Scattering for wave equations with sources in the wave zone

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joint work with

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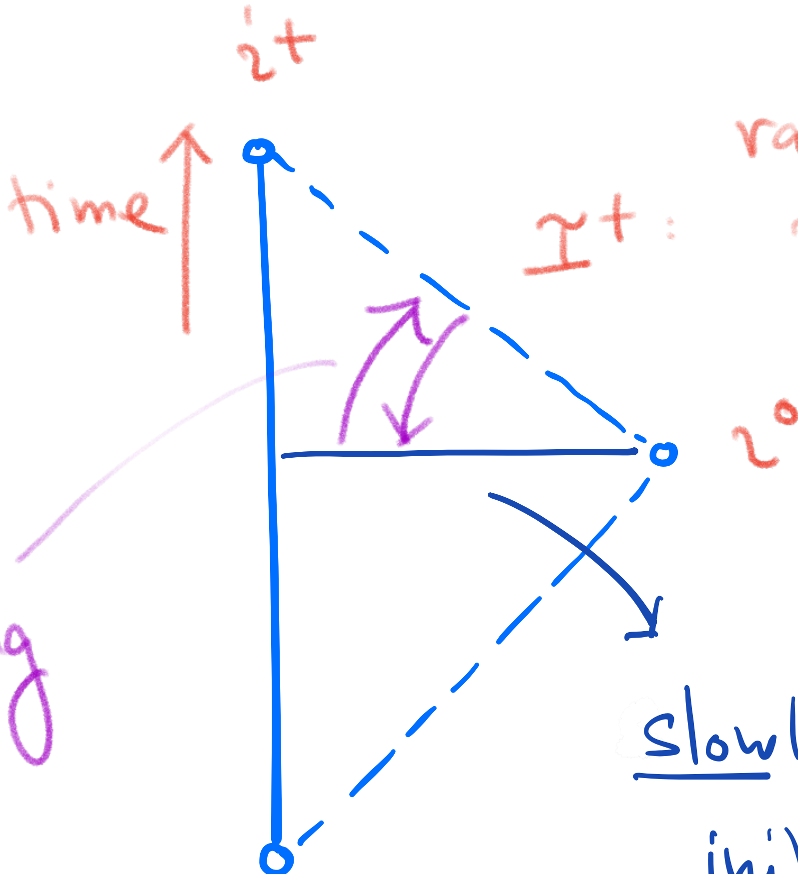
Topic of this lecture

$\Box \phi = F$

slowly decaying in time

radiation field

$\mathcal{I}^+$: may not exist.

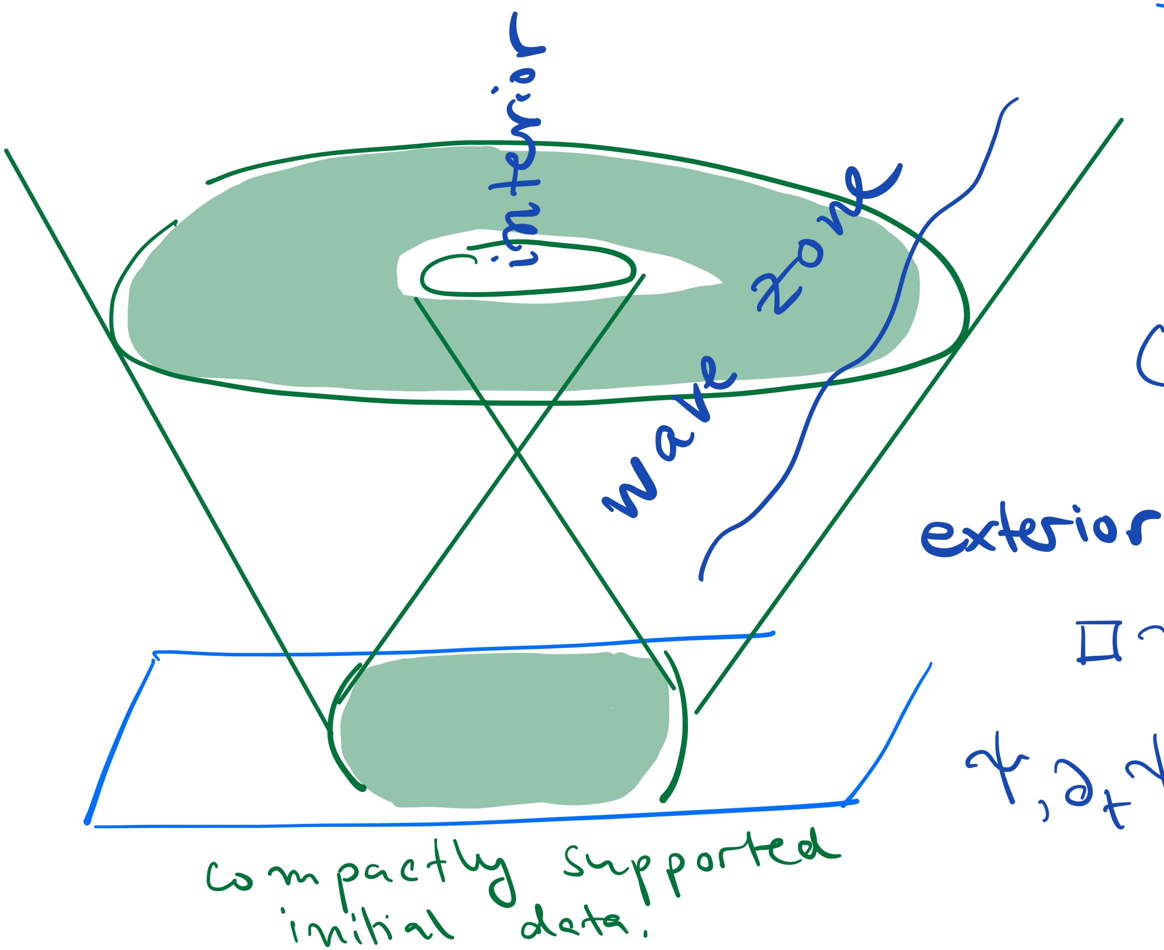


scattering map

slowly decaying initial data

Motivation: Non-linear wave eqs satisfying the weak null condition.

Friedlander radiation field (1960s)



$$\psi \sim \frac{F_0(r-t,\omega)}{r}$$

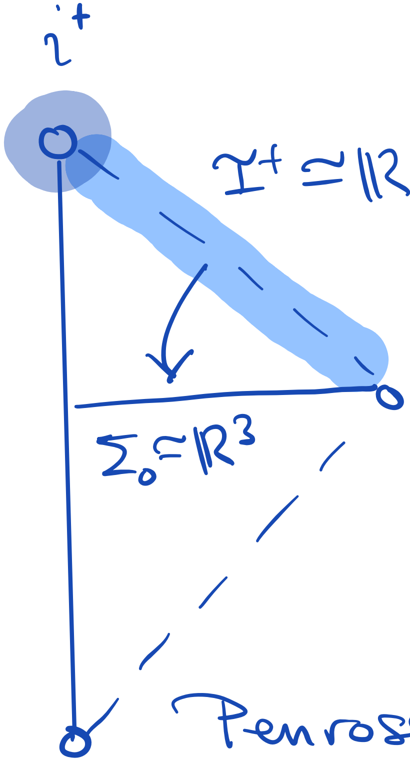
(here $r = |x|$, $\omega = \frac{x}{|x|}$)

$$\Box \psi = 0$$

$$\psi, \partial_t \psi \big|_{t=0} \in C_0^\infty(\mathbb{R}^3)$$

$$F_0(q,\omega) = \lim_{r\to\infty} (r\psi)(\overset{t}{-q+r}, \overset{x}{r\omega})$$

Scattering problem: given F_0, find ψ.



Penrose diagram.

Thm (Friedlander 80s)

For solutions to $\Box \psi = 0$,
the map

$$(\psi, \partial_t \psi)\big|_{t=0} \quad \text{in} \quad \dot{H}^1 \times L^2(\mathbb{R}^3)$$

"initial data in energy space"

$$\big\downarrow$$

"radiation field of $\partial_t\psi$ in L^2"

$$r\partial_t \psi \longrightarrow \mathcal{F}_0 \in L^2(\mathbb{R}\times S^2)$$

is an isomorphism.

Rmk.

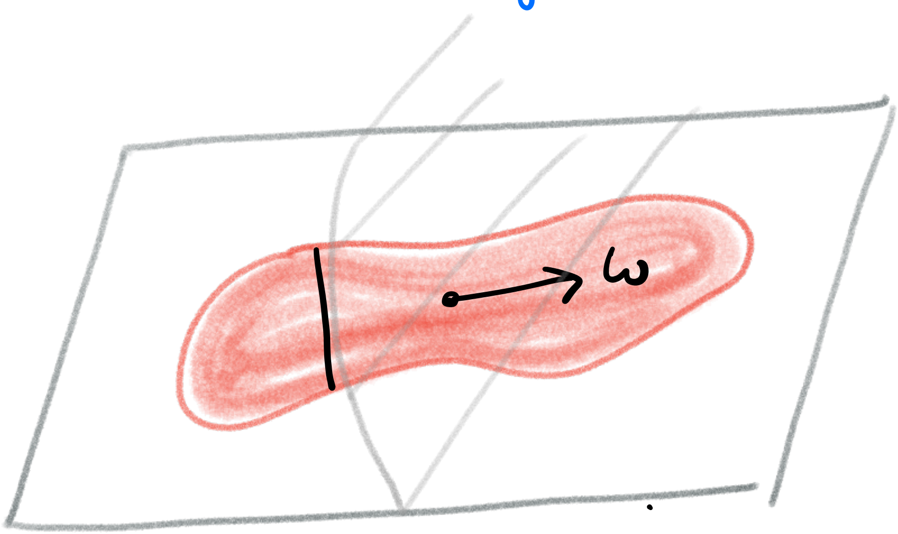
$\mathcal{F}_0$ may not exist.

Radiation field and Radon transform

$$\begin{cases} \Box \psi = 0 \\ \psi|_{t=0} = f \in C_0^\infty(\mathbb{R}^3), \quad \partial_t \psi|_{t=0} = g \in C_0^\infty(\mathbb{R}^3) \end{cases}$$

then

$$\mathcal{F}_0(q,\omega) = \mathcal{R}[g](q,\omega) - \partial_q \mathcal{R}[f](q,\omega)$$



Radon transform:

$$\mathcal{R}[g](q,\omega) = \int_{\langle\sigma,\omega\rangle = q} g \, dS(\sigma)$$

Needs fast decaying data: $|g| \leq \frac{1}{\langle x\rangle^{2+}}$, $|\nabla f| \leq \frac{1}{\langle x\rangle^{2+}}$

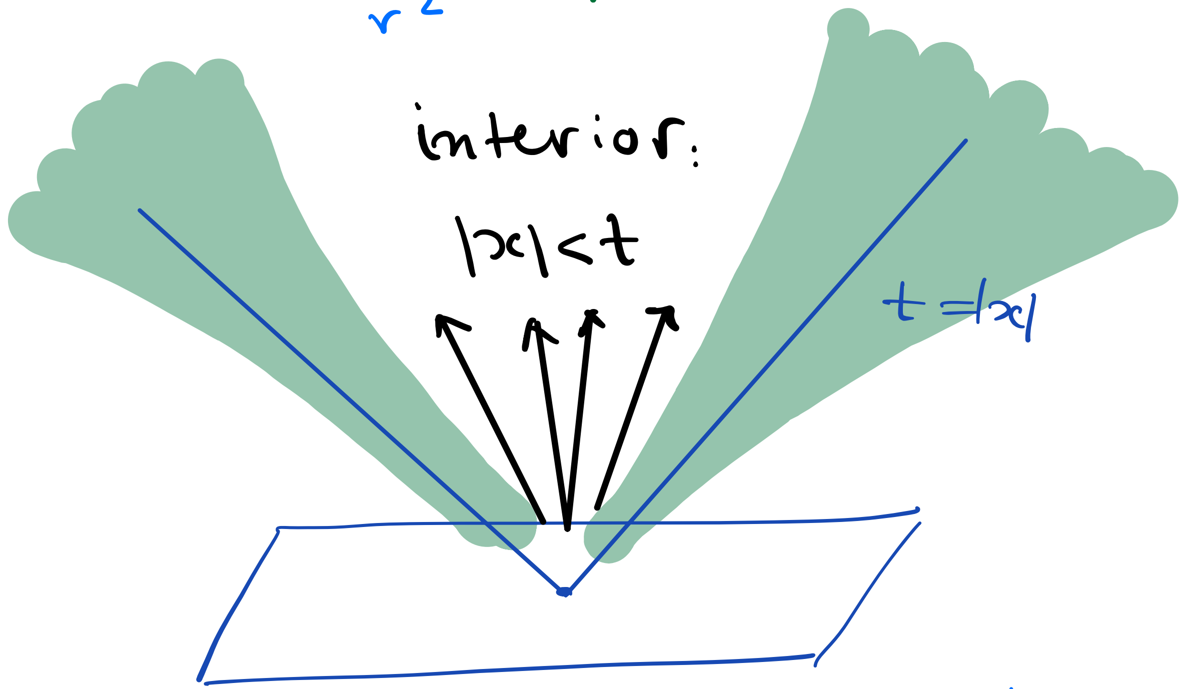
Wave eqn with sources.

$$\Box \varphi = F = \frac{n(q,w)}{r^2} \chi$$

Motivated by

$$\Box \varphi = (\partial_t \psi)^2$$

$$\psi \sim \frac{F_0}{r}$$

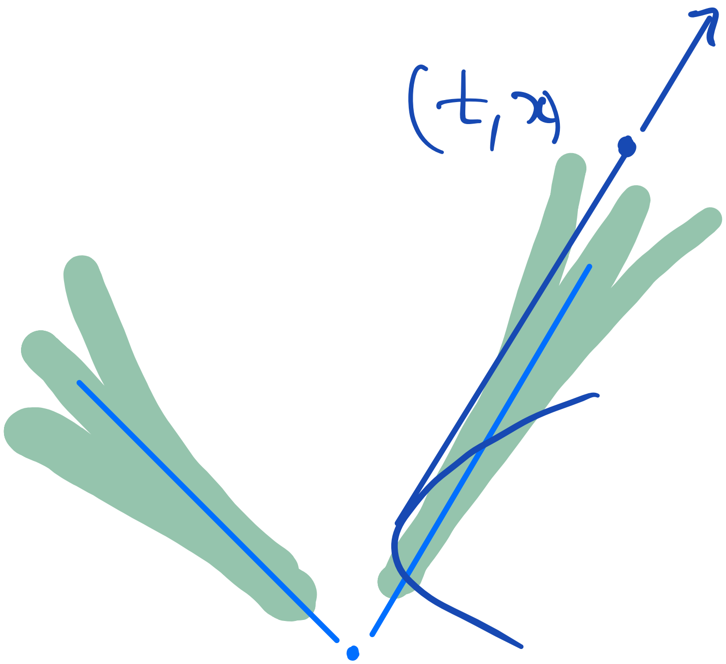


with trivial data as $t \to -\infty$.

Strong Huygen's fails:

$$\varphi(t, \overbrace{\frac{x}{t}}^{=y,\ |y|<1} t) \sim \frac{1}{t} \quad (t \to \infty)$$

Log-terms in the wave zone:



$$\varphi = E * F, \qquad F = \frac{n}{r^2}\chi$$

$$q = |x| - t,$$

$$r \to \infty$$

$$\varphi \sim \frac{1}{r}\ln\left(\frac{\langle t+r\rangle}{\langle t-r\rangle}\right)\mathcal{F}_{01}(r-t,\omega)$$

$$+ \frac{1}{r}\mathcal{F}_0(r-t,\omega)$$

where

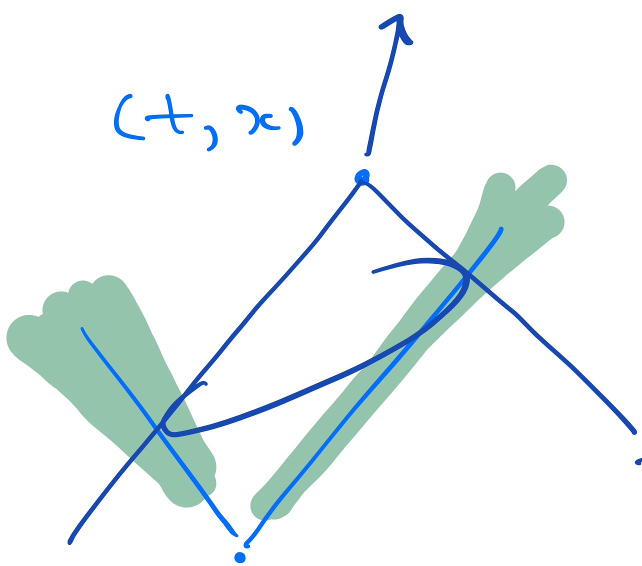
$$\mathcal{F}_{01}(q,\omega) = \int_q^\infty n(q,\omega)\,dq$$

Previous Defn of Friedlander rad. field $\mathcal{F}_0$ does not apply here.

Homogeneous asymptotics in the interior for

$$\varphi = E * F, \qquad F = \frac{n}{r^2}\, \chi$$

$$N(\omega) = \int_{-\infty}^{\infty} n(q,\omega)\, dq$$



$\varphi(t, r\omega)$

$|x| < t$,

$t \to \infty$

$$\Phi(t, r\omega) = \frac{1}{4\pi} \int_{S^2} \frac{N(\sigma)}{t - \langle r\omega, \sigma \rangle}\, dS(\sigma)$$

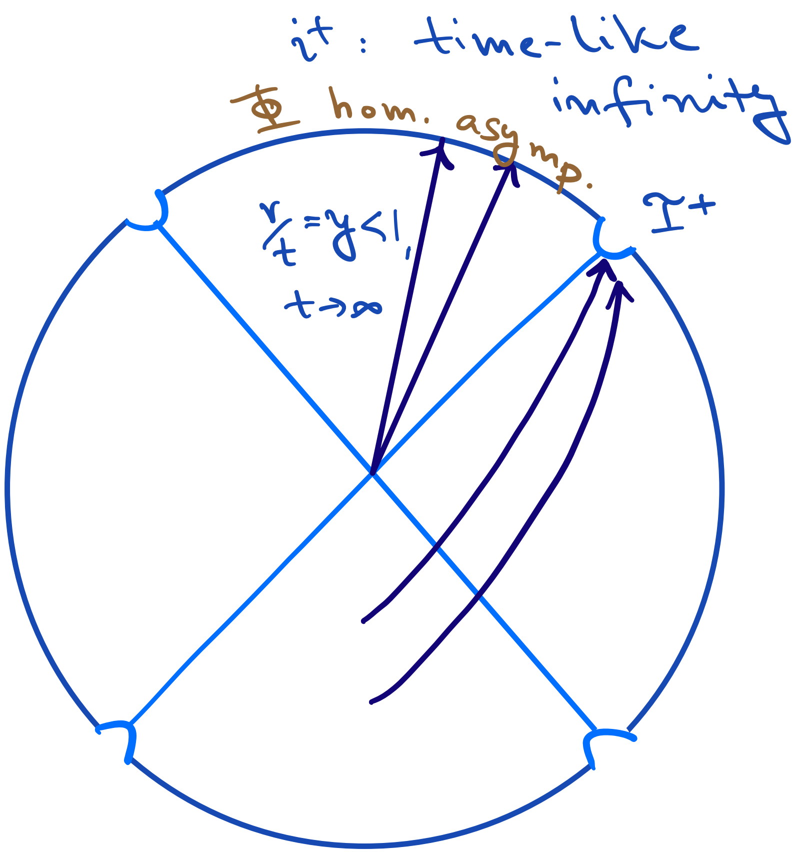
Lindblad '90

is homogeneous of degree -1,

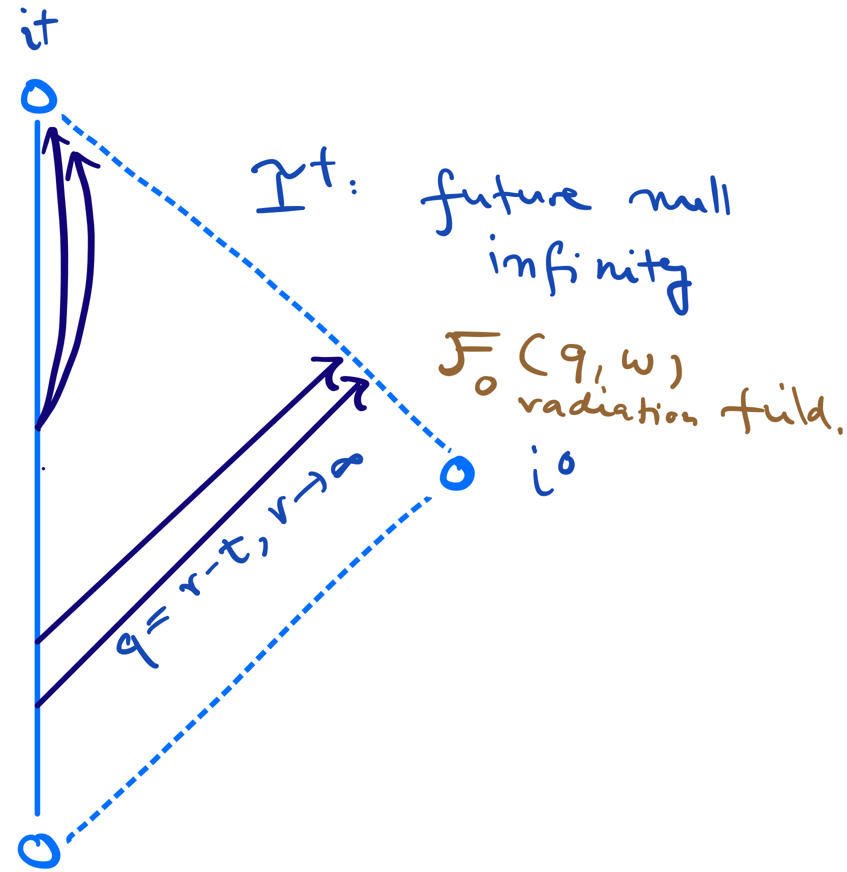
$$\Phi(at, ax) = a^{-1} \Phi(t, x),$$

and a superposition of plane waves: $\Box \Phi = 0$.

Asymptotic regimes



Melrose

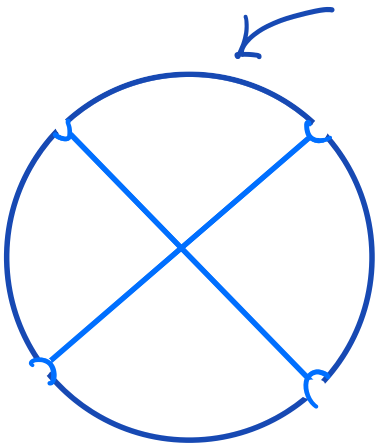


Penrose

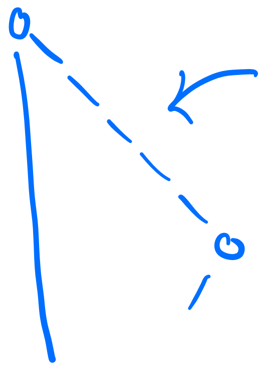
Scattering problem

for the eqn $\Box \varphi = \dfrac{n}{r^2} \chi$.

Question: What is the scattering data?



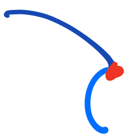
Homogeneous asymptotics; part of the scattering data?



Radiation field does not exist; Can F_0 be freely prescribed?

F_0 not completely free due to matching condition

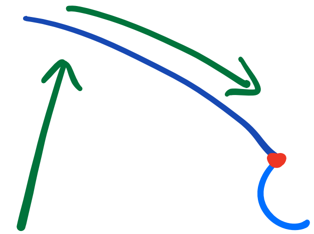
$$\lim_{q \to -\infty} F_0(q, \omega) = N(\omega)$$



How is the interior sol'n determined?

"Matching to leading order"

Homogeneous sol'n in the interior:

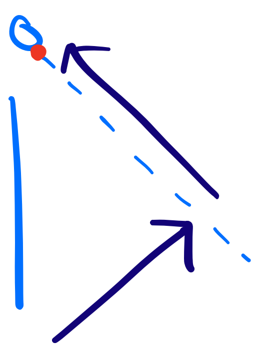


$$\Phi = \frac{1}{4\pi}\int_{S^2} \frac{N(\sigma)}{t-\langle r\omega,\sigma\rangle}\, dS(\sigma)$$

$$\left(r/t \searrow 1\right) \quad \sim \frac{1}{2r}\boxed{N(\omega)} \ln\left|\frac{2r}{t-r}\right| + \frac{1}{r} N_0(\omega)$$

Expansion in the wave zone:

$$\varphi \sim \frac{1}{2r}\ln\left(\frac{2r}{\langle t-r\rangle}\right) F_{01}(q,\omega) + \frac{1}{r}F_0(q,\omega)$$

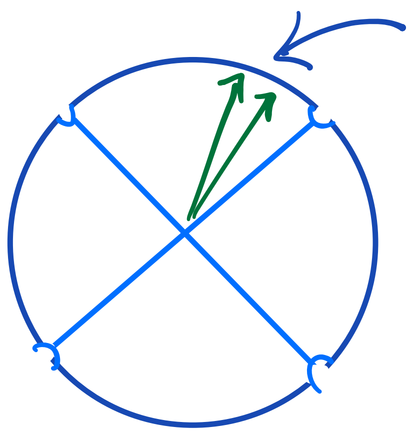


$$F_{01}(r-t,\omega) = \int_{r-t}^{\infty} n(q,\omega)\, dq \xrightarrow{t-r\to\infty} \boxed{N(\omega)}$$

Thm (Lindblad - S '23) Scattering with prescribed radiation fields for $\Box \varphi = \frac{n}{r^2}\chi$.

Given smooth functions $n(q,\omega)$,

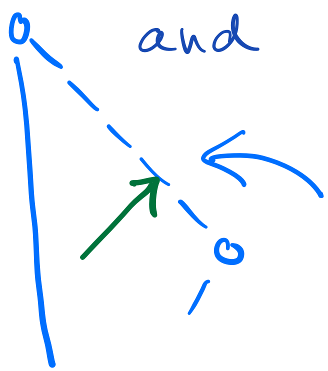
and $\mathcal{H}_0(q,\omega)$, $|\mathcal{H}_0| \leq \langle q \rangle^{-2}$



there exists a smooth sol'n φ with hom. asymptotics in the interior

$$\varphi \sim \Phi_1[N], \quad N = \int n \, dq$$

and a matching expansion in the wave zone



$$\varphi \sim \frac{1}{r} \ln\left(\frac{2r}{\langle t-r \rangle}\right) F_{01}[n](q,\omega) + \frac{1}{r} F_0(q,\omega)$$

where

$$F_0(q,\omega) = \begin{cases} N_0[n](\omega) + \frac{1}{q} M_0[n](\omega) + \mathcal{H}_0(q,\omega), & q<0 \\ \mathcal{H}_0(q,\omega), & q>0 \end{cases}$$

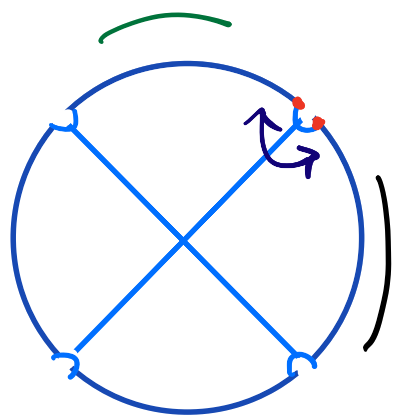
provided $\mathcal{H}_0$ satisfies a compatibility condition of the form

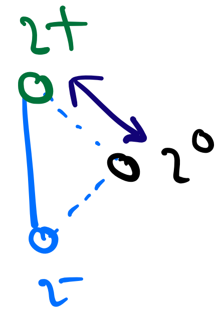
$$\int_{-\infty}^{\infty} \mathcal{H}_0(q,\omega)\, dq = \mathcal{P}[n](\omega).$$

Interior hom. sol's determined by source n.

Φ_1, Φ_2 degree 1, 2.

$\searrow N_0 \qquad \searrow M_0$





Here: No homogeneous sl'n in the exterior.

Later: $\Phi_1^{ext}, \Phi_2^{ext}$.

Remark on the compatibility condition

$$\begin{cases} \Box \psi = 0 \\ \psi|_{t=0} = f \in C_0^\infty(\mathbb{R}^3), \quad \partial_t \psi|_{t=0} = g \in C_0^\infty(\mathbb{R}^3) \end{cases}$$

then

$$F_0(q,\omega) = R[g](q,\omega) - \partial_q R[f](q,\omega)$$

hence

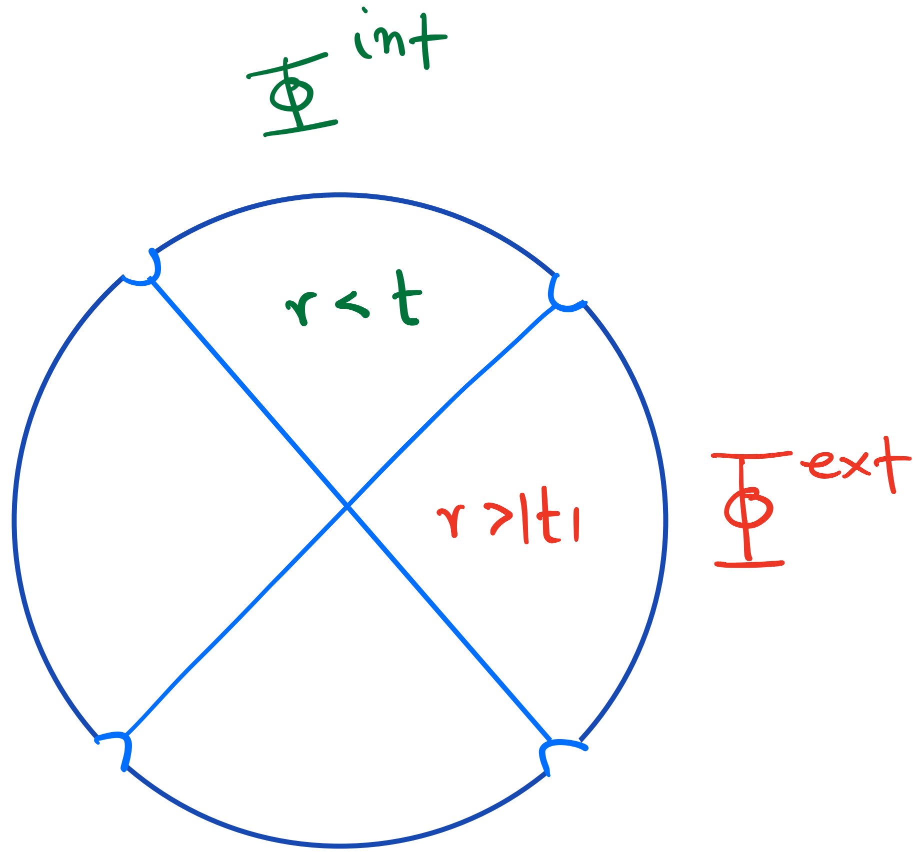
$$\underbrace{\int_{-\infty}^{\infty} F_0(q,\omega)\, dq = \int_{\mathbb{R}^3} g \neq 0.}$$

Radon transform: $R[g](q,\omega) = \displaystyle\int_{\langle \sigma, \omega\rangle = q} g \, dS(\sigma)$

Plan :

1. Homogeneous solutions in the interior

2. Homogeneous solutions in the exterior



$$\Phi(at, ax) = a^{\lambda}\, \Phi(t, x)$$

$$(\lambda = -1, -2)$$

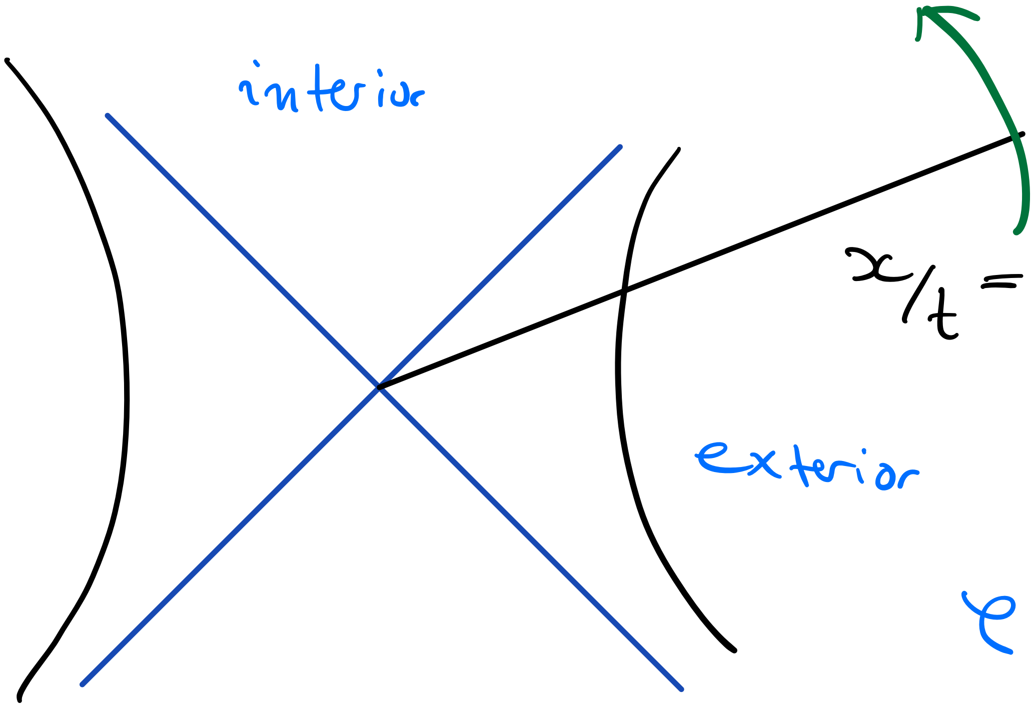
3. Scattering solutions for wave eqn's

Homogeneous solutions in the exterior

$$\Box \varphi = 0 \qquad (|x| > t)$$

Hom deg. λ:

$$\varphi(at, ax) = a^{\lambda} \varphi(t, x) \qquad (a > 0)$$



$|y| \longrightarrow 1$: Related to limits of radiation fields.

$x/t = y, \ |y| > 1$

$$\varphi(t,x) = |x|^{\lambda} \varphi\left(\frac{t}{|x|}, \omega\right)$$

$$\omega = x/|x| \in S^2.$$

Examples in spherical symmetry:

degree -1:

$$\phi_1 = \frac{M}{r}$$

constants

$$\phi_1 = \frac{N}{r} \ln \left| \frac{r+t}{r-t} \right|$$

degree -2:

$$\phi_2 = \frac{C}{r(r+t)}$$

Aim: Characterize all hom. sol'ns in terms of initial data.

Homogeneous degree -1

$$\begin{cases} \Box \Phi = 0 & (|x|>t) \\ \Phi(0, r\omega) = \dfrac{M(\omega)}{r} \\ \partial_t \Phi(0, r\omega) = \dfrac{N(\omega)}{r^2} \end{cases}$$

(Formula for radiation field in terms of Radon transform dos not apply!)

Lemma (Lindblad-S '23) As $r/t \to 1$,

$$\Phi \sim \frac{1}{2r} \ln \frac{2r}{r-t} \left(\mathcal{F}[N](\omega) + \mathcal{G}[M](\omega) \right)$$

where $\mathcal{F}[N]$ is the Funk transform.

278 P. Funk

Paul Funk 1886–1969

Thesis with Hilbert in 1911:

Über Flächen mit lauter geschlossenen geodätischen Linien.*)

Von

P. Funk in Salzburg.

Einleitung.

Mit den beiden Fragen: Wie findet man auf einer gegebenen Fläche die geschlossenen geodätischen Linien, und wie findet man Flächen, auf denen alle geodätischen Linien geschlossen sind, haben sich bereits mehrere Arbeiten beschäftigt. Für uns kommen hauptsächlich die folgenden in Betracht. Darboux**) hat zuerst in den Noten zum Cours de mécanique vom Despeyrous und später in seinen „Leçons sur la théorie générale des surfaces“ die notwendige und hinreichende Bedingung dafür angegeben, daß auf einer Rotationsfläche alle geodätischen Linien geschlossen sind. Angeregt durch Herrn Prof. Hilbert hat Zoll***) in seiner Dissertation unter Benutzung der Entwicklungen von Darboux eine geschlossene singularitätenfreie Rotationsfläche mit lauter geschlossenen geodätischen Linien angegeben. Stäckel†) hat den Verlauf der geodätischen Linien auf Liouvilleschen Flächen untersucht. Auch hierbei hat sich die Möglichkeit ... daß alle geodätischen Linien geschlossen sein können, und ... Flächenklasse angegeben, bei der dieser Fall ... ihm nicht gelungen, die Frage, wann ... in voller Allgemeinheit zu be- ... bewiesen, daß auf jeder ... schlossene geodätische ... Richtigkeit des ...

... unwesent- ... chen Titel

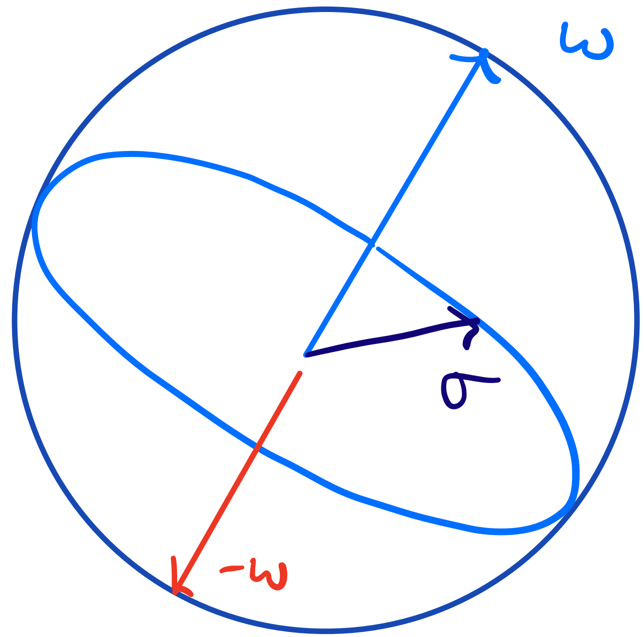
see also

Guillemin '70s

Helgason '80s

⋮

$$\mathcal{F} : C^\infty(S^2) \longrightarrow C^\infty(S^2)$$
$$M \longmapsto \mathcal{F}[M]$$



$$\mathcal{F}[M](\omega) = \frac{1}{2\pi} \int_{\langle \sigma, \omega \rangle = 0} M(\sigma)\, ds(\sigma)$$

$\mathcal{F} : \{\text{even}\} \longrightarrow \{\text{even}\}$

is invertible

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COMPLEX ANALYSIS AND THE FUNK TRANSFORM

T. N. Bailey, M. G. Eastwood, A. R. Gover, and L. J. Mason

Abstract. The Funk transform is defined by integrating a function on the two-sphere over its great circles. We use complex analysis to invert this transform.

Introduction

In 1917 Radon [19] introduced a transform $f \mapsto Rf$ for f a suitable real-valued function on $\mathbb{R}^2$ by

$$(Rf)(L) = \int_L f$$

for L a straight line in $\mathbb{R}^2$. Thus, Rf is a function defined on the set of straight lines in $\mathbb{R}^2$. See, for example, [12] for a review. There are many variations on this theme in real integral geometry.

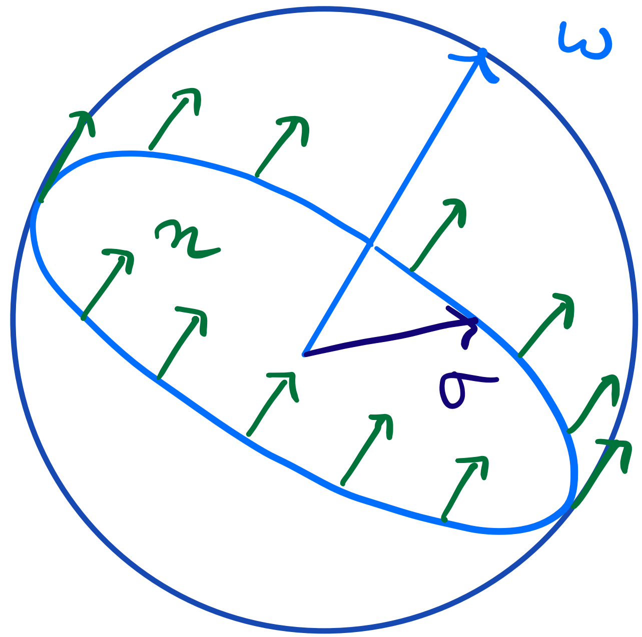
One such variation was already introduced in 1913 by Funk [10]. Its definition is just like the Radon transform except that $\mathbb{R}^2$ is replaced by the round sphere S^2 and great circles play the role of straight lines. Funk proved that a smooth function f on S^2 lies in the kernel of this transformation if and only if f is odd (see [13] for a modern treatment and a discussion of Funk's motivation in constructing Zoll metrics on the sphere).

Our aim, in this article, is to show how the Funk transform $\mathcal{F}$ acting on smooth functions may be inverted using complex analysis. More precisel… …n inverse transform $\mathcal{T}$ arises as a result … (where $\bar{\partial}$ is the Cauchy- … …it formula. Using

…05,

See also Guillemin '70s.

$$\mathcal{G} : C^\infty(S^2) \longrightarrow C^\infty(S^2)$$
$$M \longmapsto \mathcal{G}[M]$$



$$\mathcal{G}[M](\omega) = \frac{1}{2\pi}\int_{\langle\sigma,\omega\rangle=0} \frac{\partial M(\sigma)}{\partial n}\, ds(\sigma)$$

$$\mathcal{G} : \{\text{odd}\} \longrightarrow \{\text{odd}\}$$

is invertible

Thm (Lindblad-S '23)

Given smooth functions $N_{01}, N_0 \in C^\infty(S^2)$ there exists a unique homogeneous deg. -1 sol'n to $\Box\varphi = 0$ $(|x|>t)$ so that

$$\varphi \sim \frac{1}{r}\ln\frac{2r}{r-t} N_{01}(\omega) + \frac{1}{r} N_0(\omega) \quad \left(r/t \to 1\right)$$

Proof. Recall that with data $\begin{cases} \Phi|_{t=0} = \frac{M}{r} \\ \partial_t\Phi|_{t=0} = \frac{N}{r^2} \end{cases}$ we have

$$\Phi \sim \frac{1}{r}\ln\left(\frac{2r}{r-t}\right)\left(\mathcal{F}[N] + \mathcal{G}[M]\right)(\omega)$$

$$\Rightarrow N_+ = \mathcal{F}^{-1}[(N_{01})_+], \quad M_- = \mathcal{G}^{-1}[(N_{01})_-]$$

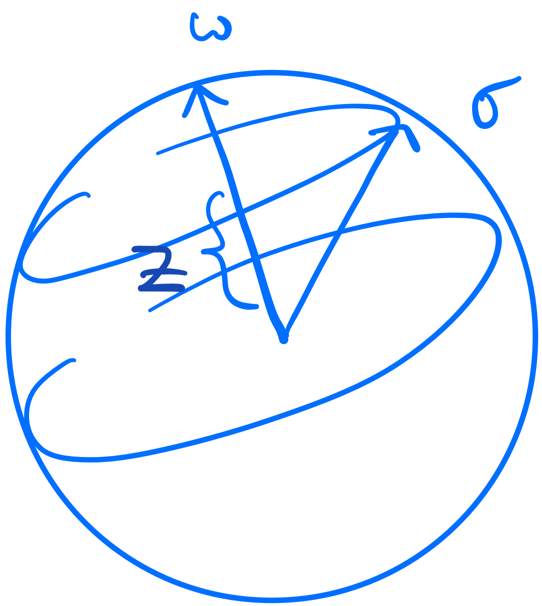
Consider
$$\begin{cases} \Box \Phi = 0 & |x| > t \\ \Phi|_{t=0} = 0, \quad \partial_t \Phi|_{t=0} = \dfrac{N}{r^2} \end{cases}$$

with N odd : $N(-\omega) = -N(\omega)$,

then
$$\Phi(t, r\omega) \sim \frac{1}{r} \underbrace{S[N](\omega)}_{\frac{1}{2}\int_{-1}^{1} \frac{N(\omega, z) - N(\omega, 0)}{z}\, dz} \quad (r/t \to 1)$$

where

$z = \langle \sigma, \omega \rangle$



and $S : \{\text{odd}\} \to \{\text{odd}\}$ is invertible:
$$S \circ \mathcal{G} = \mathcal{G} \circ S = \mathrm{id}.$$

Also
$$\begin{cases} \Box \Phi = 0 & |x| > t \\ \Phi\big|_{t=0} = \dfrac{M}{r}, \quad \partial_t \Phi\big|_{t=0} = 0 \end{cases}$$

where M is even : $M(-\omega) = M(\omega)$

then
$$\Phi \sim -\frac{1}{t}\, \omega^i\, \underbrace{S[M_i]}_{-\mathcal{T}[M] : \{\text{even}\} \circlearrowleft}$$

is invertible:

where by def'n:
$$\partial_i\left(\frac{M(\omega)}{r}\right) = \frac{M_i(\omega)}{r^2}$$

$$\mathcal{T} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{T} = id$$

COMPLEX ANALYSIS AND THE FUNK TRANSFORM

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$$\left\{\begin{array}{c}\text{smooth even functions}\\ \text{on } \mathbb{R}^3\setminus 0\\ \text{homogeneous deg } -2\end{array}\right\} \underset{\mathcal{T}}{\overset{\text{Funk}}{\rightleftarrows}} \left\{\begin{array}{c}\text{smooth even functions}\\ \text{on } \mathbb{R}^3\setminus 0\\ \text{homogeneous deg } -1\end{array}\right\}$$

$$\mathcal{T}[\phi]\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{4\pi}\int_{S^2} \log\left|\frac{x}{y}\right| \frac{\partial^2\phi}{\partial x^2}$$

$$\left\{\begin{array}{c}\text{smooth odd functions}\\ \text{on } \mathbb{R}^3\setminus 0\\ \text{homogeneous deg } -2\end{array}\right\} \underset{\mathcal{S}}{\overset{\mathcal{G}}{\leftrightarrows}} \left\{\begin{array}{c}\text{smooth odd functions}\\ \text{on } \mathbb{R}^3\setminus 0\\ \text{homogeneous deg } -1\end{array}\right\}$$

$$\mathcal{S}[\psi]\begin{pmatrix}1\\0\\0\end{pmatrix} = -\frac{1}{4\pi}\int_{S^2} \log\left|\frac{x}{y}\right| \frac{\partial\psi}{\partial x}$$

Cor. Given $M_0, M_1 \in C^\infty(S^2)$, there exists a unique sol'n Φ_2,

$$\Box \Phi_2 = 0 \quad (|x| > t),$$

$$\Phi(at, ax) = a^{-2} \Phi(t, x) \quad (a > 0)$$

such that

$$\Phi_2 \sim \frac{1}{r^2} \frac{r}{r-t} M_0(\omega) + \frac{1}{r^2} M_{11} \ln \left| \frac{2r}{r-t} \right| + \frac{1}{r^2} M_1(\omega)$$

where $M_{11}(\omega) = -\frac{1}{2} \Delta_\omega M_0(\omega)$.

Pf. Take ∂_t-derivative of Φ_1.

Thm (Lindblad-S '23)

Let N_{01}^{ext}, N_0^{ext}, $M_0^{ext} \in C^\infty(S^2)$,

and $\mathcal{H}_0(q,\omega)$ smooth, $|\mathcal{H}_0| \leq \langle q\rangle^{-2}$.

Then there exists a sol'n ϕ to

$$\Box\phi = 0 \quad \text{on } \mathbb{R}^{3+1}$$

with

$$\phi \sim \ln\left(\frac{2r}{\langle t-r\rangle}\right)\frac{F_{01}(r-t,\omega)}{r} + \frac{F_0(r-t,\omega)}{r} \quad \left(\begin{matrix} r\sim t, \\ r\to\infty \end{matrix}\right)$$

where

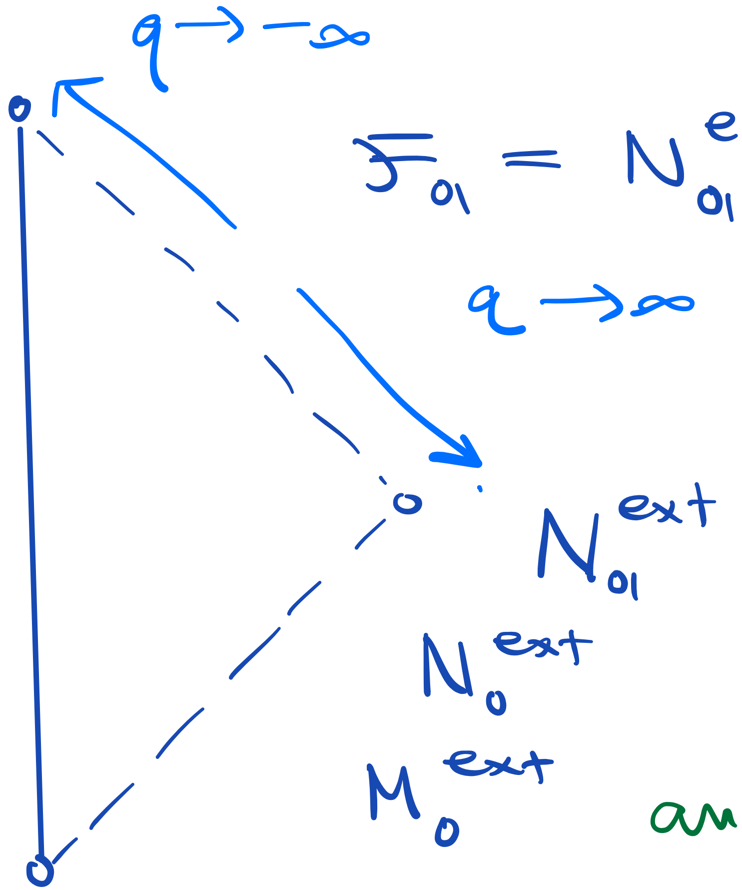
$$F_{01}(q,\omega) = N_{01}^{ext}(\omega)$$

and

$$F_0(q,\omega) = \begin{cases} N_0^{ext}(\omega) + \dfrac{M_0^{ext}}{q} + \mathcal{H}_0(q,\omega), & q>0 \\ N_0^{int}(\omega) + M_0^{int}/q + \mathcal{H}_0(q,\omega), & q<0 \end{cases}$$

N_0^{int}

M_0^{int}



Here

$$\mathcal{F}_0 \sim N_0^{int} + \frac{M_0^{int}}{q} \quad (q \to -\infty)$$

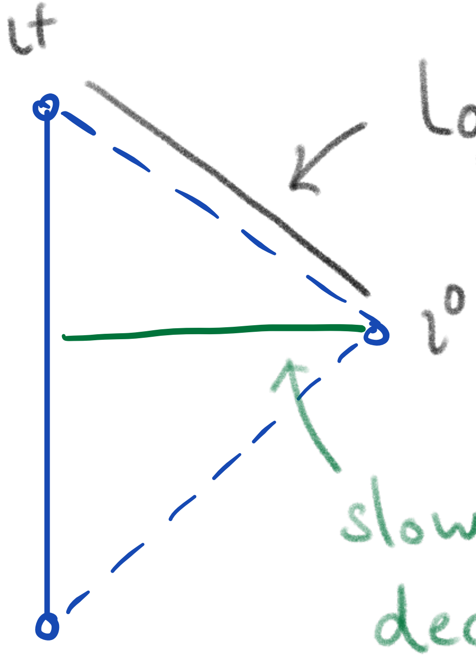
where

$$N_0^{int}(\omega) = \frac{1}{2\pi} \int_{S^2} \frac{N_{01}^{ext}(\sigma) - N_{01}^{ext}(\omega)}{1 - \langle \sigma, \omega \rangle} \, dS(\sigma)$$

and

$$M_0^{int} = M_0^{ext} + C_0 .$$

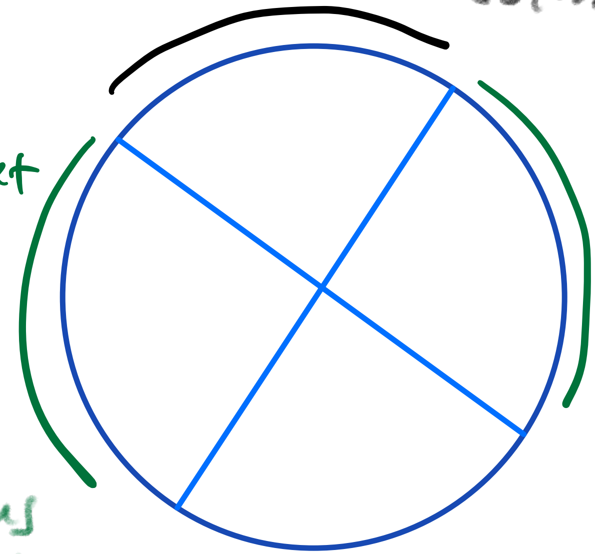
Remarks



Log-terms

slowly decaying data

Φ_1^{int} interior hom. sol'n.



Φ_1^{ext}

homogeneous sol'n in the exterior

$$\phi|_{t=0} \sim \frac{M(\omega)}{r}$$

$$\partial_t \phi|_{t=0} \sim \frac{N(\omega)}{r^2}$$

$\Rightarrow$ The limits & tails of the radiation field at i^+ and i^0 are not independent.

Thm (Lindblad-S '23)

Given a smooth source function $m(q,\omega)$, $|m| \lesssim \langle q\rangle^{-2-}$,

and $\mathcal{H}_0(q,\omega)$, $|\mathcal{H}_0| \lesssim \langle q\rangle^{-2}$,

there exists a smooth solution φ to $\Box\varphi = \frac{m}{r^3}\chi$

so that in the wave zone

$$\varphi \sim \frac{\mathcal{G}_0(r-t,\omega)}{r} \qquad (r\sim t,\ r\to\infty)$$

with

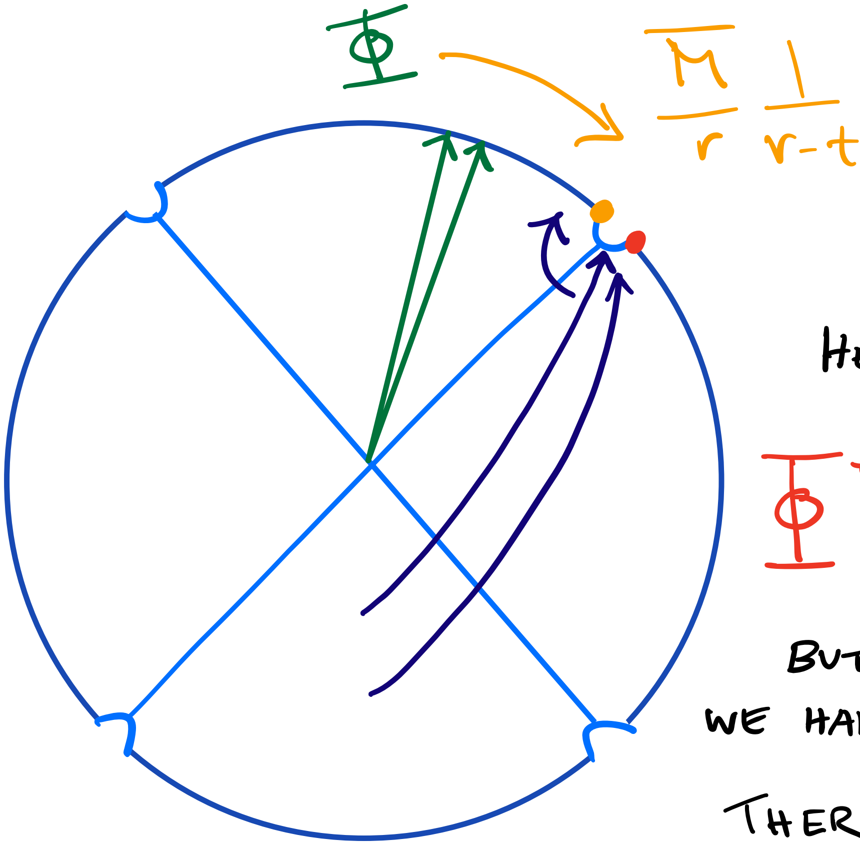
$$\mathcal{G}_0(q,\omega) = \begin{cases} \overline{M}/q + \mathcal{H}_0, & q<0 \\ \mathcal{H}_0, & q>0 \end{cases}$$

provided $\mathcal{H}_0$ satisfies a compatibility cond.

which can be removed by matching to a homogeneous sol'n in the exterior.

$$\Phi_2(at, ax) = a^{-2}\,\Phi_2(t,x)$$

$$\mathcal{F}_0 \sim \frac{M}{q}$$



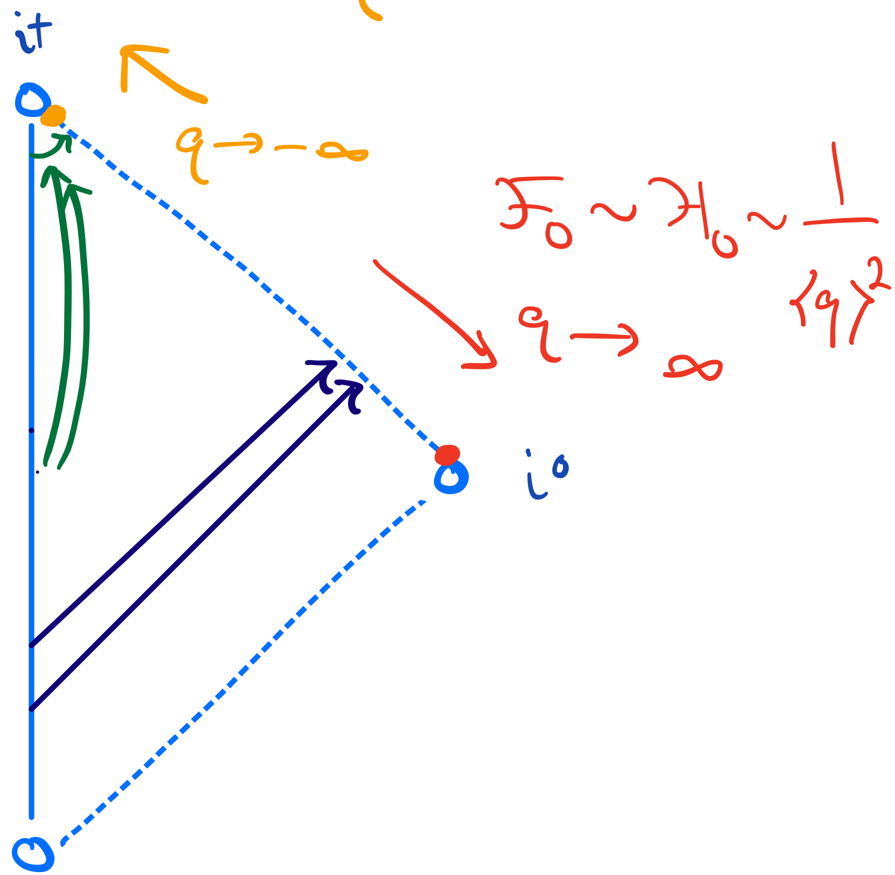
Melrose

HERE

$$\Phi^{ext} = 0$$

BUT AS WE HAVE SEEN:

THERE ARE MORE!



Penrose

Matching to trivial exterior asymptotics $\Longrightarrow$ compatibility condition.

Conclusions & Remarks

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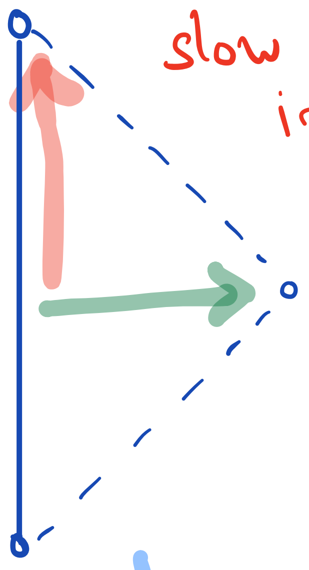


slow decay in time

slowly decaying data

Hom. sol'n $\Phi_1^{int}, \Phi_2^{int}$ ⟷ Homogeneous sol'ns $\Phi_1^{ext}, \Phi_2^{ext}$

In applications: Mass: $\frac{M(\omega)}{r}$

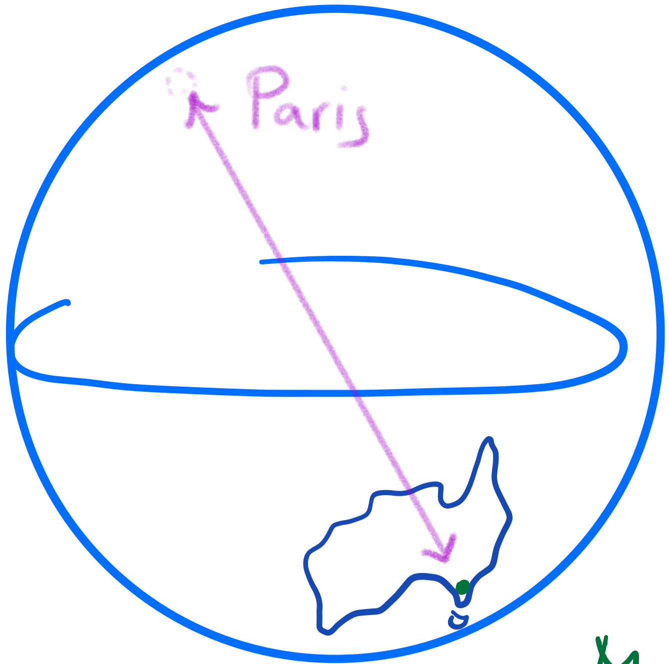
Charge

Angular momentum: $\frac{A(\omega)}{r^2}$

Matching conditions determine which part of the radiation field can be freely prescribed

$$\mathcal{F}_0(q,\omega) = N_0^{ext}(\omega)\,\chi_{q>0} + N_0^{int}(\omega)\,\chi_{q<0} + \frac{M_0^{ext}(\omega)}{q}\chi_{q>0} + \frac{M_0^{int}(\omega)}{q}\chi_{q<0} + \mathcal{H}_0(q,\omega)$$

Thank you!



Come visit!

Melbourne

Hans Lindblad, Volker Schlue, "Scattering for wave equations with sources close to the light cone and prescribed radiation fields", April 2023.

- Quasi-linear equations: E.g. Dongxiao Yu '22
- Applications to wave-Klein Gordon systems

Lili He '21, Chen-Lindblad '23.