \underline{f} + F \cdot \Gamma + \ldots$$

and

$$\Lambda = (\mathrm{div} f)_{\ell=1}, \qquad \underline{\Lambda} = (\mathrm{div} \underline{f})_{\ell=1}$$

We deduce

$$\nu(\Lambda) \sim (\mathrm{div}(\nu f))_{\ell=1} \sim (\mathrm{div}(\nabla_3' f))_{\ell=1} \sim (\mathrm{div}\eta)_{\ell=1},$$

$$\nu(\underline{\Lambda}) \sim \big(\mathrm{div}(\nu \underline{f})\big)_{\ell=1} \sim \big(\mathrm{div}(\nabla_3' \underline{f})\big)_{\ell=1} \sim (\mathrm{div}\underline{\xi})_{\ell=1}$$

ODE systems

Along the characteristic of ν:

$$\Lambda'(s) = (\mathrm{div}'\eta')_{\ell=1} + F(\Lambda, \underline{\Lambda}, \Psi),$$

$$\underline{\Lambda}'(s) = (\mathrm{div}'\underline{\xi}')_{\ell=1} + G(\Lambda, \underline{\Lambda}, \Psi),$$

$$\Psi'(s) = \overline{b} + 1 + \frac{2m'}{r'} + H(\Lambda, \underline{\Lambda}, \Psi)$$

Solving the ODE systems

$$\Lambda'(s) = F(\Lambda, \underline{\Lambda}, \Psi), \qquad \underline{\Lambda}'(s) = G(\Lambda, \underline{\Lambda}, \Psi), \qquad \Psi'(s) = H(\Lambda, \underline{\Lambda}, \Psi)$$

to find Λ, $\underline{\Lambda}$ and Ψ s.t.

$$(\mathrm{div}'\eta')_{\ell=1} = 0, \qquad (\mathrm{div}'\underline{\xi}')_{\ell=1} = 0, \qquad \overline{b} + 1 + \frac{2m'}{r'} = 0.$$

Construction of GCM hypersurfaces

Theorem [S. 22']. Let $\mathcal{R}$ be a fixed spacetime region satisfying the same conditions as [Klainerman-Szeftel 19']. Then, there exists a unique GCM hypersurface $\Sigma_* = \bigcup \mathbf{S}'(\Psi(s), s, \Lambda(s), \underline{\Lambda}(s))$, which is a combination of GCM spheres s.t.

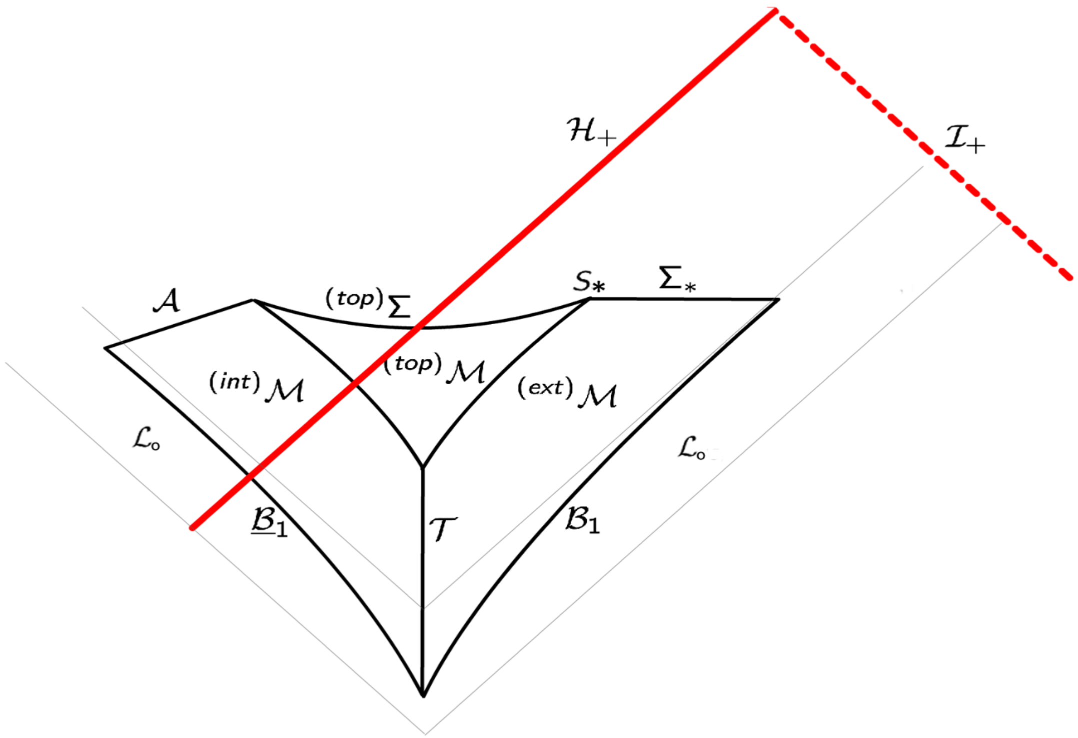
$$\operatorname{tr}\chi' - \frac{2}{r'} = 0, \quad \left(\operatorname{tr}\underline{\chi}' + \frac{2\Upsilon'}{r'}\right)_{\ell\geq 2} = 0, \quad \left(\mu' - \frac{2m'}{(r')^3}\right)_{\ell\geq 2} = 0,$$

and

$$(\operatorname{div}'\eta')_{\ell=1} = 0, \qquad (\operatorname{div}'\underline{\xi}')_{\ell=1} = 0, \qquad \overline{b} = -1 - \frac{2m'}{r'}.$$

Apply to Kerr stability

The last slice is foliated by these GCM spheres starting from the last GCM sphere S_*. In particular, the gauge is initialized from the future with no reference to the initial data. See Klainerman-Szeftel 21' (arXiv:2104.11857).



Thanks for your attention!

Appendix

Spherical harmonics

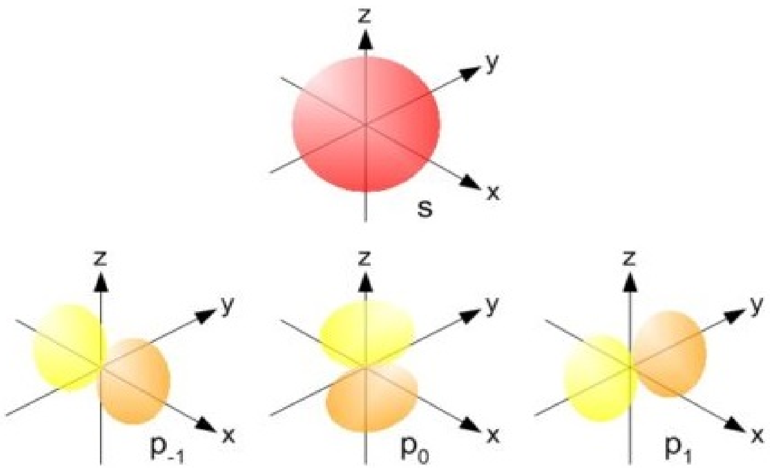
Functions Y_ℓ^m, $-\ell \leq m \leq \ell$ defined on 2–sphere S satisfying

$$\left(\Delta^S + \frac{\ell(\ell+1)}{r^2}\right) Y_\ell^m = 0$$

In the spherical coordinates coordinates:

$$Y_0^0 = 1,$$

$$Y_1^{-1} = \cos\theta, \qquad Y_1^0 = \sin\theta\cos\varphi, \qquad Y_1^1 = \sin\theta\sin\varphi$$



Basis of $\ell = 1$ modes

A basis of $\ell = 1$ modes: Scalar functions $J^{(p)} : S \to \mathbb{R}$ for $p = -, 0, +$ defined on a topological 2–sphere S satisfying:

$$
\begin{aligned}
(r^2 \Delta^S + 2) J^{(p)} &= O(\epsilon), \\
\frac{1}{|S|} \int_S J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\epsilon), \\
\frac{1}{|S|} \int_S J^{(p)} &= O(\epsilon),
\end{aligned}
$$

The $\ell = 1$ modes of scalar λ and 1–form f:

$$
\begin{aligned}
(\lambda)_{\ell=1} &:= \left\{ \int_S J^{(p)} \lambda, \quad p \in \{0, +, -\} \right\}, \\
(f)_{\ell=1} &:= \left\{ \int_S J^{(p)} \mathrm{div}^S(f), \quad p \in \{0, +, -\} \right\}
\end{aligned}
$$

Elliptic equations on Σ_*

Codazzi equations:

$$\mathrm{div}\underline{\widehat{\chi}} = \nabla \mathrm{tr}\,\underline{\chi} + \underline{\chi}\cdot\zeta - \mathrm{tr}\,\underline{\chi}\,\zeta + \underline{\beta},$$
$$\mathrm{div}\widehat{\chi} = \nabla \mathrm{tr}\,\chi - \chi\cdot\zeta + \mathrm{tr}\,\chi\,\zeta - \beta$$

Additional equations for $\underline{\omega}, \eta$ and $\underline{\xi}$:

$$2 \not{d}_2^* \not{d}_1^* \not{d}_1 \not{d}_2 \not{d}_2^* \eta = -\not{d}_2^* \not{d}_1^* \not{d}_1 \nabla_3 \nabla \mathrm{tr}\,\chi + \frac{2}{r}\nabla_3 \not{d}_2^* \not{d}_1^* \mu - \frac{4}{r}\not{d}_2^* \not{d}_1^* \mathrm{div}\underline{\beta} + \Gamma\cdot\Gamma,$$
$$2 \not{d}_2^* \not{d}_1^* \not{d}_1 \not{d}_2 \not{d}_2^* \underline{\xi} = \nabla_3 (\not{d}_2^* \not{d}_2 + 2K) \not{d}_2^* \not{d}_1^* \mathrm{tr}\,\underline{\chi} + \frac{2}{r}\nabla_3 \not{d}_2^* \not{d}_1^* \mu - \frac{4}{r}\not{d}_2^* \not{d}_1^* \mathrm{div}\underline{\beta} + \Gamma\cdot\Gamma,$$
$$2\nabla\underline{\omega} = \frac{1}{r}\underline{\xi} - \nabla_3\zeta - \underline{\beta} + \frac{1}{r}\eta + \Gamma\cdot\Gamma,$$
$$2\overline{\underline{\omega}} = r\overline{\mu} + \Gamma\cdot\Gamma$$

Curvature components α and $\underline{\alpha}$

Curvature transformation formulae:

$$\alpha' = \alpha + F \cdot \beta + O(F^2) \cdot (\rho, {}^*\rho) + \ldots$$

$$\underline{\alpha}' = \underline{\alpha} - F \cdot \underline{\beta} + O(F^2) \cdot (\rho, {}^*\rho) + \ldots$$

In Boyer-Lindquist coordinates:

$$\alpha,\ \beta,\ {}^*\rho,\ \underline{\beta},\ \underline{\alpha} = O(\epsilon), \qquad \rho + \frac{2m}{r^3} = O(\epsilon)$$

Gauge dependence of α and $\underline{\alpha}$ are higher order:

$$\alpha' - \alpha = O(\epsilon^2), \qquad \underline{\alpha}' - \underline{\alpha} = O(\epsilon^2)$$

α and $\underline{\alpha}$ can be treated independent on modulation procedure, see Giorgi-Klainerman-Szeftel 22' (arXiv:2205.14808)