Energies in Fourth Order Gravity

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Collaboration with Rodrigo Avalos (PU), Jorge H. Lira (UFC) and Paul Laurain (UP)

1. R. Avalos, J.H. Lira and N. Marque, *Energy in fourth order gravity*, arXiv:2102.00545,
2. R. Avalos, P. Laurain and J.H. Lira, *A positive energy theorem for fourth-order gravity*, by Calc. Var.,
3. R. Avalos, P. Laurain, J.H. Lira and N. Marque *Rigidity Theorems for Asymptotically Euclidean Q-singular Spaces*, arXiv:2204.03607 .

We study gravitational theories on globally hyperbolic spacetimes $(V^{n+1}, \bar{g}) = (M^n, g_t) \times \mathbb{R}$.

In GR the gravitational action in vacuum is described by the Einstein-Hilbert functional

$$EH(\bar{g}) = \int_V R_{\bar{g}} d\mathrm{vol}_{\bar{g}},$$

and the Einstein equation

$$G_{\bar{g}} := \mathrm{Ricc}_{\bar{g}} - \frac{1}{2} R_{\bar{g}} \bar{g} = 0.$$

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For α, $\beta \in \mathbb{R}$, we consider the Fourth Order Gravitational Lagrangian :

$$S(\bar{g}) = \int_V \left(\alpha R_{\bar{g}}^2 + \beta \langle \mathrm{Ric}_{\bar{g}}, \mathrm{Ric}_{\bar{g}} \rangle_{\bar{g}}\right) d\mathrm{vol}_{\bar{g}},$$

with Euler-Lagrange equations :

$$A_{\bar{g}} := \beta \Box_{\bar{g}} \mathrm{Ric}_{\bar{g}} + (\frac{1}{2}\beta + 2\alpha) \Box_{\bar{g}} R_{\bar{g}}\, \bar{g} - (2\alpha + \beta) \bar{\nabla}^2 R_{\bar{g}} - 2\beta \mathrm{Ric}_{\bar{g}}.\mathrm{Riem}_{\bar{g}}$$

$$+ 2\alpha R_{\bar{g}} \mathrm{Ric}_{\bar{g}} - \frac{1}{2}\alpha R_{\bar{g}}^2 \bar{g} - \frac{1}{2}\beta \langle \mathrm{Ric}_{\bar{g}}, \mathrm{Ric}_{\bar{g}} \rangle_{\bar{g}} \bar{g} = 0.$$

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Physical Motivations

Some observations are not coherent with GR predictions (galactic rotational curves)

Either

- We modify the data (Dark Matter)
- We modify the theory (Fourth order gravity, conformal gravity, Lovelock theories)

$S(\bar{g})$ is a good candidate :

- EH is an elastic energy, S is a higher order elastic energy
- $G_{\bar{g}} = 0 \implies A_{\bar{g}} = 0$

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Case $2\alpha + \beta = 0$

$A = 0$ becomes

$$\begin{cases} \Box_{\bar g} R_{\bar g} = 0 \\ \Box_{\bar g} G_{\bar g\,\mu\nu} + 2\left(\mathrm{Riem}_{\bar g}{}^{\tau}{}_{\mu\lambda\nu} - \frac{\mathrm{Ric}_{\bar g}{}^{\tau}_{\lambda}}{4}\bar g_{\mu\nu}\right) G_{\bar g}{}^{\lambda}_{\tau} = 0. \end{cases}$$

- Finding a solution splits into $G_{\bar g} = T$ with T an inertial energy-momentum tensor solution of $\Box_{\bar g} T + 2\left(\mathrm{Riem}_{\bar g} - \frac{\mathrm{Ric}_{\bar g}}{4}\bar g\right) T = 0$

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$3\alpha + \beta = 0$

$$S(\bar{g}) = \frac{3}{2}\int_V |W_{\bar{g}}|^2\, d\mathrm{vol}_{\bar{g}} + 48\pi^2\chi(V),$$

W is the Weyl tensor and $\chi(V)$ is topological. For $n = 3$, $B = \int_V |W_{\bar{g}}|^2\, d\mathrm{vol}_{\bar{g}}$ is a conformal invariant, and so is S.

- Conformal gravity/Bach-flat spaces
- Fiedler-Schimming-Mannheim-Kazanas (FSMK) metrics :

$$\bar{g}_{FS}(m, \Lambda, \mu) = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 g_{\mathbb{S}^2}$$

$$f(r) \doteq 1 - 3m\mu - \frac{m}{r} - \mu(3m\mu - 2)r - \frac{\Lambda}{3}r^2$$

- μ is linked to the rotational speed curves in conformal gravity.

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Working spaces

Definition

$(V, \bar{g})$ is an AM spacetime of order τ if there exists a coordinate system at ∞ $\Phi : E_i \to \mathbb{R}^n \backslash \bar{B}$ such that in those coordinates $\bar{g}_{\mu\nu} = \xi_{\mu\nu} + O(|x|^{-\tau})$.

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(M, g) is an AE space of order τ if there exists a coordinate system at $\Phi : M \backslash K \to \mathbb{R}^n \backslash \bar{B}$ such that in those coordinates $g_{ij} = \delta_{ij} + O(|x|^{-\tau})$.

- to consider curvatures issues the decays must be differentiables
- Dynamical approach
- A Lagrangian approach

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Let $(V, \bar{g})$ be an AM Einstein spacetime, and $\hat{\bar{g}}$ another AM Einstein metric with a Killing vector field ζ.
Let $\bar{h} := \bar{g} - \hat{\bar{g}}$ and $\mathcal{P}_{\hat{\bar{g}}}(\bar{h}, \zeta) \doteq (DG_{\hat{\bar{g}}} \cdot \bar{h})(\zeta, \cdot)$.
With Bianchi :

$$\begin{aligned}\mathrm{div}_{\bar{g}}(G_{\bar{g}}) = 0 = \mathrm{div}_{\hat{\bar{g}}+\bar{h}}(G_{\hat{\bar{g}}+\bar{h}}) &= \mathrm{div}_{\hat{\bar{g}}}(G_{\hat{\bar{g}}}) + \mathrm{div}_{\hat{\bar{g}}} DG_{\hat{\bar{g}}} \cdot \bar{h} + o(\bar{h}) \\ &= \mathrm{div}_{\hat{\bar{g}}} DG_{\hat{\bar{g}}} \cdot \bar{h} + o(h),\end{aligned}$$

and thus $\mathrm{div}_{\hat{\bar{g}}}\left(\mathcal{P}_{\hat{\bar{g}}}(\bar{h}, \zeta)\right) = 0$.
If $(V, \bar{g})$ is foliated by Riemannian manifolds (M_t, g_t) :

$$\int_{K_0} \langle \mathcal{P}_{\hat{\bar{g}}}(\bar{h}, \zeta), \hat{n} \rangle_{\hat{\bar{g}}} d\mathrm{vol}_{\hat{\bar{g}}} = \int_{K_1} \langle \mathcal{P}_{\hat{\bar{g}}}(\bar{h}, \zeta), \hat{n} \rangle_{\hat{\bar{g}}} d\mathrm{vol}_{\hat{\bar{g}}} + \int_{\text{exterior boundary}} \mathcal{P}_{\hat{\bar{g}}}(\bar{h}, \zeta).\hat{\nu}$$

When the exterior boundary $\to \infty$, we get the conserved quantity

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$$\mathcal{E}_{\hat{\hat{g}}}(\zeta, \bar{h}) = \int_M \langle \mathcal{P}_{\hat{\hat{g}}}(\bar{h}, \zeta), \hat{n} \rangle_{\hat{\hat{g}}} d\mathrm{vol}_{\hat{g}} = \lim_{r \to \infty} \int_{\mathbb{S}_r} \mathcal{Q}(\zeta, \hat{\nu}).$$

Taking $\zeta = \partial_t + O(|x|^{-\tau})$ (+ simplifying hypotheses) yields :

$$m_{ADM} \doteq \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j d\omega_r.$$

Taking ∂_{x_i} yields the momentum, rotations yields the angular momentum etc. . .

These quantities are geometric and control the geometry of M.

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Proposition

Given $(M \times \mathbb{R}, \hat{\hat{g}})$ an AM spacetime satisfying $A_{\hat{\hat{g}}} = 0$ and admitting a Killing field ζ, then the energy associated to a perturbation $\bar{h}$ of $\hat{\hat{g}}$ defined by

$$\mathcal{E}_{\hat{\hat{g}}}(\bar{h}) \doteq \int_M \langle \mathcal{P}_{\hat{\hat{g}}}(\bar{h}, \zeta), \hat{n} \rangle_{\hat{\hat{g}}} d\mathrm{vol}_{\hat{g}},$$

with $\mathcal{P}_{\hat{\hat{g}}}(h, \zeta) \doteq (DA_{\hat{\hat{g}}} \cdot h)(\zeta, \cdot)$,is conserved, provided it is defined. It can be written as an integral at infinity

$$\mathcal{E}_{\hat{\hat{g}}}(M, h) = -\lim_{r \to \infty} \int_{\partial K_r} \mathcal{Q}(\hat{n}, \hat{\nu}) d(\partial K_r).$$

Proposition

If $\hat{g}$ is Einstein with cosmological constant Λ , then

$$\mathcal{Q}_{\tau\mu} = -\Big\{\beta d\mathcal{P}_{\hat{g}\,\tau\mu}^{GR}(\bar{h},\zeta) + 2\beta\left(\mathrm{Ric}'_{\hat{g}}.\bar{h}_{\tau\nu}\hat{\nabla}_{\mu}\zeta^{\nu} - \mathrm{Ric}'_{\hat{g}}.\bar{h}_{\mu\nu}\hat{\nabla}_{\tau}\zeta^{\nu}\right)$$
$$-(2\alpha+\beta)\left(\hat{\nabla}_{\mu}(R'_{\hat{g}}.\bar{h})\zeta_{\tau} - \hat{\nabla}_{\tau}(R'_{\hat{g}}.\bar{h})\xi_{\mu}\right) - (2\alpha-\beta)R'_{\hat{g}}.\bar{h}\nabla_{\tau}\zeta_{\mu}$$
$$-2(4\alpha+\beta)\Lambda\mathcal{Q}_{\hat{g}\ \ \tau\mu}^{GR}(\bar{h},\zeta) - 2\beta\Lambda\left(\bar{h}_{\tau\nu}\hat{\nabla}_{\mu}\zeta^{\nu} - \bar{h}_{\mu\nu}\hat{\nabla}_{\tau}\zeta^{\nu}\right)\Big\}.$$

- Constant sectional curvature case :
  - S. Deser and B. Tekin, Energy in generic higher curvature gravity theories, Phys. Rev. D 67, 084009 (2003).

- With $\hat{\bar{g}} = \xi + O(r^{-\hat{\tau}})$, $\zeta = \partial_t + O(r^{-\hat{\tau}})$, $\bar{g} = \bar{h} + \hat{\bar{g}} = \xi + O(r^{-\tau})$ in ADM formalism (N, X, g) :

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\begin{aligned}
-\mathcal{Q}(\hat{n},\hat{\nu})|_{t=0} = & \left(\frac{3}{2}\beta + 2\alpha\right)\left(\partial_j\partial_i\partial_i g_{aa} - \partial_j\partial_u\partial_i g_{ui}\right)\hat{\nu}^j \\
& + \frac{\beta}{2}\left(\partial_i\ddot{g}_{ji} - \partial_j\ddot{g}_{ii}\right)\hat{\nu}^j \\
& + \frac{\beta}{2}\left(\partial_i\partial_j\dot{X}_i - \partial_i\partial_i\dot{X}_j\right)\hat{\nu}^j + (\beta + 2\alpha)\partial_j\partial_i\partial_i N^2\hat{\nu}^j \\
& - (\beta + 2\alpha)\partial_j\ddot{g}_{ii}\hat{\nu}^j + 2(\beta + 2\alpha)\partial_j\partial_i\dot{X}_i\hat{\nu}^j \\
& + O_1(r^{-\hat{\tau}-3}) + O_1(r^{-(\hat{\tau}+\tau)-3}).
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Definition

When $\bar{g}$ an AM solution of $A_{\bar{g}} = 0$, we define its energy as

$$\mathcal{E}_{\alpha,\beta}(\bar{g}) = \lim_{r\to\infty} \Bigg[\left(\frac{3}{2}\beta + 2\alpha\right) \int_{S_r^{n-1}} \left(\partial_j\partial_i\partial_i g_{aa} - \partial_j\partial_u\partial_i g_{ui}\right) \hat{\nu}^j d\omega_r + \frac{\beta}{2} \int_{S_r^{n-1}} \left(\partial_i \ddot{g}_{ji} - \partial_j \ddot{g}_{ii}\right) \hat{\nu}^j d\omega_r + \frac{\beta}{2} \int_{S_r^{n-1}} \left(\partial_i\partial_j \dot{X}_i - \partial_i\partial_i \dot{X}_j\right) \hat{\nu}^j d\omega_r + (\beta + 2\alpha) \left(\int_{S_r^{n-1}} \partial_j\partial_i\partial_i N^2 \hat{\nu}^j d\omega_r - \int_{S_r^{n-1}} \partial_j \ddot{g}_{ii} \hat{\nu}^j d\omega_r + 2 \int_{S_r^{n-1}} \partial_j\partial_i \dot{X}_i \hat{\nu}^j d\omega_r \right) \Bigg]$$

when the limit exists.

Testing it when $3\alpha + \beta = 0$

- FSMK metrics:

$$\bar{g}_{FS}(m, \Lambda, \mu) = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 g_{\mathbb{S}^2}$$

$$f(r) \doteq 1 - 3m\mu - \frac{m}{r} - \mu(3m\mu - 2)r - \frac{\Lambda}{3}r^2$$

- We can take $\tau < 0$! In particular $n = 3$ we can take $\tau = -1$.
- With $\hat{g} =$ Schwarzschild, $\hat{\tau} = 1$, $\tau = -1$, we compute

$$\mathcal{E}_{\alpha,\beta}(\bar{g}_{FS}) = 8\pi\mu(3m\mu - 2).$$

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If $(M \times \mathbb{R}, \bar{g})$ is stationnary, that is

$$\bar{g} = -N^2 dt^2 + g,$$

then

$$\mathcal{E}_{\alpha,\beta}(\bar{g}) = \lim_{r\to\infty} \left\{ \left(\frac{3}{2}\beta + 2\alpha\right) \int_{S_r^{n-1}} \left(\partial_j\partial_i\partial_i g_{aa} - \partial_j\partial_u\partial_i g_{ui}\right) \hat{\nu}^j d\omega_r \right.$$
$$\left. +(\beta + 2\alpha) \int_{S_r^{n-1}} \partial_j\partial_i\partial_i N^2 \hat{\nu}^j d\omega_r \right\}$$

and in the particular case $2\alpha + \beta = 0$, we get a fully Riemannian expression

$$\begin{aligned} \mathcal{E}(g) &= -\lim_{r\to\infty} \int_{S_r^{n-1}} \left(\partial_j\partial_i\partial_i g_{aa} - \partial_j\partial_u\partial_i g_{ui}\right) \hat{\nu}^j d\omega_r \\ &= -\lim_{r\to\infty} \int_{S_r^{n-1}} \partial_r R_g r^{n-1} d\omega \\ &= -\int_M \Delta_g R_g d\mathrm{vol}_g. \end{aligned}$$

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We wish to work with the Q-curvature :

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|\mathrm{Ric}_g|_g^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2.$$

Proposition

Let (M^n, g) an AE manifold such that

1. There exists a structure at infinity Φ such that $g_{ij} = \delta_{ij} + O_4(r^{-\tau})$, with $\tau > \tau_n \doteq \max\{0, \frac{n-4}{2}\}$;
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Positivity and Rigidity

Theorem

If in addition $Q_g \geq 0$ and $Y([g]) > 0$, then $\mathcal{E}(g) \geq 0$, with equality iff (M, g) is euclidien.

Idea of the proof :

- $Y([g]) > 0$: $\tilde{g} = u^{\frac{4}{n-2}} g$ s.t. $R_{\tilde{g}} = 0$.
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- Then $\frac{n-4}{2}\Phi^{\frac{n+4}{n-4}}Q_g = P_{\tilde{g}}\Phi$ yields $\Delta^2\Phi = 0$, which ensures $g = \delta$.
- For $n = 4$ only the formula for Paneitz changes.

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$$J_g \doteq \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g,$$

$$G_J \doteq J_g - \frac{1}{2} Q_g g \text{ s.t. } \mathrm{div}_g G_J = 0 \text{ and } \mathrm{Tr}_g G_J = \text{ cst } Q_g$$

2^{nd} order	4^{th} order
Scalar curvature	Q curvature
Ric	J
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$$\int_{\Omega_R} \mathrm{div}_g(G_J(X,\cdot))d\mathrm{vol}_g = \int_{\partial\Omega_R} G_J(X,\nu)d\omega_g = \frac{1}{2}\int_{\Omega_R}\langle G_J, \mathcal{L}_X g\rangle_g d\mathrm{vol}_g$$
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Theorem

If (M^n, g) is AE and $J_g = 0$, $Y([g]) > 0$ then $(M^n, g) \cong (\mathbb{R}^n, \cdot)$.

Idea: rigidity in the positive mass theorem But: the decay needs to be high enough

$$\begin{cases} Q_g = \Delta R_g + \text{ quadr. terms} \\ J_g - \frac{1}{n}\mathrm{Tr}_g(J_g) = \Delta \mathrm{Ric} + \nabla^2 R + \text{ quadr. terms} \\ \Delta g = \mathrm{Ric} + \text{ quadr. terms} \end{cases}$$

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Riemannian formalism

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With $g = \delta + h$ an AE metric, we linearize
$R_g = R_\delta + DR_\delta \cdot h + \mathcal{R}(h) = DR_\delta \cdot h + \mathcal{R}(h)$.

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