Stabilité non-linéaire des ondes de raréfaction multi-dimensionnelles

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Outline

- Riemann Problem
- Previous Works
- Main Results
- Difficulties and Novelties

Compressible Euler equations

Isentropic compressible Euler equations for polytropic gas:

$$\begin{cases} (\partial_t + v \cdot \nabla)\rho = -\rho \nabla \cdot v, \\ (\partial_t + v \cdot \nabla) v = -\rho^{-1} \nabla p, \end{cases} \Leftrightarrow \begin{cases} (\partial_t + v \cdot \nabla) c &= -\frac{\gamma-1}{2} c \nabla \cdot v, \\ (\partial_t + v \cdot \nabla) v &= -\frac{2}{\gamma-1} c \nabla c. \end{cases}$$

with $p(\rho) = k_0 \rho^\gamma$ and $\gamma \in (1,3)$ and $k_0 > 0$.

We focus on the **sound waves** by assuming $\mathrm{curl}(v) = 0$ (irrotational).

- Sound speed c given by $c = \sqrt{\frac{dp}{d\rho}} = k_0^{\frac{1}{2}} \gamma^{\frac{1}{2}} \rho^{\frac{\gamma-1}{2}}$.
- On multi-dimensions, the directions count. We use the acoustical metric (defined by the solution) to study sound waves:

$$g = -c^2 dt + \sum_{i=1}^{n} (dx^i - v^i dt)^2.$$

$n = 2$ in this talk.

Riemann's paper on plane waves of gas dynamics

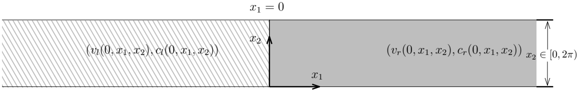
- Riemann's 1860 paper: a series of new (physical) concepts/discoveries
  - shock/rarefaction waves, Riemann invariants, hodograph transformations.
  - Mechanism of shock formations
  - Propagation of singularity
- Many other techniques used in modern (hyperbolic) PDEs.

Riemann problem

The study of the IVP with data consisting of two piecewise constant states:

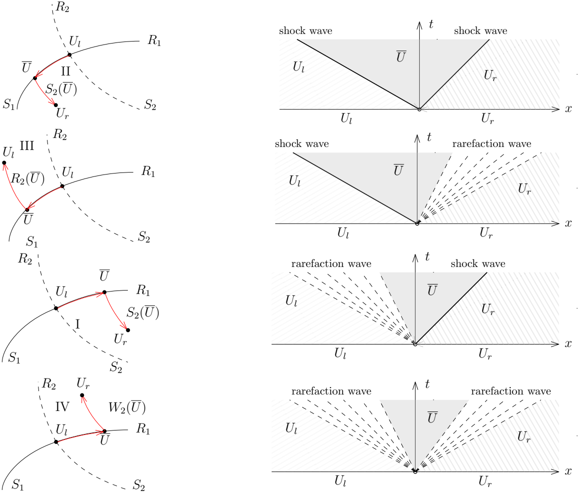
$$U(t=0,x) = \begin{cases} U_l = \binom{c_l}{v_l}, & x < 0; \\ U_r = \binom{c_r}{v_r}, & x > 0. \end{cases}$$

2-D picture:

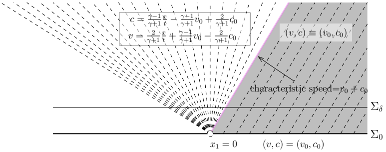


Solved in terms of shocks and rarefaction waves . It has four types of generic solutions/wave patterns.

Solutions to Riemann problem



Centered rarefaction waves (1 family theorem)



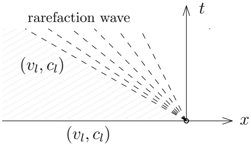
Given $U_r \equiv (v_0, c_0)$ on the right, i.e., $x_1 > 0$. There is a unique family of centered rarefaction connected to grey region on the left:

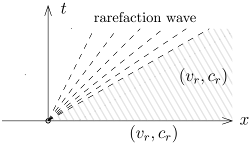
$$\begin{cases} c = \frac{\gamma-1}{\gamma+1}\frac{x}{t} - \frac{\gamma-1}{\gamma+1}v_0 + \frac{2}{\gamma+1}c_0, \\ v = \frac{2}{\gamma+1}\frac{x}{t} + \frac{\gamma-1}{\gamma+1}v_0 - \frac{2}{\gamma+1}c_0. \end{cases}$$

- The solution is continuous but singular.
- The density is decreasing when the "fan" is openning.

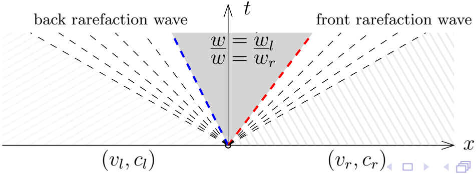
The solution with two rarefaction waves (2 families theorem)

'Even when the start of the motion is perfectly continuous, shock discontinuities may later arise automatically. Yet, under other conditions, just the opposite may happen; initial discontinuities may be smoothed out immediately'. Courant-Friedrichs, *Supersonic flow and shock waves*





Given U_l (U_r) on the left (right), we can connect them to get the fourth wave pattern:



Theory of Conservation Laws (1-D)

The theory is fairly complete:

- prove the well-posedness for initial data problem and existence of global unique weak solutions
- formation of singularities
- the interactions of elementary waves such as shocks and rarefaction waves.

Key technical tool: BV spaces (based on the understanding of Riemann's problem)

Higher dimensions

- The main technical obstacles: the breakdown of the BV space approach.
- The only effective way may through the L^2-based energy method/Sobolev spaces.
- The characteristic hypersurfaces is much more complicated than characteristic curves.
- New insights from general relativity to study characteristic hypersurfaces: (Christodoulou-Klainerman's proof of nonlinear stability of Minkowski spacetime.

Very few works in higher dimensions!

A brief review on previous works: singularity formation

- Sideris, 1985. No description of singularity.
- Alinhac, 1991-1993. Formation of singularities for 2D compressible Euler equations with radially symmetric assumptions.
- Alinhac, 1999. Formation of singularities quasilinear waves without symmetry assumptions.
  - In principle, can be extended to compressible Euler equations.
  - Derivative losses and Nash–Moser iteration scheme.
- Christodoulou, 2007.
  - stable shock formation for irrotational relativistic Euler.
  - Detailed description the geometry of the boundary of the maximal development of the data.

A brief review on previous works: singularity formations

- Works following the approach of Christodoulou
  - Christodoulou-Miao, Non-relativistic Euler;
  - Luk-Speck, with vorticity and entropy;
  - Abbrescia-Speck, up to a portion of maximal development, see also Shkoller-Vicol;
  - Holzegel-Klainerman-Speck-Wong, Holzegel-Luk-Speck-Wong, Miao-Yu, Miao, a series of works Speck, for quasilinear wave equations;
  - Q. Wang, Disconzi-Luo-Mazzone-Speck, LWP for Euler.
- Other approaches:
  - Shkoller-Vicol, Buckmaster-Shkoller-Vicol, shock formation;
  - Merle-Raphaël-Rodnianski-Szeftel, implosion.

A brief review on previous works: singularity propagation

- Shock front problem:
  - Majda, no symmetry;
  - Lisibach, shock reflection and interaction in plane symmetry;
  - ...
- Shock development problem:
  - H. Yin, Christodoulou-Lisibach, spherical symmetry;
  - Buckmaster-Drivas-Shkoller-Vicol, azimuthal symmetry;
  - Christodoulou, restricted problem without symmetry.

The open problem asked by Majda

- Discuss the rigorous existence of rarefaction fronts for the physical equations
- Elucidate the differences in multi-D rarefaction phenomena when compared with the 1-D case'.

According to him, the problem is much harder due to

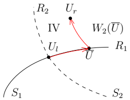
- Surfaces bounding the rarefaction wave regions are characteristic: lack of uniform stability condition.
- Linearized equations lose derivatives.
- Coupled with initial singularity further complicating the analysis.

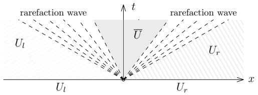
A brief review on previous works: rarefaction waves

- Alinhac, local existence and uniqueness of multi-dimensional rarefaction waves for a general hyperbolic system. He introduced several innovative techniques:
  - The celebrated 'good unknown' for the linearized equations.
  - Use approximate characteristic coordinate system which blows up the singularity.
  - Nash–Moser scheme based on non-isotropic Littlewood–Paley decomposition to overcome the derivative loss (also treat the characteristic boundary).
  - Finding an approximate ansatz for rarefaction waves up to sufficiently large order near the singularity.
- Z.Wang-H.Yin, for steady supersonic flow around a sharp corner.
- Coulombel-Secchi, $\cdots$, for other elementary wave patterns such as contact discontinuities.

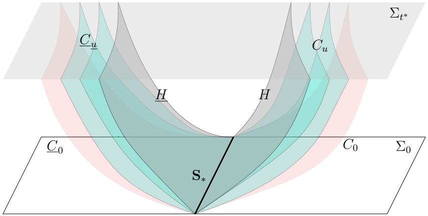
Main results (joint with Tian-Wen Luo)

- U_l is first connected $\overline{U}$ by a back rarefaction wave and then connected to U_r by a front rarefaction wave. The initial discontinuity is resolved by two families of rarefaction waves.

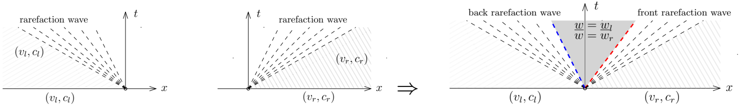




The pictures are stable under small perturbations (in H^k-norms) in higher dimensions without any symmetry assumptions.

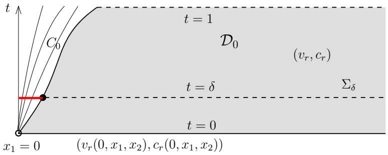


Strategy



- Construct single family of rarefaction waves
  - Construct data for approximate solution
  - Derive a priori energy estimates for approximate solution
  - Pass to limit to construct real solution
- Assemble to obtain the solution
- Uniqueness

The single family case

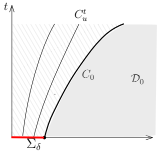


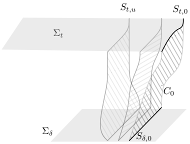
We assume that $\left(v\big|_{t=0} - (\mathring{v}_r, 0), c\big|_{t=0} - \mathring{c}_r\right)$ is small.

- Connect $\mathcal{D}_0$ by a rarefaction wave in a unique way on the left.
- Construct initial data "at" the singularity.
  - Admissible conditions at the "corner".
  - The data should provide the rarefaction waves (not the smooth extension).
  - Use the last slice argument from Christodoulou-Klainerman.

The main estimates

- Use the acoustical coordinate (t, u, ϑ) for the rarefaction region
- C_u=level sets of u=characteristic hypersurfaces.
- $\kappa = |\nabla u|^{-1} \approx t$=inverse density.





We have (independent of δ!)

$$\boxed{\mathcal{E}_{\leqslant n}|_{\Sigma_t} + \mathcal{F}_{\leqslant n}|_{C_u} \leqslant \mathcal{E}_{\leqslant n}|_{\Sigma_\delta} + \mathcal{F}_{\leqslant n}|_{C_0} + \text{error}}$$

The energies are defined for the Riemann invariants:

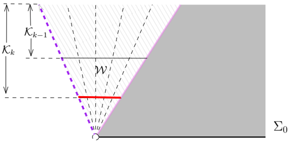
$$w = \frac{1}{2}\Big(\frac{2}{\gamma-1}c - v_1\Big),\ \underline{w} = \frac{1}{2}\Big(\frac{2}{\gamma-1}c + v_1\Big)$$

through wave equations:

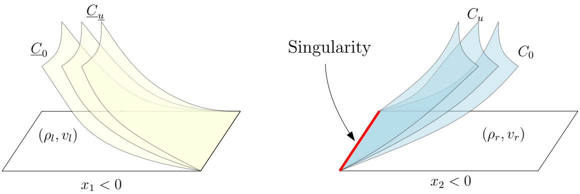
$$\Box_g w = c^{-1}(g^{\mu\nu}\partial_\mu \underline{w} \partial_\nu \underline{w} + \cdots)$$

Stability of Riemann problem 1

Construction of centered rarefaction waves:limiting argument for $\delta = 2^{-k}$



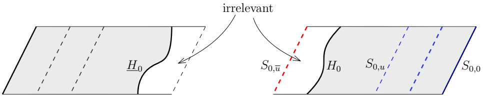
Obtain two families of centered rarefaction waves:



Stability of Riemann problem 2

At the singularity $S_* = (u, \vartheta) \in [0, u^*] \times [0, 2\pi]$ where $\vartheta = x_2$:

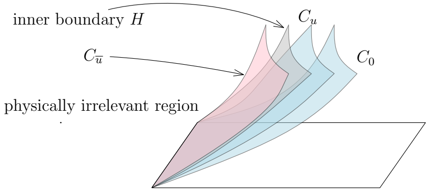
$$\begin{cases} \underline{w}(u,\vartheta) = \underline{w}_r(0,\vartheta) - \frac{2}{\gamma+1}u, \\ w(u,\vartheta) = w_r(0,\vartheta), \\ \psi_2(u,\vartheta) = -v_r^2(0,\vartheta). \end{cases}$$



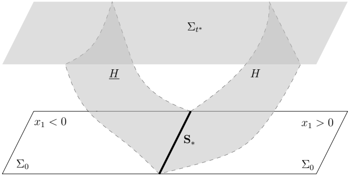
On $\underline{H}_0$, we have $\underline{w} = \underline{w}_l$; on H_0, we have $w = w_r$.

Stability of Riemann problem 3

From H_0, it determines a characteristic hypersurface H (not C_u in general) which cuts the universal family of front rarefaction waves.



We then solve a smooth Goursat problem:



Comparison with Alinhac's results

- Use the Nash-Moser iteration scheme to save the loss of regularity (even in the linear estimates).

  Energy estimates in standard Sobolev space H^s with $s \geqslant 6$.

- Requires the compatibility conditions and solutions in different regions must be iterated simultaneously to correct the boundaries.

  No boundary condition on the left boundary $u = u_*$. Describe all rarefaction waves which can be connected to C_0.

- Loss of normal derivatives due to the the degeneration of weight (even for $1D$). Estimates do not degenerate on boundaries (flux). Quantify the perturbations relative to $1D$ case in terms of the small parameter ε (asymptotical stability).

We use wave equations associated a given Lorentzian metric.

Remarks on Christodoulou's work on shock formations

- The degeneration of angular derivatives near shocks.
  - Disparity in κ: Near-shock, $\kappa \to 0$. Angular part of the energy $\approx \int_{\mathcal{D}} \mu |\nabla\!\!\!/ \psi|^2$ but the error integrals have $\nabla\!\!\!/ \psi$ components *without* κ factor.
  - Error integrals can not be bounded by the energy integrals (even on linear level).
- Christodoulou's solution:
  - Initially $\kappa \approx 1$ and near shocks $\kappa \approx 0 \;\Rightarrow\; L(\kappa) < 0$.
  - Error integrals without factor μin the form $\int_{\mathcal{D}} L(\kappa) \cdot |\nabla\!\!\!/ \psi|^2$.
  - The negative sign of $L(\kappa)$ manifests a miraculous coercivity.
- The sign of $L(\kappa)$ in the rarefaction wave region is positive. We need new mechanism.

Remarks on Christodoulou's work: control the second fundamental form

- A direct integration along L would cause a loss of one derivative:

$$L\left(Z^N(\mathrm{tr}(\chi))\right) = Z^{N+2}(\psi) + \cdots .$$

- Renormalization: $Z^{N+2}(\psi) = L\big(Z^{N+1}(\psi')\big)$ to derive

$$L\big\|Z^N(\mathrm{tr}(\chi)) - \big(Z^{N+1}(\psi')\big)\big\|_{L^2} = \frac{L(\kappa)}{\kappa}\big\|Z^N(\mathrm{tr}\chi) - \big(Z^{N+1}(\psi')\big)\big\|_{L^2} + \cdots .$$

- For shock formation $L\kappa < 0$, the blue term can be dropped.
- For rarefaction waves, $L(\kappa) \approx 1 \Rightarrow$ loss in t. (most challenging part)

Dealing with initial data at singularities

- Linear estimates: bound $\int_{\mathcal{D}} L(\kappa) \cdot |\nabla\!\!\!/ \psi|^2$.
  - small-data-global-existence problem for nonlinear wave equations:
    $$E(t) \leqslant E(0) + \int_0^t \frac{C_0}{\tau} E(\tau) d\tau.$$
    Gronwall's inequality to show that $E(t) = O(\log(t))$.
    $\log(t)$-loss but long time lifespan of size $O(e^{\frac{1}{\varepsilon}})$.
  - Rarefaction waves:
    $$E(t) \leqslant \left(\frac{t}{\delta}\right)^{C_0} E(\delta).$$
    When $\delta \to 0$, unless the initial energy $E(\delta)$ decays in the correct way,

The decay hierarchy of the Riemann invariants

- Correct ansatz and Riemann invariants:$\{\underline{w}, w, v_2\}$ (can diagonalize Euler equations)
  - If $\psi \neq \underline{w}$ or $k \geqslant 1$, $L(Z^k\psi)$ and $\widehat{X}(Z^k\psi)$ are of size ε^2; $\underline{L}(Z^k\psi)$ are of size $t^2\varepsilon^2$. (in L^2-norms)
  - The $\underline{Lw}$ is of size 1, generate most of the linear terms (main enemies) in the energy estimates.
  - (Believe) Unique energy ansatz (from last slice argument)
- New Gronwall and the use of flux $F(t,u)$:

$$E(t,u) + F(t,u) \leqslant At^2 + B\int_0^u F(t,u')du' + C\int_\delta^t \frac{E(t',u)}{t'}dt'.$$

  We have

$$E(t,u) + F(t,u) \leqslant 3Ae^{Bu}t^2$$

  provided $e^{Bu^*}C \leqslant 1$.
- No degeneration at the boundaries of the rarefaction wave region.
- Flux also controls the geometry of rarefaction fronts.

Null structure

- The nonlinear term enjoys the null structure through the Riemann invariants:$\{\underline{w}, w, v_2\}$.
- The source terms are all in the covariant form $g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi'$.
- No terms of the type $\underline{L}\psi \cdot \underline{L}\psi'$. We notice that there is no smallness in $\underline{Lw}$. Therefore, the worst contribution in the energy estimates from the source terms are at least linear hence borderline terms.
- The flux term on the characteristic hypersurfaces C_u contains no $\underline{L}$-derivative components, these null structures allow us to deal with most of the error terms.

Two null frames

Commutation $\Box_g(Z^N\psi) = Z^N\chi + \cdots$ require the bound on χ:

$$L\big\|Z^N(\chi) - (Z^{N+1}(\psi'))\big\|_{L^2} = \frac{L(\kappa)}{\kappa}\big\|Z^N(\chi) - (Z^{N+1}(\psi'))\,\big\|_{L^2} + \cdots.$$

- Covariant nature of the equation.
- Similar to the GCM construction in the stability of Kerr family (Klainerma-Szeftel): the new null frame (non-integrable) $\{\mathring{L}, \underline{\mathring{L}}, \mathring{X}\}$ explicitly written in $\psi \in \{\underline{w}, w, v_2\}$, commutation only contributes $\mathring{Z}^k(\psi)$. (better chance to be controlled by Gronwall). Indeed, $\mathring{X} = \partial_2, \underline{\mathring{L}} = \partial_t + (v_1 + c)\partial_1 + v_2\partial_2$.
- Price to pay: worst possible error terms are related to ${}^{(\mathring{Z})}\pi_{\mathring{L}\mathring{L}}$ (vanishing for the old frame).

A new coercivity

$$(\gamma+1)\int_{D(t,w)} \underbrace{\frac{\mu}{\mathring{\mu}}}_{\approx 1} \cdot \frac{\underline{\mathring{L}}(\underline{w})}{\mathring{\kappa}} |\mathring{T}\mathring{Z}^{\underline{w}}\psi|^2 + \cdots$$

- Out of the scope of refined Gronwall.
- $\underline{\mathring{L}}(\underline{w}) < 0 \Rightarrow$ This term can be ignored.
- This is due to the expansion nature of rarefaction waves and it reflects the fact that along the transversal direction the density of the gas is decreasing.

An extra vanishing

We encounter the following terms:

$$t^{-1}\,{}^{(\mathring{X})}\pi_{\mathring{L}\mathring{L}} = \frac{\mathring{X}(v_1+c)}{t}, \quad t^{-1}\,{}^{(\mathring{T})}\pi_{\mathring{L}\mathring{L}} = t^{-1}\Big(1+\frac{\gamma+1}{2}\mathring{T}(\underline{w})+\frac{\gamma-3}{2}\mathring{T}(w)\Big).$$

- Energy ansatz suggests size $O(t^{-1}\varepsilon)$ and $O(t^{-1})$. The t^{-1} factor is out of reach for the energy estimates.
- In fact, these two terms are of size $O(\varepsilon)$ and $O(1)$. It comes from the delicate choice of the initial data near singularity (last slice argument). The geometry of initial rarefaction wave fronts must be matched in an exact way on Σ_δ.
- For example, this forces $\mathring{T}(\underline{w}) = -\frac{\gamma+1}{2}$.

Thank you very much!