On the Linear (In)stability of Extremal Reissner-Nordström

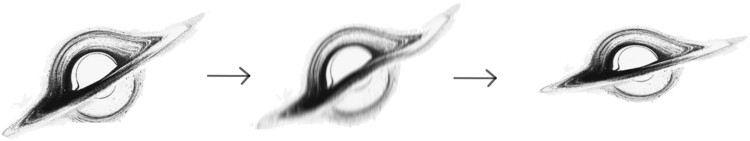
Marios Antonios Apetroaie

University of Münster

Seminar on Mathematical General Relativity
Sorbonne Université

The Stability Problem

Evolution of perturbed stationary solutions to the Einstein equations.



The Stability Problem

Evolution of perturbed stationary solutions to the Einstein equations.

Let $\{g_a\}$ be a family of stationary solutions to Einstein equations (EE). Then, $\{g_a\}$ is **stable** under small perturbations iff for any solution g to EE with initial data "close" to that of g_a, $\quad g \xrightarrow{t\to\infty} g_{a'}, \quad$ for some $\quad a' = a + \delta$.

The Stability Problem

Evolution of perturbed stationary solutions to the Einstein equations.

Let $\{g_a\}$ be a family of stationary solutions to Einstein equations (EE). Then, $\{g_a\}$ is **stable** under small perturbations iff for any solution g to EE with initial data "close" to that of g_a, $\quad g \xrightarrow{t\to\infty} g_{a'}, \quad$ for some $\quad a' = a + \delta.$

Conjecture: Kerr(-Newmann) black holes are stable

Initial data, $(\Sigma_0, \bar{g}, k)$, for the Einstein equation which are sufficiently close to a Kerr(-Newman) black hole evolve asymptotically in time to another member of the Kerr(-Newman) family.

Einstein–Maxwell Equations

Gravitational field interacts with electromagnetic radiation

$$Ric(g) = 2F \cdot F - \frac{1}{2} |F|^2 \; g$$
$$dF = 0, \qquad div \; F = 0$$

$F_{\mu\nu}$: electromagnetic tensor ($F = d \; A$).

Reissner-Nordström spacetime (1917): $|Q| \leq M$

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \gamma_{\mathbb{S}^2}$$

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

What about the linear stability of **extremal** RN spacetime, $|Q| = M$?

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

What about the linear stability of **extremal** RN spacetime, $|Q| = M$?

(Physics)

- About 70% of the stellar black holes are near-extremal (rotating very fast).
- Observational signatures
- Supersymmetry, holography and quantum gravity (Zero entropy)

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

What about the linear stability of **extremal** RN spacetime, $|Q| = M$?

(Physics)

- About 70% of the stellar black holes are near-extremal (rotating very fast).
- Observational signatures
- Supersymmetry, holography and quantum gravity (Zero entropy)

(Math)

- Degeneracy of the red-shift effect on the event horizon $\mathcal{H}^+$.

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

What about the linear stability of **extremal** RN spacetime, $|Q| = M$?

(Physics)

- About 70% of the stellar black holes are near-extremal (rotating very fast).
- Observational signatures
- Supersymmetry, holography and quantum gravity (Zero entropy)

(Math)

- Degeneracy of the red-shift effect on the event horizon $\mathcal{H}^+$.
- Trapping effect on degenerate horizons.

Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi © Springer

Department of Mathematics, Princeton University, Princeton, USA.

What about the linear stability of **extremal** RN spacetime, $|Q| = M$?

(Physics)

- About 70% of the stellar black holes are near-extremal (rotating very fast).
- Observational signatures
- Supersymmetry, holography and quantum gravity (Zero entropy)

(Math)

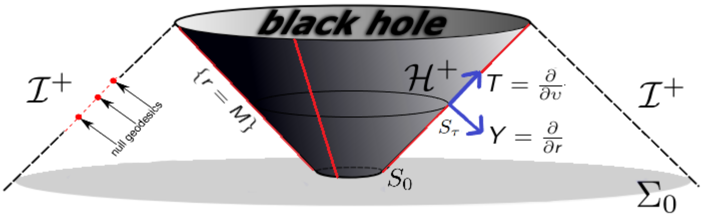
- Degeneracy of the red-shift effect on the event horizon $\mathcal{H}^+$.
- Trapping effect on degenerate horizons.
- Horizon instabilities manifest for the scalar wave equation (Aretakis Instability).

ERN Geometry

Ingoing Eddington-Finkelstein coordinates: $(\upsilon, r, \theta, \phi)$

$$g_{ERN} = -D(r)d\upsilon^2 + 2d\upsilon dr + r^2 \cdot \gamma, \qquad D(r) := \left(1 - \frac{M}{r}\right)^2$$

- $T := \frac{\partial}{\partial \upsilon}$ is normal to the event horizon $\mathcal{H}^+$, tangent to it's null geodesics/generators.
- $Y := \frac{\partial}{\partial r}$ is transversal to $\mathcal{H}^+$



Aretakis Instability

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010)

For all solutions ψ_ℓ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, i.e. $\not{\Delta}\psi_\ell = -\frac{\ell(\ell+1)}{r^2}\psi_\ell$, there exists a quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i \cdot \partial_r^i \psi_\ell$$

which is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, generically on $\mathcal{H}^+$.)

Aretakis Instability

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010)

For all solutions ψ_ℓ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, i.e. $\Delta\!\!\!/ \psi_\ell = -\frac{\ell(\ell+1)}{r^2}\psi_\ell$, there exists a quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i \cdot \partial_r^i \psi_\ell$$

which is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, generically on $\mathcal{H}^+$.)

Idea: In $(\upsilon, r, \theta, \phi)$ coordinates,

$$\Box\psi = D\partial_r\partial_r\psi + 2\partial_\upsilon\partial_r\psi + \frac{2}{r}\partial_\upsilon\psi + R\partial_r\psi + \Delta\!\!\!/\psi$$

For spherical symmetric solutions ψ, i.e. $\Delta\!\!\!/\psi = 0$, along the horizon $\mathcal{H}^+$ we have $D(r = M) = R(r = M) = 0$, thus

$$T\left(2\partial_r\psi + \frac{2}{M}\psi\right) = 0$$

Aretakis Instability

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010)

For all solutions ψ_ℓ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, i.e. $\not{\Delta}\psi_\ell = -\frac{\ell(\ell+1)}{r^2}\psi_\ell$, there exists a quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i \cdot \partial_r^i \psi_\ell$$

which is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, generically on $\mathcal{H}^+$.)

- **Decay:** For all $k \leq \ell$, we have
$$\left|\partial_r^k \psi_\ell\right| \xrightarrow{\tau \to \infty} 0$$
- **Non-decay:**
$$\partial_r^{\ell+1}\psi_\ell \xrightarrow{\tau \to \infty} H_\ell[\psi_\ell]$$
- **Blow-up:** For all $k \in \mathbb{N}$, we have
$$\partial_r^{\ell+1+k}\psi_\ell \sim_k H_\ell \cdot \tau^k$$

Linearized Einstein–Maxwell equations

Represent the Einstein–Maxwell equations with the non-linear operator

$$\mathcal{E}[\varphi] = 0 \tag{2}$$

Let φ_λ be a family of stationary solutions to (2).

$$\frac{\delta \mathcal{E}[\psi]}{\delta \psi}\Big|_{\varphi_\lambda} = 0$$

Study solutions ψ asymptotically in time.

Linearized Einstein–Maxwell equations

$$\begin{aligned}
\slashed{\nabla}_3\underline{\widehat{\chi}} + (\underline{\kappa} + 2\underline{\omega})\,\underline{\widehat{\chi}} &= -2\,\slashed{\mathcal{D}}_2^\star\underline{\xi} - \underline{\alpha},\\
\slashed{\nabla}_4\widehat{\chi} + (\kappa + 2\omega)\ \widehat{\chi} &= -2\,\slashed{\mathcal{D}}_2^\star\xi - \alpha,\\
\slashed{\nabla}_3\widehat{\chi} + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right)\widehat{\chi} &= -2\slashed{\mathcal{D}}_2^\star\eta - \frac{1}{2}\kappa\underline{\widehat{\chi}}\\
\slashed{\nabla}_4\underline{\widehat{\chi}} + \left(\frac{1}{2}\kappa - 2\omega\right)\underline{\widehat{\chi}} &= -2\slashed{\mathcal{D}}_2^\star\underline{\eta} - \frac{1}{2}\underline{\kappa}\widehat{\chi},
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\underline{\kappa} + \frac{1}{2}\underline{\kappa}^2 + 2\underline{\omega}\ \underline{\kappa} &= 2\slashed{\mathrm{div}}\,\underline{\xi},\\
\slashed{\nabla}_4\kappa + \frac{1}{2}\kappa^2 + 2\omega\ \kappa &= 2\slashed{\mathrm{div}}\,\xi,\\
\slashed{\nabla}_3\kappa + \frac{1}{2}\kappa\underline{\kappa} - 2\underline{\omega}\kappa &= 2\slashed{\mathrm{div}}\,\eta + 2\rho,\\
\slashed{\nabla}_4\underline{\kappa} + \frac{1}{2}\kappa\underline{\kappa} - 2\omega\underline{\kappa} &= 2\slashed{\mathrm{div}}\,\underline{\eta} + 2\rho,
\end{aligned}$$

$$\begin{aligned}
\slashed{\mathrm{div}}\,\underline{\widehat{\chi}} &= -\frac{1}{2}\underline{\kappa}\zeta - \frac{1}{2}\,\slashed{\mathcal{D}}_1^\star(\underline{\kappa}, 0) + \underline{\beta} - {}^{(F)}\rho\,{}^{(F)}\underline{\beta},\\
\slashed{\mathrm{div}}\,\widehat{\chi} &= \frac{1}{2}\kappa\zeta - \frac{1}{2}\,\slashed{\mathcal{D}}_1^\star(\kappa, 0) - \beta + {}^{(F)}\rho\,{}^{(F)}\beta
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_4\underline{\omega} + \slashed{\nabla}_3\omega &= 4\omega\underline{\omega} + \rho + {}^{(F)}\rho^2,\\
\slashed{\mathrm{curl}}\,\eta &= \sigma,\\
\slashed{\mathrm{curl}}\,\underline{\eta} &= -\sigma
\end{aligned}$$

$$K = -\frac{1}{4}\kappa\underline{\kappa} - \rho + {}^{(F)}\rho^2$$

$$\begin{aligned}
\slashed{\nabla}_3\,{}^{(F)}\rho + \underline{\kappa}\,{}^{(F)}\rho &= -\slashed{\mathrm{div}}\,{}^{(F)}\underline{\beta}\\
\slashed{\nabla}_4\,{}^{(F)}\rho + \kappa\,{}^{(F)}\rho &= \slashed{\mathrm{div}}\,{}^{(F)}\beta
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\,{}^{(F)}\sigma + \underline{\kappa}\ {}^{(F)}\sigma &= -\slashed{\mathrm{curl}}\ {}^{(F)}\underline{\beta}\\
\slashed{\nabla}_4\,{}^{(F)}\sigma + \kappa\ {}^{(F)}\sigma &= \slashed{\mathrm{curl}}\ {}^{(F)}\beta
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\zeta + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right)\zeta &= 2\,\slashed{\mathcal{D}}_1^\star(\underline{\omega}, 0) - \left(\frac{1}{2}\underline{\kappa} + 2\underline{\omega}\right)\eta + \left(\frac{1}{2}\kappa + 2\omega\right)\underline{\xi} - \underline{\beta} - {}^{(F)}\rho\,{}^{(F)}\underline{\beta},\\
\slashed{\nabla}_4\zeta + \left(\frac{1}{2}\kappa - 2\underline{\omega}\right)\zeta &= -2\,\slashed{\mathcal{D}}_1^\star(\underline{\omega}, 0) + \left(\frac{1}{2}\kappa + 2\omega\right)\underline{\eta} - \left(\frac{1}{2}\underline{\kappa} + 2\underline{\omega}\right)\xi - \beta - {}^{(F)}\rho\,{}^{(F)}\beta,\\
\slashed{\nabla}_4\underline{\xi} - \slashed{\nabla}_3\underline{\eta} &= -\frac{1}{2}\underline{\kappa}\,(\eta - \underline{\eta}) + 4\omega\underline{\xi} - \underline{\beta} - {}^{(F)}\rho\,{}^{(F)}\underline{\beta},\\
\slashed{\nabla}_3\xi - \slashed{\nabla}_4\eta &= \frac{1}{2}\kappa\,(\eta - \underline{\eta}) + 4\underline{\omega}\xi + \beta + {}^{(F)}\rho\,{}^{(F)}\beta,
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\,{}^{(F)}\beta + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right){}^{(F)}\beta &= -\slashed{\mathcal{D}}_1^\star({}^{(F)}\rho, {}^{(F)}\sigma) + 2\,{}^{(F)}\rho\eta,\\
\slashed{\nabla}_4\,{}^{(F)}\underline{\beta} + \left(\frac{1}{2}\kappa - 2\omega\right){}^{(F)}\underline{\beta} &= \slashed{\mathcal{D}}_1^\star({}^{(F)}\rho, -{}^{(F)}\sigma) - 2\,{}^{(F)}\rho\underline{\eta}
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\alpha + \left(\frac{1}{2}\underline{\kappa} - 4\underline{\omega}\right)\alpha &= -2\,\slashed{\mathcal{D}}_2^\star\,\beta - 3\rho\widehat{\chi} - 2\,{}^{(F)}\rho\ \left(\slashed{\mathcal{D}}_2^\star\,{}^{(F)}\beta + {}^{(F)}\rho\widehat{\chi}\right),\\
\slashed{\nabla}_4\underline{\alpha} + \left(\frac{1}{2}\kappa - 4\omega\right)\underline{\alpha} &= 2\,\slashed{\mathcal{D}}_2^\star\,\underline{\beta} - 3\rho\underline{\widehat{\chi}} + 2\,{}^{(F)}\rho\ \left(\slashed{\mathcal{D}}_2^\star\,{}^{(F)}\underline{\beta} - {}^{(F)}\rho\underline{\widehat{\chi}}\right),
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\beta + (\underline{\kappa} - 2\underline{\omega})\,\beta &= \slashed{\mathcal{D}}_1^\star(-\rho, \sigma) + 3\rho\eta + {}^{(F)}\rho\left(\slashed{\mathcal{D}}_1^\star(-{}^{(F)}\rho, {}^{(F)}\sigma) - \kappa\,{}^{(F)}\underline{\beta} - \frac{1}{2}\underline{\kappa}\,{}^{(F)}\beta\right),\\
\slashed{\nabla}_4\underline{\beta} + (\kappa - 2\omega)\,\underline{\beta} &= \slashed{\mathcal{D}}_1^\star(\rho, \sigma) - 3\rho\underline{\eta} + {}^{(F)}\rho\left(\slashed{\mathcal{D}}_1^\star({}^{(F)}\rho, {}^{(F)}\sigma) - \underline{\kappa}\,{}^{(F)}\beta - \frac{1}{2}\kappa\,{}^{(F)}\underline{\beta}\right),\\
\slashed{\nabla}_3\underline{\beta} + (2\underline{\kappa} + 2\underline{\omega})\,\underline{\beta} &= -\slashed{\mathrm{div}}\,\underline{\alpha} - 3\rho\underline{\xi} + {}^{(F)}\rho\left(\slashed{\nabla}_3\,{}^{(F)}\underline{\beta} + 2\underline{\omega}\,{}^{(F)}\underline{\beta} + 2\,{}^{(F)}\rho\ \underline{\xi}\right),\\
\slashed{\nabla}_4\beta + (2\kappa + 2\omega)\,\beta &= \slashed{\mathrm{div}}\,\alpha + 3\rho\xi + {}^{(F)}\rho\ \left(\slashed{\nabla}_4\,{}^{(F)}\beta + 2\omega\,{}^{(F)}\beta - 2\,{}^{(F)}\rho\xi\right)
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\rho + \frac{3}{2}\underline{\kappa}\rho &= -\underline{\kappa}\,{}^{(F)}\rho^2 - \slashed{\mathrm{div}}\,\underline{\beta} - {}^{(F)}\rho\ \slashed{\mathrm{div}}\ {}^{(F)}\underline{\beta},\\
\slashed{\nabla}_4\rho + \frac{3}{2}\kappa\rho &= -\kappa\,{}^{(F)}\rho^2 + \slashed{\mathrm{div}}\,\beta + {}^{(F)}\rho\ \slashed{\mathrm{div}}\ {}^{(F)}\beta,
\end{aligned}$$

$$\begin{aligned}
\slashed{\nabla}_3\sigma + \frac{3}{2}\underline{\kappa}\sigma &= -\slashed{\mathrm{curl}}\ \underline{\beta} - {}^{(F)}\rho\ \slashed{\mathrm{curl}}\ {}^{(F)}\underline{\beta},\\
\slashed{\nabla}_4\sigma + \frac{3}{2}\kappa\sigma &= -\slashed{\mathrm{curl}}\ \beta - {}^{(F)}\rho\ \slashed{\mathrm{curl}}\ {}^{(F)}\beta
\end{aligned}$$

Linearized Einstein–Maxwell equations

$$\nabla_3 \alpha + \left(\frac{1}{2}\underline{\kappa} - 4\underline{\omega}\right)\alpha = -2\,\mathcal{D}_2^\star \beta - 3\rho\widehat{\chi} - 2\,{}^{(F)}\rho\,\left(\mathcal{D}_2^\star\,{}^{(F)}\beta + {}^{(F)}\rho\widehat{\chi}\right),$$

$$\nabla_4 \underline{\alpha} + \left(\frac{1}{2}\kappa - 4\omega\right)\underline{\alpha} = 2\,\mathcal{D}_2^\star \underline{\beta} - 3\rho\underline{\widehat{\chi}} + 2\,{}^{(F)}\rho\,\left(\mathcal{D}_2^\star\,{}^{(F)}\underline{\beta} - {}^{(F)}\rho\underline{\widehat{\chi}}\right),$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\Box_{g_{Sch}}\alpha + c(r)\nabla_T\alpha + d(r)\nabla_Y\alpha + V(r)\alpha = 0$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\mathcal{T}(\alpha) := \Box_{g_{Sch}}\alpha + c(r)\nabla_T\alpha + d(r)\nabla_Y\alpha + V(r)\alpha = 0$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\mathcal{T}(\alpha) := \Box_{g_{Sch}}\alpha + c(r)\nabla_T\alpha + d(r)\nabla_Y\alpha + V(r)\alpha = 0$$

For **Reissner–Nordström**

$$\mathcal{T}(\alpha) = S_1(\mathfrak{f}), \qquad \mathfrak{f} := -\frac{1}{2}\nabla\widehat{\otimes}{}^{(F)}\overset{(1)}{\beta} + {}^{(F)}\rho\,\overset{(1)}{\widehat{\chi}}$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\mathcal{T}(\alpha) := \Box_{g_{Sch}} \alpha + c(r)\nabla_T \alpha + d(r)\nabla_Y \alpha + V(r)\alpha = 0$$

For **Reissner–Nordström**

$$(+2 \text{ spin}) \quad \begin{cases} \mathcal{T}(\alpha) = S_1(\mathfrak{f}), & \mathfrak{f} := -\frac{1}{2}\nabla \widehat{\otimes} {}^{(F)}\overset{(1)}{\beta} + {}^{(F)}\rho \, \overset{(1)}{\widehat{\chi}} \\ \mathcal{T}(\mathfrak{f}) = S_2(\alpha, \mathfrak{b}), & \mathfrak{b} := 2\,{}^{(F)}\rho \, \overset{(1)}{\beta} - 3\rho \, {}^{(F)}\overset{(1)}{\beta} \end{cases}$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\mathcal{T}(\alpha) := \Box_{g_{Sch}}\alpha + c(r)\nabla_T\alpha + d(r)\nabla_Y\alpha + V(r)\alpha = 0$$

For **Reissner–Nordström**

$$(+2 \text{ spin}) \quad \begin{cases} \mathcal{T}(\alpha) = S_1(\mathfrak{f}), & \mathfrak{f} := -\frac{1}{2}\nabla\widehat{\otimes}{}^{(F)}\overset{(1)}{\beta} + {}^{(F)}\rho\,\overset{(1)}{\widehat{\chi}} \\ \mathcal{T}(\mathfrak{f}) = S_2(\alpha,\mathfrak{b}), & \mathfrak{b} := 2\,{}^{(F)}\rho\,\overset{(1)}{\beta} - 3\rho\,{}^{(F)}\overset{(1)}{\beta} \end{cases}$$

$$(+1 \text{ spin}) \quad \mathcal{T}(\mathfrak{b}) = S_3(\alpha,\mathfrak{f})$$

Teukolsky system of $\pm$ spin

For **Schwarzschild** (D.H.R., 2016)

$$\mathcal{T}(\alpha) := \Box_{g_{Sch}}\alpha + c(r)\nabla_T\alpha + d(r)\nabla_Y\alpha + V(r)\alpha = 0$$

For **Reissner–Nordström**

$$(+2\text{ spin})\quad \begin{cases}\mathcal{T}(\alpha) = S_1(\mathfrak{f}), & \mathfrak{f} := -\frac{1}{2}\nabla\widehat{\otimes}{}^{(F)}\overset{(1)}{\beta} + {}^{(F)}\rho\,\overset{(1)}{\widehat{\chi}}\\ \mathcal{T}(\mathfrak{f}) = S_2(\alpha,\mathfrak{b}), & \mathfrak{b} := 2{}^{(F)}\rho\,\overset{(1)}{\beta} - 3\rho\,{}^{(F)}\overset{(1)}{\beta}\end{cases}$$

$$(+1\text{ spin})\quad \mathcal{T}(\mathfrak{b}) = S_3(\alpha,\mathfrak{f})$$

Teukolsky system in RN. (Elena Giorgi, 2018)

$$\mathcal{T}(\alpha) = s_1\ \nabla_{\partial_v}\mathfrak{f} + w_1\ \mathfrak{f} \qquad \mathcal{T}(\underline{\alpha}) = s_{-1}\nabla_{\partial_r}\underline{\mathfrak{f}} + w_{-1}\underline{\mathfrak{f}}$$

$$\mathcal{T}(\mathfrak{f}) = s_2\ \mathfrak{b} + w_2\ \alpha \qquad \mathcal{T}(\underline{\mathfrak{f}}) = s_{-2}\underline{\mathfrak{b}} + w_{-2}\underline{\alpha}$$

$$\mathcal{T}(\mathfrak{b}) = s_3\ \mathrm{d\!\!/iv}\,\mathfrak{f} + w_3\ \mathrm{d\!\!/iv}\,\alpha \qquad \mathcal{T}(\underline{\mathfrak{b}}) = s_{-3}\ \mathrm{d\!\!/iv}\,\underline{\mathfrak{f}} + w_{-3}\ \mathrm{d\!\!/iv}\,\underline{\alpha}$$

Studying the above system directly is not possible*

Regge–Wheeler System

Transformation:

$$\boldsymbol{q}^F = r\nabla_{\partial_r}\Big(c_1(r)\cdot \mathfrak{f}\Big)$$
$$\boldsymbol{p} = r\nabla_{\partial_r}\Big(c_2(r)\cdot \mathfrak{b}\Big)$$

$$\Box_{g_{ERN}}\boldsymbol{q}^F - A_1(r)\cdot\boldsymbol{q}^F = h_1(r)\nabla\!\!\!/\,\widehat{\otimes}\boldsymbol{p}$$
$$\Box_{g_{ERN}}\boldsymbol{p} - A_2(r)\cdot\boldsymbol{p} = h_2(r)\,d\!\!/iv\,\boldsymbol{q}^F$$

Regge–Wheeler System

Transformation:

$$\boldsymbol{q}^F = r\nabla_{\partial_r}\Big(c_1(r)\cdot \mathfrak{f}\Big)$$
$$\boldsymbol{p} = r\nabla_{\partial_r}\Big(c_2(r)\cdot \mathfrak{b}\Big)$$

$$\Box_{g_{ERN}}\boldsymbol{q}^F - A_1(r)\cdot\boldsymbol{q}^F = h_1(r)\nabla\!\!\!/\,\widehat{\otimes}\boldsymbol{p}$$
$$\Box_{g_{ERN}}\boldsymbol{p} - A_2(r)\cdot\boldsymbol{p} = h_2(r)\,\mathrm{d}\!\!/\mathrm{iv}\,\boldsymbol{q}^F$$
$$\downarrow$$
$$\Box_{g_{ERN}}\phi + V_1(r)\phi = -\frac{1}{2r}\Delta\!\!\!/\,\psi - \frac{1}{r^3}\psi$$
$$\Box_{g_{ERN}}\psi + V_2(r)\psi = \frac{8M^2}{r^3}\phi$$

$$\phi = r^2\left(\mathrm{d}\!\!/\mathrm{iv}\,\mathrm{d}\!\!/\mathrm{iv}\,\boldsymbol{q}^F, \mathrm{c}\!\!/\mathrm{url}\,\mathrm{d}\!\!/\mathrm{iv}\,\boldsymbol{q}^F\right)$$
$$\psi = r\ \left(\mathrm{d}\!\!/\mathrm{iv}\,\boldsymbol{p}, \mathrm{c}\!\!/\mathrm{url}\,\boldsymbol{p}\right)$$

Regge–Wheeler System

Transformation:

$$\boldsymbol{q}^F = r\nabla_{\partial_r}\Big(c_1(r)\cdot\mathfrak{f}\Big)$$
$$\boldsymbol{p} = r\nabla_{\partial_r}\Big(c_2(r)\cdot\mathfrak{b}\Big)$$

$$\Box_{g_{ERN}}\boldsymbol{q}^F - A_1(r)\cdot\boldsymbol{q}^F = h_1(r)\nabla\!\!\!/\widehat{\otimes}\boldsymbol{p}$$
$$\Box_{g_{ERN}}\boldsymbol{p} - A_2(r)\cdot\boldsymbol{p} = h_2(r)\,d\!\!/iv\,\boldsymbol{q}^F$$

$$\downarrow$$

$$\Box_{g_{ERN}}\phi + V_1(r)\phi = -\frac{1}{2r}\Delta\!\!\!/\psi - \frac{1}{r^3}\psi$$
$$\Box_{g_{ERN}}\psi + V_2(r)\psi = \frac{8M^2}{r^3}\phi$$

$$\phi = r^2\left(d\!\!/iv\,d\!\!/iv\,\boldsymbol{q}^F, c\!\!/url\,d\!\!/iv\,\boldsymbol{q}^F\right)$$
$$\psi = r\;\left(d\!\!/iv\,\boldsymbol{p}, c\!\!/url\,\boldsymbol{p}\right)$$

$$\downarrow$$

Scalar Regge–Wheeler equations (RW)

$$\left(\Box_{g_{ERN}} - V_i^{(\ell)}(r)\right)\Psi_i^{(\ell)} = 0, \quad \ell \geq i,\ i = 1,2$$

$$\Psi_i^{(\ell)} = a_i(\ell, M)\phi_\ell + b_i(\ell, M)\psi_\ell$$
$$V_i^{(\ell)}(r) = \mathcal{O}(\tfrac{1}{r^3}).$$

Qualitative behavior of solutions to (RW)

Conservation Laws along $\mathcal{H}^+$. (A. 2022)

Let $\Psi_i^{(\ell)}$, $\ell \geq i$, be a solution to (RW), i.e. $\Box\Psi_i^{(\ell)} - V_i^{(\ell)}\Psi_i^{(\ell)} = 0$, then the quantities

$$H_\ell[\Psi_1] \;=\; \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)}$$

$$H_\ell[\Psi_2] \;=\; \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)}$$

are conserved along the null generators of $\mathcal{H}^+$.

Qualitative behavior of solutions to (RW)

Conservation Laws along $\mathcal{H}^+$. (A. 2022)

Let $\Psi_i^{(\ell)}$, $\ell \geq i$, be a solution to (RW), i.e. $\Box\Psi_i^{(\ell)} - V_i^{(\ell)}\Psi_i^{(\ell)} = 0$, then the quantities

$$H_\ell[\Psi_1] \; = \; \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)}$$

$$H_\ell[\Psi_2] \; = \; \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)}$$

are conserved along the null generators of $\mathcal{H}^+$.

Decay along $\mathcal{H}^+$:

- $\left|\partial_r^k \Psi_1^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell + 2$
- $\left|\partial_r^k \Psi_2^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell$

Qualitative behavior of solutions to (RW)

Conservation Laws along $\mathcal{H}^+$. (A. 2022)

Let $\Psi_i^{(\ell)}$, $\ell \geq i$, be a solution to (RW), i.e. $\Box\Psi_i^{(\ell)} - V_i^{(\ell)}\Psi_i^{(\ell)} = 0$, then the quantities

$$H_\ell[\Psi_1] \;=\; \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)}$$

$$H_\ell[\Psi_2] \;=\; \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)}$$

are conserved along the null generators of $\mathcal{H}^+$.

Decay along $\mathcal{H}^+$:

- $\left|\partial_r^k \Psi_1^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell + 2$
- $\left|\partial_r^k \Psi_2^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell$

Non-decay and Blow-up along $\mathcal{H}^+$:

- $\partial_r^{(\ell+2)+k}\Psi_1^{(\ell)}(\tau) \sim_k H_\ell[\Psi_1] \cdot \tau^k, \quad k \in \mathbb{N}$
- $\partial_r^{\ell+k}\Psi_2^{(\ell)}(\tau) \sim_k H_\ell[\Psi_2] \cdot \tau^k, \quad k \in \mathbb{N}$

Qualitative behavior of solutions to (RW)

Conservation Laws along $\mathcal{H}^+$. (A. 2022)

Let $\Psi_i^{(\ell)}$, $\ell \geq i$, be a solution to (RW), i.e. $\Box\Psi_i^{(\ell)} - V_i^{(\ell)}\Psi_i^{(\ell)} = 0$, then the quantities

$$H_\ell[\Psi_1] \; = \; \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)}$$

$$H_\ell[\Psi_2] \; = \; \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)}$$

are conserved along the null generators of $\mathcal{H}^+$.

Decay along $\mathcal{H}^+$:

- $\left|\partial_r^k \Psi_1^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell + 2$
- $\left|\partial_r^k \Psi_2^{(\ell)}\right| \xrightarrow{\tau\to\infty} 0, \quad k < \ell$

Non-decay and Blow-up along $\mathcal{H}^+$:

- $\partial_r^{(\ell+2)+k}\Psi_1^{(\ell)}(\tau) \sim_k H_\ell[\Psi_1] \cdot \tau^k, \quad k \in \mathbb{N}$
- $\partial_r^{\ell+k}\Psi_2^{(\ell)}(\tau) \sim_k H_\ell[\Psi_2] \cdot \tau^k, \quad k \in \mathbb{N}$

Note,the most dominant along $\mathcal{H}^+$ is $\Psi_2^{(\ell=2)}$, with $\partial_r^2\Psi_2^{(2)} \xrightarrow{\tau\to\infty} H_{\ell=2}[\Psi_2]$

On the horizon instability of an extreme Reissner-Nordström black hole

James Lucietti[a*], Keiju Murata[b,c†], Harvey S. Reall[b‡]
and Norihiro Tanahashi[d§]

April 24, 2013

- Moncrief's formalism.

	$l = 1$ odd	$l > 1$ odd	$l = 1$ even	$l > 1$ even
ψ	P_f	$P_\pm$	H	$R_\pm$
$W\|_{r=M}$	6	$l(l+1)+1\pm(2l+1)$	6	$l(l+1)+1\pm(2l+1)$
p	2	$l\pm1$	2	$l\pm1$

Table 1: Conserved quantities $H_p[\psi]$ for Moncrief's perturbations.

On the horizon instability of an extreme Reissner-Nordström black hole

James Lucietti[a*], Keiju Murata[b,c†], Harvey S. Reall[b‡] and Norihiro Tanahashi[d§]

April 24, 2013

- Moncrief's formalism.

	$l=1$ odd	$l>1$ odd	$l=1$ even	$l>1$ even
ψ	P_f	$P_\pm$	H	$R_\pm$
$W\|_{r=M}$	6	$l(l+1)+1\pm(2l+1)$	6	$l(l+1)+1\pm(2l+1)$
p	2	$l\pm 1$	2	$l\pm 1$

Table 1: Conserved quantities $H_p[\psi]$ for Moncrief's perturbations.

- For $\ell = 2$ odd parity perturbations the conserved quantities are

$$\begin{aligned} H_1[P_-] &= \left(\partial_r^2 P_- + \frac{2}{M}\partial_r P_-\right)_{r=M} \\ H_3[P_+] &= \left(\partial_r^4 P_+ + \ldots\right)_{r=M} \end{aligned}$$

- $P_+ \sim \Psi_1^{(\ell)}$ and $P_- \sim \Psi_2^{(\ell)}$

Estimates for the Regge–Wheeler tensorial system

- Using the estimates for $\Psi_i^{(\ell)}$, we control ϕ_ℓ, ψ_ℓ,

$$\phi_\ell = c_1 \cdot \Psi_1^{(\ell)} + c_2 \cdot \Psi_2^{(\ell)}, \qquad \psi_\ell = d_1 \cdot \Psi_1^{(\ell)} + d_2 \cdot \Psi_2^{(\ell)}$$

  They both inherit the behavior of $\Psi_2^{(\ell)}$ asymptotically on $\mathcal{H}^+$.

Estimates for the Regge–Wheeler tensorial system

- Using the estimates for $\Psi_i^{(\ell)}$, we control ϕ_ℓ, ψ_ℓ,

$$\phi_\ell = c_1 \cdot \Psi_1^{(\ell)} + c_2 \cdot \Psi_2^{(\ell)}, \qquad \psi_\ell = d_1 \cdot \Psi_1^{(\ell)} + d_2 \cdot \Psi_2^{(\ell)}$$

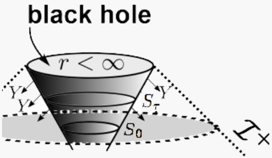
  They both inherit the behavior of $\Psi_2^{(\ell)}$ asymptotically on $\mathcal{H}^+$.
- Using standard elliptic identities we obtain
  - Decay: $\|\boldsymbol{p}\|_{S^2_{\tau,M}} \lesssim_M \tau^{-\frac{3}{4}}, \qquad \|\nabla_{\partial_r}\boldsymbol{p}\|_{S^2_{\tau,M}} \lesssim_M \tau^{-\frac{1}{4}}$
  - Non-Decay: $\left\|\nabla^2_{\partial_r}\boldsymbol{p}\right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} c\,\|H_2[\Psi_2]\|_{S^2_{\tau,M}}$
  - Blow-up: $\left\|\nabla^k_{\partial_r}\boldsymbol{p}\right\|_{S^2_{\tau,M}} \gtrsim_M \|H_2[\Psi_2]\|_{S^2_{\tau,M}} \cdot \tau^{k-2}, \quad k \geq 3.$

  where $\|\xi\|^2_{S^2_{v,r}} := \int_{S^2_{v,r}} r^2 \sin\theta d\theta d\phi\, |\xi|^2$.
- Similar estimates hold for $\boldsymbol{q}^F$ (Same order of instability).

The $\pm$ spin Teukolsky solutions

Theorem (A. 2022).

Let $\alpha, \mathfrak{f}, \mathfrak{b}$ and $\underline{\alpha}, \underline{\mathfrak{f}}, \underline{\mathfrak{b}}$ be solutions to the Teukolsky system, then for generic initial data, they all **decay** away from the event horizon $\mathcal{H}^+$, i.e. $\{r \geq r_0\}$, $r_0 > M$.



The $\pm$ spin Teukolsky solutions

Theorem (A. 2022).

Let $\alpha, \mathfrak{f}, \mathfrak{b}$ and $\underline{\alpha}, \underline{\mathfrak{f}}, \underline{\mathfrak{b}}$ be solutions to the Teukolsky system, then for generic initial data, they all **decay** away from the event horizon $\mathcal{H}^+$, i.e. $\{r \geq r_0\}$, $r_0 > M$. In addition, along $\mathcal{H}^+$ we have

The $\pm$ spin Teukolsky solutions

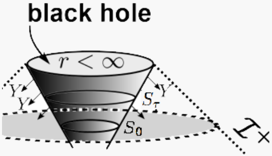
$\|\xi\|_{S_\tau} := \int_{\mathbb{S}^2} r^2 \sin\theta d\theta d\phi\, |\xi|^2, \;\; \|\xi\|_\infty(\tau) := \|\xi\|_{L^\infty(S_\tau)}$

Theorem (A. 2022).

Let $\alpha, \mathfrak{f}, \mathfrak{b}$ and $\underline{\alpha}, \underline{\mathfrak{f}}, \underline{\mathfrak{b}}$ be solutions to the Teukolsky system, then for generic initial data, they all **decay** away from the event horizon $\mathcal{H}^+$, i.e. $\{r \geq r_0\}$, $r_0 > M$. In addition, along $\mathcal{H}^+$ we have

Positive spin:

- **Decay:** $\left\|\nabla_Y^{\leq 2}\mathfrak{f}\right\|_\infty(\tau)$, $\left\|\nabla_Y^{\leq 2}\mathfrak{b}\right\|_\infty(\tau)$, and $\left\|\nabla_Y^{\leq 4}\alpha\right\|_\infty(\tau)$
- **Non-decay:** $\left\|\nabla_Y^3\mathfrak{f}\right\|_{S_\tau}$, $\left\|\nabla_Y^3\mathfrak{b}\right\|_{S_\tau}$, and $\left\|\nabla_Y^5\alpha\right\|_{S_\tau}$
- **Blow-up:** $\left\|\nabla_Y^{k+3}\xi\right\|_{S_\tau} \sim_k \tau^k$, and $\left\|\nabla_Y^{k+5}\alpha\right\|_{S_\tau} \sim_k \tau^k$, $\xi \in \{\mathfrak{f}, \mathfrak{b}\}$.



The $\pm$ spin Teukolsky solutions

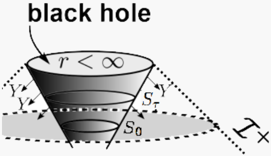
$\|\xi\|_{S_\tau} := \int_{\mathbb{S}^2} r^2 \sin\theta d\theta d\phi \, |\xi|^2, \;\; \|\xi\|_\infty(\tau) := \|\xi\|_{L^\infty(S_\tau)}$

Theorem (A. 2022).

Let $\alpha, \mathfrak{f}, \mathfrak{b}$ and $\underline{\alpha}, \underline{\mathfrak{f}}, \underline{\mathfrak{b}}$ be solutions to the Teukolsky system, then for generic initial data, they all **decay** away from the event horizon $\mathcal{H}^+$, i.e. $\{r \geq r_0\}$, $r_0 > M$. In addition, along $\mathcal{H}^+$ we have

Positive spin:

- **Decay:** $\left\|\nabla_Y^{\leq 2}\mathfrak{f}\right\|_\infty(\tau)$, $\left\|\nabla_Y^{\leq 2}\mathfrak{b}\right\|_\infty(\tau)$, and $\left\|\nabla_Y^{\leq 4}\alpha\right\|_\infty(\tau)$
- **Non-decay:** $\left\|\nabla_Y^3\mathfrak{f}\right\|_{S_\tau}$, $\left\|\nabla_Y^3\mathfrak{b}\right\|_{S_\tau}$, and $\left\|\nabla_Y^5\alpha\right\|_{S_\tau}$
- **Blow-up:** $\left\|\nabla_Y^{k+3}\xi\right\|_{S_\tau} \sim_k \tau^k$, and $\left\|\nabla_Y^{k+5}\alpha\right\|_{S_\tau} \sim_k \tau^k$, $\xi \in \{\mathfrak{f}, \mathfrak{b}\}$.



Negative spin:

- **Decay:** $\left\|\underline{\mathfrak{f}}\right\|_\infty(\tau)$, $\|\underline{\mathfrak{b}}\|_\infty(\tau)$
- **Non-decay:** $\left\|\nabla_Y\underline{\mathfrak{f}}\right\|_{S_\tau}$, $\|\nabla_Y\underline{\mathfrak{b}}\|_{S_\tau}$, and $\|\underline{\alpha}\|_{S_\tau}$
- **Blow-up:** $\left\|\nabla_Y^{k+1}\xi\right\|_{S_\tau} \sim_k \tau^k$, and $\left\|\nabla_Y^{k}\alpha\right\|_{S_\tau} \sim_k \tau^k$, $\xi \in \left\{\underline{\mathfrak{f}}, \underline{\mathfrak{b}}\right\}$.

Estimates for the +spin $\mathfrak{f}, \mathfrak{b}$ Teukolsky solutions

Recall

$$(\star) \qquad \boldsymbol{q}^F = r\nabla_{\partial_r}\left(c_1(r)\cdot\mathfrak{f}\right)$$

Thus,

$$\nabla_{\partial_r}^{k+1}\mathfrak{f} = \tilde{c}_1(r)\cdot\nabla_{\partial_r}^{k}\boldsymbol{q}^F + \mathcal{L}^{k-1}[\boldsymbol{q}^F] + k_1(r)\cdot\mathfrak{f}$$

Using $(\star)$ we show decay estimates for $\mathfrak{f}$. For higher order, we have

- Decay: $\left\|\nabla_{\partial_r}^{k}\mathfrak{f}\right\|_{S^2_{\tau,M}} \lesssim_M \dfrac{1}{\tau^{\left(\frac{4-k}{4}\right)^k}}, \qquad 0 \leq k \leq 2.$
- Non-Decay: $\left\|\nabla_{\partial_r}^{3}\mathfrak{f}\right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} c\left\|H_{\ell=2}[\Psi_2]\right\|_{S^2_{\tau,M}}$
- Blow-up: $\left\|\nabla_{\partial_r}^{k}\mathfrak{f}\right\|_{S^2_{\tau,M}} \gtrsim_{k,M} \left\|H_{\ell=2}[\Psi_2]\right\|_{S^2_{\tau,M}}\cdot\tau^{k-3}, \qquad \forall\ k \geq 4.$

Identical estimates hold for the gauge invariant quantity $\mathfrak{b}$ as well.

Estimates for the +2-spin α Teukolsky solution

Linearized Bianchi equation induces the relating transport equation

$$\nabla_{\partial_r}(r\alpha) = c_1(r) \cdot \nabla\!\!\!/ \widehat{\otimes} \mathfrak{b} + c_2(r) \cdot \mathfrak{f}$$

- $\left\| \nabla\!\!\!/^{k}_{\partial_r} \alpha \right\|_{L^\infty(S^2_{\tau,M})} \lesssim_M \frac{1}{\tau}, \qquad 0 \leq k \leq 2.$

  $\left\| \nabla\!\!\!/^{k+2}_{\partial_r} \alpha \right\|_{L^\infty(S^2_{\tau,M})} \lesssim_M \frac{1}{\tau^{\left(\frac{4-k}{4}\right)^k}}, \qquad 1 \leq k \leq 2.$
- $\left\| \nabla\!\!\!/^{5}_{\partial_r} \alpha \right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} \left(\tilde{c}_2 \left\| H_{\ell=2}[\Psi_2] \right\|^2_{S^2_{\tau,M}} + \tilde{c}_3 \left\| H_{\ell=3}[\Psi_2] \right\|^2_{S^2_{\tau,M}} \right)^{\frac{1}{2}}$
- $\left\| \nabla\!\!\!/^{k}_{\partial_r} \alpha \right\|_{S^2_{\tau,M}} \gtrsim_{k,M} \tau^{k-5}, \qquad \forall\ k \geq 6.$

The negative spin case

The transformation equations leading to Regge–Wheeler solutions are

$$\underline{\boldsymbol{q}}^F = 2r^3\nabla_{\partial_v}(\underline{\mathfrak{f}}) + r\nabla_{\partial_r}(r^2 D\underline{\mathfrak{f}}),$$
$$\underline{\boldsymbol{p}} = 2r^5\nabla_{\partial_v}(\underline{\mathfrak{b}}) + r\nabla_{\partial_r}(r^4 D\underline{\mathfrak{b}}).$$

The negative spin case

The transformation equations leading to Regge–Wheeler solutions are

$$\underline{\boldsymbol{q}}^F = 2r^3 \nabla_{\partial_v}(\underline{\mathfrak{f}}) + r\nabla_{\partial_r}(r^2 D\underline{\mathfrak{f}}),$$
$$\underline{\boldsymbol{p}} = 2r^5 \nabla_{\partial_v}(\underline{\mathfrak{b}}) + r\nabla_{\partial_r}(r^4 D\underline{\mathfrak{b}}).$$

Linearized Bianchi equation for $\underline{\alpha}$ induces the relating equation

$$2\nabla_{\partial_v}\underline{\alpha} + D(r)\nabla_{\partial_r}\underline{\alpha} + D'(r)c(r)\cdot\underline{\alpha} = c_1(r)\cdot\nabla\widehat{\otimes}\underline{\mathfrak{b}} + c_2(r)\cdot\underline{\mathfrak{f}}$$

The negative spin case

The transformation equations leading to Regge–Wheeler solutions are

$$\underline{\boldsymbol{q}}^F = 2r^3\nabla_{\partial_v}(\underline{\mathfrak{f}}) + r\nabla_{\partial_r}(r^2 D\underline{\mathfrak{f}}),$$
$$\underline{\boldsymbol{p}} = 2r^5\nabla_{\partial_v}(\underline{\mathfrak{b}}) + r\nabla_{\partial_r}(r^4 D\underline{\mathfrak{b}}).$$

Linearized Bianchi equation for $\underline{\alpha}$ induces the relating equation

$$2\nabla_{\partial_v}\underline{\alpha} + D(r)\nabla_{\partial_r}\underline{\alpha} + D'(r)c(r)\cdot\underline{\alpha} = c_1(r)\cdot\nabla\widehat{\otimes}\underline{\mathfrak{b}} + c_2(r)\cdot\underline{\mathfrak{f}}$$

Along the event horizon $\mathcal{H}^+$ we have

$$\nabla_{\partial_v}(\underline{\mathfrak{f}}) \sim \underline{\boldsymbol{q}}^F, \qquad \nabla_{\partial_v}(\underline{\mathfrak{b}}) \sim \underline{\boldsymbol{p}}$$
$$\nabla_{\partial_v}\underline{\alpha} \sim \nabla\widehat{\otimes}\underline{\mathfrak{b}} + \underline{\mathfrak{f}}$$

The negative spin case

The transformation equations leading to Regge–Wheeler solutions are

$$\underline{\boldsymbol{q}}^F = 2r^3\nabla_{\partial_v}(\underline{\mathfrak{f}}) + r\nabla_{\partial_r}(r^2 D\underline{\mathfrak{f}}),$$
$$\underline{\boldsymbol{p}} = 2r^5\nabla_{\partial_v}(\underline{\mathfrak{b}}) + r\nabla_{\partial_r}(r^4 D\underline{\mathfrak{b}}).$$

Linearized Bianchi equation for $\underline{\alpha}$ induces the relating equation

$$2\nabla_{\partial_v}\underline{\alpha} + D(r)\nabla_{\partial_r}\underline{\alpha} + D'(r)c(r)\cdot\underline{\alpha} = c_1(r)\cdot\nabla\widehat{\otimes}\underline{\mathfrak{b}} + c_2(r)\cdot\underline{\mathfrak{f}}$$

Along the event horizon $\mathcal{H}^+$ we have

$$\nabla_{\partial_v}(\underline{\mathfrak{f}}) \sim \underline{\boldsymbol{q}}^F, \qquad \nabla_{\partial_v}(\underline{\mathfrak{b}}) \sim \underline{\boldsymbol{p}}$$
$$\nabla_{\partial_v}\underline{\alpha} \sim \nabla\widehat{\otimes}\underline{\mathfrak{b}} + \underline{\mathfrak{f}}$$

Idea: $\nabla_Y\nabla_T\underline{\mathfrak{f}} \sim \nabla_Y\underline{\boldsymbol{q}}^F$ and $\nabla_Y\nabla_T\underline{\mathfrak{b}} \sim \nabla_Y\underline{\boldsymbol{p}}$. We use Teukolsky and arrive at

$$\underline{\mathfrak{f}} \sim \underline{\mathfrak{b}} \sim (\nabla_Y\underline{\boldsymbol{q}}^F + \nabla_Y\underline{\boldsymbol{p}})$$

Compare with the positive spin case

$$\boldsymbol{q}^F \sim \nabla_Y(\mathfrak{f})$$

Nonlinear Scalar Perturbations of Extremal Reissner–Nordström Spacetimes

Y. Angelopoulos[1] · S. Aretakis[2,3] · D. Gajic[4]

$$\begin{cases} \Box_{g_M}\psi = A(x,\psi)\cdot g^{\alpha\beta}\cdot\partial_\alpha\psi\cdot\partial_\beta\psi + \mathcal{O}(|\psi|^k, |T\psi|^k) k \geq 3, \\ \psi|_{\Sigma_{\tau_0}} = \epsilon f, \quad n_{\Sigma_{\tau_0}^{int}}\psi|_{\Sigma_{\tau_0}^{int}} = \epsilon h, \end{cases}$$

$$|Y^2\psi|(v, M, \omega) \simeq \epsilon v \text{ on } \mathcal{H}^+$$

Nonlinear Scalar Perturbations of Extremal Reissner–Nordström Spacetimes

Y. Angelopoulos[1] · S. Aretakis[2,3] · D. Gajic[4]

$$\begin{cases} \Box_{g_M}\psi = A(x,\psi)\cdot g^{\alpha\beta}\cdot\partial_\alpha\psi\cdot\partial_\beta\psi + \mathcal{O}(|\psi|^k, |T\psi|^k) k \geq 3, \\ \psi|_{\Sigma_{\tau_0}} = \epsilon f, \quad n_{\Sigma_{\tau_0}^{int}}\psi|_{\Sigma_{\tau_0}^{int}} = \epsilon h, \end{cases}$$

$$|Y^2\psi|(v, M, \omega) \simeq \epsilon v \ on\ \mathcal{H}^+$$

What happens at the horizon(s) of an extreme black hole?

Keiju Murata[1], Harvey S Reall[2] and Norihiro Tanahashi[3]

Abstract

..... We study numerically the nonlinear evolution of this instability for spherically symmetric perturbations of an extreme Reissner–Nordstrom (RN) black hole. .....

$\partial_r\phi$ decays

$\partial_r^2\phi$ does not decay

Gravitational instability of an extreme Kerr black hole

James Lucietti[a*] and Harvey S. Reall[b†]

Furthermore, we learn that if $\delta\Psi_4$ decays then a transverse derivative of $\delta\Psi_4$ generically does not decay along $\mathcal{H}^+$ and certain second transverse derivatives will blow up along $\mathcal{H}^+$. If $\delta\Psi_0$ and its first 4 derivatives decay then a 5th transverse derivative generically will not decay, and a 6th transverse derivative will blow up. It appears that the Weyl component perturbation $\delta\Psi_4$ exhibits worse behaviour that $\delta\Psi_0$.

$$\delta\Psi_4 \longleftrightarrow \text{extreme curvature component, } \underline{\alpha}$$

$$\delta\Psi_0 \longleftrightarrow \text{extreme curvature component, } \alpha$$

Similar numeric results have been produced by other authors including: Khanna, Burko, Sabhawal, Zimmerman, Gralla, Casals, ...

Gravitational instability of an extreme Kerr black hole

James Lucietti[a*] and Harvey S. Reall[b†]

Furthermore, we learn that if $\delta\Psi_4$ decays then a transverse derivative of $\delta\Psi_4$ generically does not decay along $\mathcal{H}^+$ and certain second transverse derivatives will blow up along $\mathcal{H}^+$. If $\delta\Psi_0$ and its first 4 derivatives decay then a 5th transverse derivative generically will not decay, and a 6th transverse derivative will blow up. It appears that the Weyl component perturbation $\delta\Psi_4$ exhibits worse behaviour that $\delta\Psi_0$.

$$\delta\Psi_4 \longleftrightarrow \text{extreme curvature component, } \underline{\alpha}$$

$$\delta\Psi_0 \longleftrightarrow \text{extreme curvature component, } \alpha$$

Similar numeric results have been produced by other authors including: Khanna, Burko, Sabhawal, Zimmerman, Gralla, Casals, ...

Gravitational instability of an extreme Kerr black hole

James Lucietti[a*] and Harvey S. Reall[b†]

Furthermore, we learn that if $\delta\Psi_4$ decays then a transverse derivative of $\delta\Psi_4$ generically does not decay along $\mathcal{H}^+$ and certain second transverse derivatives will blow up along $\mathcal{H}^+$. If $\delta\Psi_0$ and its first 4 derivatives decay then a 5th transverse derivative generically will not decay, and a 6th transverse derivative will blow up. It appears that the Weyl component perturbation $\delta\Psi_4$ exhibits worse behaviour that $\delta\Psi_0$.

$$\delta\Psi_4 \quad \longleftrightarrow \quad \text{extreme curvature component, } \underline{\alpha}$$

$$\delta\Psi_0 \quad \longleftrightarrow \quad \text{extreme curvature component, } \alpha$$

Similar numeric results have been produced by other authors including: Khanna, Burko, Sabhawal, Zimmerman, Gralla, Casals, ...

- **+2 Spin:** The same asymptotic behavior is realized along the horizon $\mathcal{H}^+$, when compared to ERN.

Gravitational instability of an extreme Kerr black hole

James Lucietti[a*] and Harvey S. Reall[b†]

Furthermore, we learn that if $\delta\Psi_4$ decays then a transverse derivative of $\delta\Psi_4$ generically does not decay along $\mathcal{H}^+$ and certain second transverse derivatives will blow up along $\mathcal{H}^+$. If $\delta\Psi_0$ and its first 4 derivatives decay then a 5th transverse derivative generically will not decay, and a 6th transverse derivative will blow up. It appears that the Weyl component perturbation $\delta\Psi_4$ exhibits worse behaviour that $\delta\Psi_0$.

$\delta\Psi_4 \longleftrightarrow$ extreme curvature component, $\underline{\alpha}$

$\delta\Psi_0 \longleftrightarrow$ extreme curvature component, α

Similar numeric results have been produced by other authors including: Khanna, Burko, Sabhawal, Zimmerman, Gralla, Casals, ...

- **+2 Spin:** The same asymptotic behavior is realized along the horizon $\mathcal{H}^+$, when compared to ERN.
- **-2 Spin:** The linearized gravity in ERN exhibits more "unstable" behavior than this of EK (axisymmetric-linear perturbations).

Gravitational instability of an extreme Kerr black hole

James Lucietti[a*] and Harvey S. Reall[b†]

Furthermore, we learn that if $\delta\Psi_4$ decays then a transverse derivative of $\delta\Psi_4$ generically does not decay along $\mathcal{H}^+$ and certain second transverse derivatives will blow up along $\mathcal{H}^+$. If $\delta\Psi_0$ and its first 4 derivatives decay then a 5th transverse derivative generically will not decay, and a 6th transverse derivative will blow up. It appears that the Weyl component perturbation $\delta\Psi_4$ exhibits worse behaviour that $\delta\Psi_0$.

$$\delta\Psi_4 \longleftrightarrow \text{extreme curvature component, } \underline{\alpha}$$

$$\delta\Psi_0 \longleftrightarrow \text{extreme curvature component, } \alpha$$

Similar numeric results have been produced by other authors including: Khanna, Burko, Sabhawal, Zimmerman, Gralla, Casals, ...

- **+2 Spin:** The same asymptotic behavior is realized along the horizon $\mathcal{H}^+$, when compared to ERN.
- **-2 Spin:** The linearized gravity in ERN exhibits more “unstable” behavior than this of EK (axisymmetric-linear perturbations).
  - In the non-axisymmetric case, the Hartle-Hawking Weyl scalar ψ_4 seems to blows up asymptotically along $\mathcal{H}^+$. (Z.G.C)

Thank You
For Your Attention