On the linear stability of Kerr black holes

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Einstein vacuum equation

We are interested in the global behavior of solutions of

$$\text{Ric}(g) = 0,$$

where $g$ is a Lorentzian metric $(+−−−)$ on a 4-manifold $M$.

Here: study perturbations of special solutions.
Special solutions of $\text{Ric}(g) = 0$

1. Minkowski space.

$$M = \mathbb{R}_t \times \mathbb{R}^3,$$

$$g_{(0,0)} = dt^2 - dx^2 = dt^2 - dr^2 - r^2 g_{S^2}.$$ 

2. Schwarzschild black holes (mass $m > 0$).

$$M = \mathbb{R}_t \times (0, \infty)_r \times S^2,$$

$$g_{(m,0)} = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 g_{S^2}$$

$$= g_{(0,0)} + O(r^{-1}).$$
Illustration of the Schwarzschild metric

\[ b_0 = (m_0, 0), \quad g_{b_0} = \left(1 - \frac{2m_0}{r}\right) dt^2 - \left(1 - \frac{2m_0}{r}\right)^{-1} dr^2 - r^2 g_{S^2}. \]

Regge-Wheeler coordinate: \( r_* = r + 2m_0 \log(r - 2m_0) \).
\( t_* = t + r_* \) near \( \mathcal{H}^+ (r = 2m_0) \), \( t_* = t - r_* \) near \( \mathcal{I}^+ \).
\( M = \mathbb{R}_{t_*} \times X, \quad X = [r_-, \infty) \times S^2, \ r_\in (0, 2m_0). \)
Special solutions of $\text{Ric}(g) = 0$, continued

3. Kerr black holes

$(\text{mass } m > 0, \text{ angular momentum } a \in \mathbb{R}^3, a = |a|)$.

$$g_{(m,a)} = \frac{\Delta_b - a^2 \sin^2 \theta}{\varrho_b^2} dt^2 + \frac{4amr \sin^2 \theta}{\varrho_b^2} dtd\varphi$$

$$- \frac{(r^2 + a^2)^2 - \Delta_b a^2 \sin^2 \theta}{\varrho_b^2} \sin^2 \theta d\varphi^2 - \frac{r^2 + a^2 \cos^2 \theta}{\Delta_b} dr^2 - d\theta^2$$

$$= g_{(m,0)} + \mathcal{O}(r^{-2}),$$

$$\Delta_{(m,a)} = r^2 - 2mr + a^2, \quad \varrho_{(m,a)}^2 = r^2 + a^2 \cos^2 \theta.$$

Consider slowly rotating Kerr black holes:

$$b := (m, a) \approx b_0 = (m_0, 0) \text{ on } M = \mathbb{R}_{t_*} \times X.$$

$g_b$ for such $b$ is a smooth family of stationary metrics on $M$. 

Kerr solution continued

- The Kerr metric is asymptotically Minkowskian.
- There exist trapped null geodesics. $r$-normally hyperbolic trapping for each $r$ (stable property with respect to perturbations).

- There doesn’t exist any global timelike Killing vector field outside the black hole. Consequence: no conserved positive quantity for the wave equation.

Analogous solution for positive cosmological constant: De Sitter Kerr metric. The De Sitter Kerr metric is asymptotically De Sitter.

Only special solutions or real?
Initial value problem for $\text{Ric}(g) = 0$

Given on $\Sigma = t^{-1}(0) \subset M$:

- $\gamma$: Riemannian metric on $\Sigma$,
- $k$: symmetric 2-tensor on $\Sigma$.

Find:

- Lorentzian metric $g$ on $M$, $\text{Ric}(g) = 0$,
- $\tau(g) := (-g|_{\Sigma}, \Pi^g_{\Sigma}) = (\gamma, k)$.

Necessary and sufficient for local existence: constraint equations on $(\gamma, k)$. (Choquet-Bruhat '52.)

Example

For $(\gamma, k) = (\gamma_b, k_b) := \tau(g_b)$, the solution of the initial value problem is $g_b$. 

Kerr black hole stability

Kerr:

Theorem (Klainerman, Szeftel, Giorgi, Shen '22)

The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a Kerr \((a_0, m_0) = b_0\) initial data set, for sufficiently small \(\frac{|a_0|}{m_0}\), has a complete future null infinity \(\mathcal{I}^+\) and converges in its causal past \(J^- (\mathcal{I}^+)\) to another nearby Kerr spacetime with parameters \(b_f\) close to the initial ones \(b_0\).

De Sitter Kerr:

Theorem (Hintz, Vasy '16)

In an equivalent situation for De Sitter Kerr there exists \(g\) such that

\[ g = g_{bf} + \tilde{g}, \quad |\tilde{g}| \lesssim e^{-\beta t_*}, \quad \beta > 0. \]
\[ \Sigma = t^{-1}(0) \]

\[ \approx g_{b_0} \]

\[ t_\ast = \infty \]

\[ g = g_{b_f} \]
$\Lambda > 0$ versus $\Lambda = 0$.

Toy model of linearized Einstein equations around (De Sitter) Schwarzschild: wave equation on scalars:

$$(\partial_t^2 + P)u = 0.$$ 

- De Sitter Schwarzschild: $P$ similar to a Laplace Beltrami operator on a manifold with two asymptotically hyperbolic ends: meromorphic extension of the resolvent on suitable spaces (Mazzeo-Melrose, Guillarmou).

- Schwarzschild: one asymptotically euclidean end: close to zero, resolvent only $H^k$ down to the real axis (Bony-H., Guillarmou-Hassell, Wunsch-Vasy, Vasy,...)
Kerr stability: main issues

1. Find final black hole parameters \((m_f, a_f)\).
2. Diffeomorphism invariance: \(\text{Ric}(g) = 0 \Rightarrow \text{Ric}(\Phi^*g) = 0\).
   - gauge fixing;
   - track location of black hole in chosen gauge.
3. Because of the weak decay for the linearized problem, the precise structure of the nonlinearity is needed.

In Hintz–Vasy: use Newton-type iteration scheme; naively \(g_0 = g_{b_0}\),

\[
\begin{cases}
D_{g_0} \text{Ric}(h_0) = -\text{Ric}(g_0), \\
\text{initial data for } h_0 \text{ on } \Sigma,
\end{cases}
\Rightarrow g_1 = g_0 + h_0, \text{ etc.}
\]

Idea: read off improved guess of final black hole parameters and location/velocity from asymptotic behavior of \(h_0\).
Linear stability (modulo gauge)

Consider black hole parameters $b \approx b_0 = (m_0, 0)$.

Theorem (H.–Hintz–Vasy '19)

Let $\gamma', k'$ be symmetric 2-tensors on $\Sigma = t^{-1}(0)$ satisfying the linearized constraint equations and

$|\gamma'| \lesssim r^{-1-\alpha}, |k'| \lesssim r^{-2-\alpha}, 0 < \alpha < 1$,

(and similar bounds for derivatives). Then there exists a symmetric 2-tensor $h$ on $M$ such that

$$D_{g_b} \text{Ric}(h) = 0, \quad D_{g_b} \tau(h) = (\gamma', k'),$$

which decays to a linearized Kerr metric,

$$h = g'_b(b') + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}, \quad \left( g'_b(b') := \frac{d}{ds} g_{b+sb'} \bigg|_{s=0} \right).$$
Gauge fixing

Eliminate diffeomorphism invariance: impose extra condition on $g$:

$$W(g) = \Box_{g, gb} 1 \ (= 1\text{-form in } g, \partial g) = 0.$$  

Then (‘DeTurck trick’):

$$\begin{cases} 
\text{Ric}(g) = 0, \\
W(g) = 0, \quad \iff \quad \text{IVP for } P(g) := \text{Ric}(g) - \delta^*_g W(g) = 0.
\end{cases}$$

initial data

Linearized version:

$$\begin{cases} 
D_{gb} \text{Ric}(h) = 0, \\
D_{gb} W(h) = 0, \quad \iff \quad \text{IVP for } D_{gb} P(h) \left( \approx \frac{1}{2} \Box_{gb} h \right) = 0.
\end{cases}$$

initial data
Main theorem

Let $L_b := D_{g_b} P$. Study $L_b h = 0$ with general initial data.

Theorem (H.–Hintz–Vasy ’19)
Let $\alpha \in (0, 1)$, and let $h_0, h_1 \in C^\infty(\Sigma; S^2 T^*_\Sigma M)$,

$$|h_0| \lesssim r^{-1-\alpha}, \quad |h_1| \lesssim r^{-2-\alpha}, \quad 0 < \alpha < 1$$

(and similar bounds for derivatives). Let

$$\begin{cases} 
L_b h = 0, \\
(h|_{\Sigma}, \mathcal{L}_{\partial_t} h|_{\Sigma}) = (h_0, h_1).
\end{cases}$$

Then there exist $b' \in \mathbb{R} \times \mathbb{R}^3$ and $V \in$ fixed 8-dimensional space of vector fields on $M$ such that

$$h = g'_b(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}.$$
Main theorem, continued

\[ h = g'_b(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha^+}. \]

Here,

\[ V \in \text{span}\{ \text{asymptotic translations: } \partial_{x^i} + \mathcal{O}(r^{-1}), \]
\[ \text{asymptotic boosts: } t\partial_{x^i} - x^i\partial_t + \text{l.o.t.} \} \]
\[ + \text{two non geometric vector fields.} \]

Can read off:

- change of black hole parameters,
- movement of black hole in chosen gauge.

Remark: upon given up one order of decay one can choose \( V \) in a 6 dimensional space consisting only of the vector fields with clear geometric interpretation.
Prior work

- Andersson–Bäckdahl–Blue–Ma ’19: initial data with fast decay ($\alpha > 5/2$; then $b' = 0$)
- Dafermos–Holzegel–Rodnianski ’16: $b = b_0$
- Johnson, Hung–Keller–Wang ’16–’18: $b = b_0$
- Finster–Smoller ’16 (no decay rate).
Strategy of proof

Recast as

\[ L_b h = f. \]

Work on spectral side:

\[ h(t_*, x) = \frac{1}{2\pi} \int_{\text{Im } \sigma = C} e^{-i\sigma t_*} \hat{L}_b(\sigma)^{-1} \hat{f}(\sigma, x) \, d\sigma. \]

Shift contour to \( C = 0. \)

Uses/is related to: Melrose, Vasy–Zworski, Vasy; Wunsch–Zworski, Dyatlov, Hintz, Vasy, Guillarmou–Hassell, Bony–H.
1st step: Fredholm setting.

Uses Melrose ’93, radial point estimates Melrose ’94, Vasy ’13, non elliptic Fredholm framework Vasy ’13.

Proposition

Let $0 \leq a < m$. There exists $s_2 > 0$ with the following property.

1. Let $s > s_2$, $\ell < -1/2$, $s + \ell > -1/2$. Then the operators

$$\widehat{L}_b(\sigma) : \{ u \in \bar{H}^{s,\ell}_b(X) : \widehat{L}_g(\sigma) u \in \bar{H}^{s-1,\ell+2}_b(X) \} \rightarrow \bar{H}^{s-1,\ell+2}_b(X)$$

are Fredholm operators of index 0 for $\text{Im} \sigma \geq 0$, $\sigma \neq 0$.

2. If moreover $\ell \in (-3/2, -1/2)$, then

$$\widehat{L}_b(0) : \{ u \in \bar{H}^{s,\ell}_b(X) : \widehat{L}_g(0) u \in \bar{H}^{s-1,\ell+2}_b(X) \} \rightarrow \bar{H}^{s-1,\ell+2}_b(X)$$

is Fredholm of index 0.
2nd step. Mode analysis for the gauged fixed linearized Einstein equations

- $\widehat{L}_b(\sigma)$ invertible for $\text{Im}\sigma \geq 0$, $\sigma \neq 0$.
- Precise description of the kernel of $\widehat{L}_b(0)$, $\ell \in (-3/2, -1/2)$.

The mode analysis of $\widehat{L}_b(\sigma)$ follows from the mode analysis of
- ungauged linearized Einstein equations,
- the 1—form wave operator.

One also has to consider generalized mode solutions

$$h = \sum_{j=0}^{d} t^j h_j, \ h_j \in \overline{H}_{-1/2}^{-}, \ L_{b_0} h = 0.$$ 

There exist such solutions with $h_d \neq 0$, $d \geq 2$. 
Mode stability without gauge

Theorem (Andersson-H-Whiting '22)

Let $0 \leq a < m$, $\sigma \in \mathbb{C}$, $\text{Im}\, \sigma \geq 0$, and suppose

$$\dot{g} = e^{-i\sigma t} h, \ h \in \tilde{H}_b^{\infty, \ell}(X; S^2 \tilde{sc} T^* X), \ \ell \in (-3/2, -1/2)$$

is an outgoing mode solution of the linearized Einstein equation

$$D_g\text{Ric}(\dot{g}) = 0.$$

If $\sigma = 0$, then there exist parameters $\dot{m} \in \mathbb{R}$, $\dot{a} \in \mathbb{R}^3$, and a 1-form $\omega \in \tilde{H}_b^{\infty, \ell-1}(X; \tilde{sc} T^* X)$, such that

$$\dot{g} - \dot{g}_{(m,a)}(\dot{m}, \dot{a}) = \delta^*_g \omega.$$

If $\sigma \neq 0$, then $\dot{g} = \delta^*_g \omega$ for a suitable outgoing 1-form $\omega$. 
Mode stability with gauge

**Theorem (Andersson-H-Whiting ’22)**

Let $0 < a < m$.

1. **For** $\text{Im} \, \sigma \geq 0$, $\sigma \neq 0$, the operator 
   
   $$ \widehat{L}_{gb}(\sigma) : \{ u \in \tilde{H}_b^{s,\ell}(X; S^2 \, \text{sc} \, T^* X) : \widehat{L}_{gb}(\sigma)u \in \tilde{H}_b^{s-1,\ell+2}(X; S^2 \, \text{sc} \, T^* X) \} $$
   
   $$ \rightarrow \tilde{H}_b^{s-1,\ell+2}(X; S^2 \, \text{sc} \, T^* X) $$

   **is invertible** when $s > s_2$, $\ell < -\frac{1}{2}$, $s + \ell > -\frac{1}{2}$.

2. **If** moreover $\ell \in (-\frac{3}{2}, -\frac{1}{2})$, the **zero energy operator**

   $$ \widehat{L}_{gb}(0) : \{ u \in \tilde{H}_b^{s,\ell}(X; S^2 \, \text{sc} \, T^* X) : \widehat{L}_{gb}(0)u \in \tilde{H}_b^{s-1,\ell+2}(X; S^2 \, \text{sc} \, T^* X) \} $$

   $$ \rightarrow \tilde{H}_b^{s-1,\ell+2}(X; S^2 \, \text{sc} \, T^* X) $$

   **has 7-dimensional kernel and cokernel.**
Mode analysis with and without gauge

\[ \sigma \neq 0 \]

\[ L_b h = D_{gb} \text{Ric}(h) - \delta^*_{gb} D_{gb} W(h) = 0, \quad h = e^{-i\sigma t*} h_0, \quad (1) \]

\[ h_0 = h_0(r, \omega) \] fulfills outgoing radiation condition. Apply linearized second Bianchi identity

\[ \Box_{b,1}(D_{gb} W(h)) = 0. \]

We show absence of modes for the \( 1 \)-form wave operator:

\[ D_{gb} W(h) = 0. \quad (2) \]

Put into (1):

\[ D_{gb} \text{Ric}(h) = 0. \]

\[ \Rightarrow h \text{ is pure gauge } h = \delta^*_{gb} \omega. \] Putting this into (2) gives

\[ \Box_{b,1} \omega = 0. \]

It follows \( \omega = 0. \)
Proof of mode stability

$\sigma = 0$.

- The Teukolsky scalars are zero (Whiting).
- **Gauge invariants.** For linearized perturbations of Kerr with vanishing Teukolsky scalars, the only non vanishing gauge invariants are $\mathbb{I}_\xi, \mathbb{I}_\zeta$ and they have in this case exactly the form they have for the linearized Plebanski-Demianski family.
- The set of gauge invariants \{Teukolsky scalars, linearized Ricci tensor, $\mathbb{I}_\xi, \mathbb{I}_\zeta$\} is complete (Aksteiner, Andersson, Bäckdahl, Khavkine, Whiting '19), it follows that up to gauge the perturbation is a Plebanski-Demianski line element.
- For all $\dot{a} \in \mathbb{R}^3$, there exists $\lambda(\dot{a}) \in \mathbb{R}$, a 1–form $\omega \in \bar{H}_b^{\infty,\ell-1}(X; \widetilde{S^{2\text{sc}} T^* X})$ and $g_0 \in C^\infty(\partial X; S^{2\text{sc}} T^{* \partial X} X)$ such that
  \[
  \dot{g} - \delta_{gb}^{* \omega} - \dot{g}_b(\lambda(\dot{a}), \dot{a}) - \frac{1}{r} g_0 \in O(r^{-2+}).
  \]
- Asymptotic behavior considerations eliminate the nut and acceleration parameters.
3rd step: Constraint dumping

Gauge freedom (Constraint dumping)
Gundlach et al ’05.
Zeroth order modification of $\delta^*_g$:

$$\tilde{\delta}^*_g = \delta^*_g + E, \quad E\omega = \gamma(2c \otimes_s \omega - g^{-1}(c, \omega)g),$$

$c$ being a stationary 1–form with compact spatial support.
Replacing $\delta^*_g$ by $\tilde{\delta}^*_g$ gives a modified gauge fixed linear operator.
For $\gamma \neq 0$ small no quadratically growing generalized modes exist.
All other important properties of $L_b$ remain unchanged.
4\textsuperscript{th} step: Regularity of the resolvent at high frequencies

Proposition

Let $\ell < -\frac{1}{2}$, and $s > \frac{5}{2}$, $s + \ell > -\frac{1}{2} + m$, $m \in \mathbb{N}$. Let $\sigma_0 > 0$. Then for $\text{Im} \sigma \geq 0$, $|\sigma| > \sigma_0$, $h = |\sigma|^{-1}$, the operator

$$\partial^m_{\sigma} \hat{L}_b(\sigma)^{-1}: \bar{H}^{s,\ell+1}_{b,h} \to h^{-m} \bar{H}^{s-m,\ell}_{b,h}$$

is uniformly bounded.

Main issue at high frequencies ($|\text{Re} \sigma| \to \infty$): Trapping, see Wunsch-Zworski '11, Dyatlov '16, Hintz '17.

Remark: the trapping is $r-$ normally hyperbolic for $0 \leq a < m$ (Dyatlov '15), therefore the above proposition should hold for $0 \leq a < m$. 
5th step: Structure of the resolvent at low frequencies

▶ There exists $V \in \Psi^{-\infty}(X^o; S^2 T^* X^o)$,

$$\tilde{L}_b(\sigma) := \hat{L}_b(\sigma) + V : \mathcal{X}_{b^*}^{s,\ell}(\sigma) \to \bar{H}^{s-1,\ell+2}_b$$

is invertible for $\sigma \in \mathbb{C}$, $\text{Im} \sigma \geq 0$, $b$ close to $b_0$.

▶ Kernel of $\hat{L}_b(0)$ :

$$\mathcal{K}_b = \mathcal{K}_{b,s} \oplus \mathcal{K}_{b,v}, \quad \tilde{\mathcal{K}}_{b,o} = \tilde{L}_b(0)\mathcal{K}_{b,o}(0), \ o = s, v.$$

▶ There exists $\Pi_b^\perp : \bar{H}^{s-1,\ell+2}_b \to \bar{H}^{s-1,\ell+2}_b$ of rank 7 which depends continuously on $b$ near $b_0$, and satisfies

$$\langle \Pi_b f, h^* \rangle = 0 \quad \forall \ h^* \in \mathcal{K}_b^*, \ \Pi_b = 1 - \Pi_b^\perp.$$

▶ Consider $\hat{L}_b(\sigma)\tilde{L}_b(\sigma)^{-1} : \bar{H}^{s-1,\ell+2}_b \to \bar{H}^{s-1,\ell+2}_b$.

Decomposition

domain: $\bar{H}^{s-1,\ell+2}_b \cong \tilde{\mathcal{K}}^\perp \oplus \tilde{\mathcal{K}}_{b,s} \oplus \tilde{\mathcal{K}}_{b,v}$,

target: $\bar{H}^{s-1,\ell+2}_b \cong \text{ran} \ \Pi_b \oplus \mathcal{R}_s^\perp \oplus \mathcal{R}_v^\perp$. 
\[
\hat{L}_b(\sigma)\hat{L}_b(\sigma)^{-1} = \begin{pmatrix}
L_{00} & \sigma \tilde{L}_{01} & \sigma \tilde{L}_{02} \\
\sigma \tilde{L}_{10} & \sigma^2 \tilde{L}_{11} & \sigma^2 \tilde{L}_{12} \\
\sigma \tilde{L}_{20} & \sigma^2 \tilde{L}_{21} & \sigma \tilde{L}_{22}
\end{pmatrix}.
\]

We then obtain:

\[
\hat{L}_b(\sigma)^{-1} = \sigma^{-2} R_2 + \sigma^{-1} R_1 + L_b^-,
\]

where the range of $R_1$, $R_2$ is explicit (linearized Kerr, pure gauge).

Regularity of $L_b^-$ (uses Vasy'19):

**Proposition**

*Let* $\ell \in (-\frac{3}{2}, -\frac{1}{2})$, $\epsilon \in (0, 1)$, $\ell + \epsilon \in (-\frac{1}{2}, \frac{1}{2})$, *and* $s - \epsilon > \frac{7}{2}$. *Then we have*

\[
L_b^- \in H^{3/2-\epsilon}((-\sigma_0, \sigma_0); \mathcal{L}(\bar{H}^{s-1,\ell+2}_b, \bar{H}^{s-\text{max}(\epsilon,1/2),\ell+\epsilon-1}_b))
\]

**Main issue**: behavior of the metric at infinity.
Thank you for your attention!